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AN OPTIMAL EIGHTH ORDER OSTROWSKI'S - BASED ITERATIVE SCHEME FOR NONLINEAR SYSTEMS ARISING IN BIOLOGICAL AND NEUROSCIENCE MODELS

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Abstract: This paper presents an optimal eighth order multi-step iterative method based on Ostrowski's fourth-order approach, designed for solving nonlinear equations and systems with high efficiency and accelerated convergence. The method requires only three function evaluations and one derivative evaluation per iteration, achieving high precision while reducing computational costs. It satisfies the Kung–Traub optimality conjecture for $n = 4$, ensuring theoretically optimal convergence. This method is particularly suitable for applications in mathematical biology and neuroscience, such as modeling interacting with neuronal populations, gene regulatory networks, and multi-compartment physiological systems. Numerical examples demonstrate that the proposed approach provides fast, accurate, and reliable solutions, outperforming classical methods and offering a versatile tool for analyzing complex biological and neural systems.

Keywords: Iterative methods, efficiency, non-linear equations, system of non-linear equations, interpolation technique, Kung–Traub conjecture.

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1. INTRODUCTION

The nonlinear algebraic and transcendental equations play a central role in numerous scientific and engineering projects as elaborate tools of representation and understanding of many complex phenomena [1-7]. The equations of nonlinear systems may easily have several roots, and therefore, direct methods do not work any longer, which creates the need to resort to numerical methods. Numerous methods have been devised to process such systems with greater accuracy. One of the most popular and successfully used iterative schemes is that of Newton's. Its iterative formula

$$y_k = x_k - [F'(x_k)]^{-1} F(x_k) \tag{1}$$

In which the Jacobian matrix of F is denoted by $F\ '\ (x)$. New developments in numerical techniques have been aimed at improving the speed of convergence with a minimum amount of computation. One way of doing this is to use divided differences instead of the Jacobian matrix. These include Steffensen's method [8], which employs frozen differences as an alternative to the Jacobian matrix. Parametric methods of Ostrowski's and Chun type [9-11], which combine divided differences. The approach of Ostrowski's is regarded as the best based on the Kung-Traub conjecture.

The immediate application of divided differences can however decrease the order of convergence. To reduce this, scientists such as Amiri [12] came up with methods of maintaining the convergence order by choosing a good order m. Kung and Traub [13] reasoned that non-memory multipoint iteration methods have the optimal order of 2^{n-1}, so the family conjectures as Kung-Traub did in the special case n = 4.

Instead, Taylor series expansion-based methods have been suggested to obtain appropriate expressions in place of the Jacobian matrix [14-17]. The methods are stable, accurate, and efficient in terms of computation.

More recently, new methods have been developed to improve convergence of iterative processes with little extra work. To illustrate, [18] described how to raise the convergence order of an iterative scheme of p to $p+2$ by the addition of a Newton-type step requiring just possess one extra function evaluation. These developments demonstrate the ongoing research to improve the numerical methods of solving nonlinear systems. The iterative formulation is as follows.

$$\left.\begin{array}{c} y_k = x_k - [F'(x_k)]^{-1}F(x_k) \\ z_k = \phi(x_k, y_k) \\ x_{k+1} = z_k - [F'(z_k)]^{-1}F(z_k) \end{array}\right\} \quad (2)$$

Any two-step method can be enhanced by introducing an additional step, which increases its convergence order by two units. Increasing the order of convergence in three units of iterative methods for solving nonlinear systems by Cardero [19] introduces another useful technique, which increases the order by 3 units. The first two steps can be taken from any existing method, and the third step enhances the convergence by 3 units.

In our work, the proposed method is based on an efficient three-step method that is designed for higher accuracy and lower complexity. The first two steps follow Ostrowski's method, while the third step applies Newton's method. We use the Hermite technique [20] in the third step, so order of convergence increases from $p+3$ to $p+4$, while maintaining low computational costs.

For numerical comparison, a set of recently introduced iterative algorithms with different orders of convergence is discussed in the next subsection. Notably, Xiao and Yin [21] developed a three-stage iterative scheme achieving fifth-order convergence, which is denoted by (M1) in this study.

$$\left.\begin{array}{l} y_k = x_k - F'(x_k)^{-1}F(x_k) \\ z_k = y_k - F'(x_k)^{-1}F(y_k) \\ x_{k+1} = z_k + [F'(y_k)^{-1} - 2F'(x_k)^{-1}]F(z_k) \end{array}\right\}(\text{M1}) \quad (3)$$

The sixth-order method proposed by Soleymani et al. [22], identified by us as M2 is a three-step Jarratt-type method, the iterative expression of which is:

$$\left.\begin{array}{l} y_k = x_k - \frac{2}{3}F'(x_k)^{-1}F(x_k) \\ z_k = x_k - \frac{1}{2}SF'(x_k)^{-1}F(x_k) \\ x_{k+1} = z_k - \frac{1}{4}S^2F'(x_k)^{-1}F(z_k) \end{array}\right\}(\text{M2}) \quad (4)$$

Where $S = [3F'(y_k) - F'(x_k)]^{-1}[3F'(y_k) + F'(x_k)]$

The seventh-order method designed by Alicia Cordero [19], which we denote by M3, has the iterative expression:

$$\left.\begin{array}{c} y_k = x_k - F'(x_k)^{-1}F(x_k) \\ z_k = x_k - [(\frac{1}{6}G(n)^3 + 2G(n)^2 + G(n) + I)]F'(x_k)^{-1}F(x_k) \\ x_{k+1} = z_k - [(\frac{7}{2}I + L(n))(-4I + \frac{3}{2}L(n))]F'(x_k)^{-1}F(z_k) \end{array}\right\}(\text{M3}) \quad (5)$$

Where, $G(n) = [I - F'(x_k)]^{-1}[x_k, y_k; F]$

$$L(n) = [F'(x_k)]^{-1} F'(y_k)$$

Xiao and Yin [23-25] took a sixth-order approach of Sharma and Arora and created an eighth-order approach, which is going to be applied to this paper in comparison.

$$\left.\begin{aligned} y_k &= x_k - \tfrac{2}{3} F'(x_k)^{-1} F(x_k) \\ z_k &= x_k - [\ \tfrac{23}{8} I + \bigl(F'(x_k)^{-1} F'\,(y_k)\bigr) \Bigl(-3I + \tfrac{9}{8} (F'(x_k)^{-1} F'(y_k)\ )\Bigr] F'(x_k)^{-1} F(x_k) \\ w_k &= z_k - [\ \tfrac{5}{2} I - (\tfrac{3}{2} F'(x_k)^{-1} F'(y_k)] F'(x_k)^{-1} F(z_k) \\ x_{k+1} &= w_k - \tfrac{1}{2} (3\, F'(y_k)^{-1} - \ F'({x_k}^{k})^{-1})\, F(w_k) \end{aligned}\right\} \text{(M4)} \qquad (6)$$

The structure of the paper is as follows: Section 2 begins with the design of iterative techniques for nonlinear equations. The extension of the results to systems of nonlinear equations using the Hermite interpolation technique is the focus of Section 3. In Section 4, we validate the theoretical results after testing and comparing proposed method with other well-known approaches on various problems. In Section 5, we wrap up the paper with a few closing thoughts.

2. THE PROPOSED METHOD'S DESIGN FOR NONLINEAR EQUATIONS

The proposed method is based on an efficient three-step method that is designed for higher accuracy and lower complexity.

Now consider the first two steps to follow Ostrowski's method [9-10].

$$\left.\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f'(x_k)} \\ z_k &= \ y_k - \frac{f(x_k)}{f(x_k) - 2f(y_k)} \frac{f(y_k)}{f'(x_k)} \\ k &= 1,2,3\ \ldots \end{aligned}\right\} \qquad (7)$$

The method achieves optimal fourth-order convergence with an efficiency index of $I \approx 1.5874$. In the third step of iterative method (7), Newton's method is applied as follows. Consequently, equation (8) represents a sixth-order three-step iterative method with five evaluations.

$$\left.\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f'(x_k)} \\ z_k &= \ y_k - \frac{f(x_k)}{f(x_k) - 2f(y_k)} \frac{f(y_k)}{f'(x_k)} \\ x_{k+1} &= z_k - \frac{f(z_k)}{f'(z_k)} \\ k &= 1,2,3\ \ldots \end{aligned}\right\} \qquad (8)$$

Furthermore, to derive a method with a higher efficiency index and order, we will replace $f'(z_k)$ by the Hermite interpolation technique $f'(z_k) \approx h_3'(z_k)$ [20]. We get our Proposed Method:

$$\left.\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f'(x_k)} \\ z_k &= y_k - \frac{f(x_k)}{f(x_k)-2f(y_k)}\frac{f(y_k)}{f'(x_k)} \\ x_{k+1} &= z_k - \frac{f(z_k)}{\left(2.\left[\frac{f(z_k)-f(x_k)}{z_k-x_k}\right]-\left[\frac{f(y_k)-f(x_k)}{y_k-x_k}\right]\right)+\left[\frac{f(z_k)-f(y_k)}{z_k-y_k}\right]+\frac{y_k-z_k}{y_k-x_k}\left(\left[\frac{f(y_k)-f(x_k)}{y_k-x_k}\right]-f'(x_k)\right)} \end{aligned}\right\} \tag{9}$$

This is an Efficient Eight-Order Method to solve nonlinear equations based on divided differences with Order: 8, Functional Evaluations: 4, and EI $\approx$ 1.6817

2.1 Convergence Analysis

Theorem 1. Suppose a is a simple root of a smooth real function f on an open interval $D \subset \mathbb{R}$ containing the initial approximation x_0. Then the algorithm in (9) converges at order eight and, for each iteration, requires three function evaluations and a single first-derivative evaluation, without using higher derivatives.

Proof: Using a Taylor series expansion, $f(x_k)$ can be written in the form:

$$f(x_k) = \sum_{m=0}^{\infty} \frac{f^m(a)}{m!}(x_k - a)^m$$

$$f(x_k) = f(a) + f'(a)(x_k - a) + \frac{f''(a)}{2!}(x_k - a)^2 + \frac{f'''(a)}{3!}(x_k - a)^3 + \ldots\ldots \tag{10}$$

For simplicity, we assume that $c_k = \left(\frac{1}{k!}\right)\frac{f^k(a)}{f'(a)}, k \geq 2.$

and assume that $e_k = x_k - a$. Thus, we have

1st Step:

$$f(x_k) = f'(a)\left(\left(e_k + c_2e_k^2 + c_3e_k^3 + c_4e_k^4 + c_5e_k^5 + c_6e_k^6 + c_7e_k^7 + c_8e_k^8 + O(e_k^9)\right)\right) \tag{11}$$

And its derivative

$$f'(x_k) = f'(a)(1 + 2c_2e_k + 3c_3e_k^2 + 4c_4e_k^3 + 5c_5e_k^4 + 6c_6e_k^5 + 7c_7e_k^6 + 8c_8e_k^7 + \cdots + O(e_k^9)) \tag{12}$$

Divide eq (11) by eq (12), we have

$$\frac{f(x_k)}{f'(x_k)} = e_k - c_2{e_k}^2 + 2{c_2}^2{e_k}^3 - 2c_3{e_k}^3 - 4{c_2}^3{e_k}^4 + 7c_2c_3{e_k}^4 - 3c_4{e_k}^4 + \cdots + O(e_k^9) \tag{13}$$

$$y_k = x_k - \frac{f(x_k)}{f'(x_k)} = c_2{e_k}^2 - 2{c_2}^2{e_k}^3 + 2c_3{e_k}^3 + \cdots + O(e_k^9) \tag{14}$$

2nd Step:

$$f(y_k) = f'(a)(c_2{e_k}^2 - 2{c_2}^2{e_k}^3 + 2c_3{e_k}^3 + 5{c_2}^3{e_k}^4 - 7c_2c_3{e_k}^4 + 3c_4 + \cdots + O(e_k^9) \tag{15}$$

$$\frac{f(x_k)}{f(x_k)-2f(y_k)} = 1 + 2c_2e_k - 2{c_2}^2{e_k}^2 + 4c_3{e_k}^2 - 4c_2c_3{e_k}^3 + 6c_4{e_k}^3 + \cdots + O\left(e_k^9\right) \tag{16}$$

$$\frac{f(y_k)}{f'(x_k)} \ = \ c_2{e_k}^2 - 4{c_2}^2{e_k}^3 + 2c_3{e_k}^3 + 13{c_2}^3{e_k}^4 - 14c_2c_3{e_k}^4 + 3c_4{e_k}^4 \ + \cdots + O\left(e_k^9\right) \tag{17}$$

$$z_k = \ y_k - \frac{f(x_k)}{f(x_k)-2f(y_k)}\frac{f(y_k)}{f'(x_k)} = ({c_2}^3 - c_2c_3){e_k}^4 + \cdots + O\left(e_k^9\right) \tag{18}$$

3rd Step:

$$(z_k) = f'(a)(({c_2}^3 - c_2c_3){e_k}^4 - 2(2{c_2}^4 - 4{c_2}^2c_3 + {c_3}^2 + c_2c_4){e_k}^5 + \cdots + O\left(e_k^9\right) \tag{19}$$

$$\left[\frac{f(z_k)-f(x_k)}{z_k-x_k}\right] = 1 + c_2e_k + c_3{e_k}^2 + c_4{e_k}^3 + {c_2}^4{e_k}^4 - {c_2}^2c_3{e_k}^4 + c_5{e_k}^4 + \cdots + O\left(e_k^9\right) \tag{20}$$

$$\left[\frac{f(z_k)-f(y_k)}{z_k-y_k}\right] = 1 + {c_2}^2{e_k}^2 - 2{c_2}^3{e_k}^3 + 2c_2c_3{e_k}^3 + 5{c_2}^4{e_k}^4 - 7{c_2}^2c_3{e_k}^4 + \cdots + O\left(e_k^9\right) \tag{21}$$

$$\left[\frac{f(z_k)-f(y_k)}{z_k-y_k}\right] = 1 + {c_2}^2{e_k}^2 - 2{c_2}^3{e_k}^3 + 2c_2c_3{e_k}^3 + 5{c_2}^4{e_k}^4 - 7{c_2}^2c_3{e_k}^4 + \cdots + O\left(e_k^9\right) \tag{22}$$

$$\frac{y_k-z_k}{y_k-x_k} = -c_2e_k + {c_2}^2{e_k}^2 - 2c_3{e_k}^2 + 2c_2c_3{e_k}^3 - 3c_4{e_k}^3 - 2{c_2}^4{e_k}^4 + 3{c_2}^2c_3{e_k}^4 + \cdots + O\left(e_k^9\right) \tag{23}$$

$$\frac{f(z_k)}{\left(2.\left[\frac{f(z_k)-f(x_k)}{z_k-x_k}\right]-\left[\frac{f(y_k)-f(x_k)}{y_k-x_k}\right]\right)+\left[\frac{f(z_k)-f(y_k)}{z_k-y_k}\right]+\frac{y_k-z_k}{y_k-x_k}.\left(\left[\frac{f(y_k)-f(x_k)}{y_k-x_k}\right]-f'(x_k)\right)} =$$

$$= 1 + 2{c_2}^4{e_k}^4 - 2{c_2}^2c_3{e_k}^4 + c_2c_4{e_k}^4 + \cdots + O\left(e_k^9\right) \tag{24}$$

$$x^{k+1} = z^k - \frac{f(z^k)}{\left(2.\left[\frac{f(z^k)-f(x^k)}{z^k-x^k}\right]-\left[\frac{f(y^k)-f(x^k)}{y^k-x^k}\right]\right)+\left[\frac{f(z^k)-f(y^k)}{z^k-y^k}\right]+\frac{y^k-z^k}{y^k-x^k}.\left(\left[\frac{f(y^k)-f(x^k)}{y^k-x^k}\right]-f'(x)\right)}$$

$$= ({c_2}^7 - 2{c_2}^5c_3 + {c_2}^3{c_3}^2 + {c_2}^4c_4 - {c_2}^2c_3c_4){e_k}^8 + O\left(e_k^9\right) \tag{25}$$

It is worth noting that the iterative method defined in equation (9) exhibits an eighth-order rate of convergence while demanding only three function evaluations and one evaluation of the first derivative per iteration, leading to an efficiency index of 1.6817 [26].

3. EXTENSION TO SYSTEMS OF NONLINEAR EQUATIONS

In this section, we generalize the above method to compute multiple roots of a system of nonlinear equations. The proposed method is described in the following three steps.

$$\left.\begin{array}{c} y_k = x_k - F^{'}(x_k)^{-1}F(x_k) \\ z_k = y_k - [2[x_k, y_k; F] - F^{'}(x_k)]^{-1}F(y_k) \\ x_{k+1} = z_k - \left((2[x_k, z_k; F] - [x_k, y_k; F]) + [y_k, z_k; F] + \frac{y_k-z_k}{y_k-x_k}\left([x_k, y_k; F] - F^{'}(x_k)\right)\right)^{-1} F(z_k) \end{array}\right\} \tag{26}$$

This is generalization of efficient eight order Method to system of nonlinear equations based on divided differences with Order: 8, Functional evaluations: $4n^2$ and Cost: $\frac{1}{3}n^3 + 8n^2 - \frac{1}{3}n$.

3.1 Convergence Analysis

Theorem 2. Let $F: D \subset \mathbb{R}^m \to \mathbb{R}^m$be sufficiently differentiable on a convex set D, and assume $a \in D$satisfies $F(x) = 0$. Provided that $F'(x)$is continuous and non-singular at a, the iterative sequence $\{x_k\}_{k\geq 0}$ converges to a with at least eight order, with the following error formula:

$$e_{k+1} = A_2^4 e_k^8 + O(e_n^9) \tag{27}$$

Let α be a simple root of $F(x) = 0$.

Where,

$e_k = x_k - \alpha, e_y = y_k - \alpha, e_z = z_k - \alpha.$

$e_{k+1} = x_{k+1} - a$,

Proof: Through the Taylor series representation of $F(x)$ and its derivatives at $\bar{x}$,

$$F(x_k) = F'(\alpha)(e_k + A_2 e_k^2 + A_3 e_k^3 + A_4 e_k^4 + A_5 e_k^5 + A_6 e_k^6 + A_7 e_k^7 + A_8 e_k^8 + O(e_k^9) \tag{28}$$

$$F'(x_k) = F'(\alpha)(I + 2A_2 e_k + 3A_3 e_k^2 + 4A_4 e_k^3 + \cdots O(e_k^9)) \tag{29}$$

where $A_p = \frac{1}{p!} F^{(p)}(\alpha)\ p \geq 2$

Let us consider

$$F'(x)^{-1} = I - 2A_2 e_k + (4A_2^2 - 3A_3){e_k}^2 + (-8A_2^3 + 12A_2A_3 - 4A_4){e_k}^3 + O({e_k}^4) \tag{30}$$

$\therefore F'(x_k)^{-1} F(x_k) = I$

1st Step:

$$y_k = x_k - F'(x_k)^{-1} F(x_k) \tag{31}$$

Using the above series (31) after simplification, we get.

$$e_y = e_k - (I - 2A_2 e_k + (4A_2^2 - 3A_3){e_k}^2 + \cdots)(e_k + A_2 {e_k}^2 + A_3 {e_k}^3 + A_4 {e_k}^4 + \cdots) \tag{32}$$

$$e_y = A_2 {e_k}^2 + 2(A_3 - A_2^2){e_k}^3 + (3A_2^3 - 5A_2A_3 + 2A_4){e_k}^4 \dots\dots + O({e_k}^6) \tag{33}$$

$$e_y = A_2 {e_k}^2 + O({e_k}^3) \tag{34}$$

with the leading second-order constant

2nd Step:

$$z_k = y_k - [2[x_k, y_k; F] - F'(x_k)]^{-1} F(y_k) \tag{35}$$

let $M_k = 2[x_k, y_k; F] - F'(x_k)$

Using the Genochi–Hermite representation

$$[x_k, y_k; F] = \int_0^1 F'\big(x + t(y_k - x_k)\big)\ dt \tag{36}$$

$$[x_k, y_k; F] = F'(\alpha)(I + A_2\big(e_k + e_y\big) + (A_2^2 + A_3)e_k{}^2 + \theta_3 e_k{}^3 + \theta_4 e_k{}^4 + O(e_k{}^5)) \tag{37}$$

where θ_3, θ_4 are multilinear combinations of $A_2, \ldots, A_5$.

And

$$M_k = I + (2A_2^2 - A_3)e_k{}^2 + \cdots + O(e_k{}^4) \tag{38}$$

$$M_k = I + m_2 e_k{}^2 + m_3 e_k{}^3 + O(e_k{}^4) \tag{39}$$

Where $\quad m_2 = 2A_2^2 - A_3$

hence

$$M_k^{-1} = I - m_2 e_k{}^2 - (m_3 - m_2^2)e_k{}^3 + O(e_k{}^4) \tag{40}$$

$$F(y_k) = e_y + A_2 e_y^2 + A_3 e_y^3 + O(e_k{}^6) \tag{41}$$

$$= A_2 e_k{}^2 + B_3 e_k{}^3 + (C_4 + A_2^3)e_k{}^4 + O(e_k{}^5) \tag{42}$$

Then

$$e_z = e_y - M_1^{-1} F(y_k) \tag{43}$$

$$e_z = (A_2 e_k{}^2 + B_3 e_k{}^3 + C_4 e_k{}^4 + \cdots) - (A_2 e_k{}^2 + B_3 e_k{}^3 + (D_4 - m_2 A_2)e_k{}^4 + \cdots) + O(e_k{}^5) \tag{44}$$

$$e_z = (C_4 - D_4 + m_2 A_2)e_k{}^4 + O(e_k{}^5) \tag{45}$$

Using $D_4 = C_4 + A_2^3$ and $m_2 = 2A_2^2 - A_3$

Therefore,

$$e_z = K_4 e_k{}^4 + K_5 e_k{}^5 + K_6 e_k{}^6 + K_7 e_k{}^7 + O(e_k{}^8) \tag{46}$$

with the leading fourth-order constant

Where $\quad K_4 = (C_4 - (C_4 + A_2^3) + (2A_2^2 - A_3)A_2) = (A_2^2 - A_3)A_2$

$$e_z = K_4 e_k{}^4 + O(e_k{}^5) \tag{47}$$

The higher coefficients K_5, K_6, K_7 are multilinear polynomials in $A_2, \ldots, A_5$.

3rd Step:

$$x_{k+1} = z_k - (H_k)^{-1} F(z_k) \tag{48}$$

Where $\quad H_k = \big(2[x_k, z_k; F] - [x_k, y_k; F]\big) + [y_k, z_k; F] + \frac{y_k - z_k}{y_k - x_k}\Big([x_k, y_k; F] - F'(x_k)\Big)$

Let us again consider the first-order divided difference operator of Fas a mapping

$\mathbb{R}^m \times \mathbb{R}^m \to L(\mathbb{R}^m)$,

$$[x_k + h, x_k; F] = \int_0^1 F'(x_k + th)\, dt, \forall (x, h) \in \mathbb{R}^m \times \mathbb{R}^m.$$

Expanding $F'(x + th)$about x via a Taylor series and integrating leads to

$$\int_0^1 F'(x + t(y_k - x_k))dt = [x_k, y_k; F] = F'(\alpha)(I + A_2(e_k + e_y) + O(e_k^2)) \tag{49}$$

$$\int_0^1 F'\big(x + t(y_k - z_k)\big)dt = [y_k, z_k; F] = F'(\alpha)(I + A_2\big(e_y + e_z\big) + O(e_k^3)) \tag{50}$$

And

$$\int_0^1 F'\big(x + t(z_k - x_k)\big)dt = [x_k, z_k; F] = F'(\alpha)(I + A_2(e_k + e_z) + O(e_k^2)) \tag{51}$$

Substituting $[x_k, y_k; F], [y_k, z_k; F]$ *and* $[x_k, z_k; F]$ in (28) and setting the third step of the proposed method, we get,

$$H_k = F'(\alpha)(I + A_2e_z + A_3e_y^2 + \cdots + O(e_k^9)) \tag{52}$$

Hence

$$H_k^{-1} = (I - A_2e_z - A_3e_y^2)F'(\alpha)^{-1} \ldots + O(e_k^9)) \tag{53}$$

$$F(z_k) = F'(\alpha)(e_z + A_2e_z^2) + \cdots + O(e_k^9)) \tag{54}$$

$$e_{k+1} = e_z - H_k^{-1}F(z_k) \tag{55}$$

Substitute the expansions of $H_k^{-1}$and $F\big(z^k\big)$ in (35). We get,

$$e_{k+1} = e_z - (I - A_2e_z - A_3e_y^2)(e_z + A_2e_z^2) + O(e_k^9) \tag{56}$$

$$e_{k+1} = A_2^4e_k^8 + O(e_k^9) \tag{57}$$

$e_y = O(e_k{}^2), e_z = O(e_k{}^4)$ and $e_{x+1} = O(e_k{}^8)$

Hence, the method given in equation (26) possesses eighth-order convergence for the three-step approach [27]. The proof is therefore complete.

3.2 Basic Definitions

The performance of iterative schemes is often assessed using efficiency indices that relate the convergence rate to computational cost. Ostrowski (1966) [10] introduced an efficiency index expressed as:

$$IE = p^{1/d}, \tag{58}$$

where $p$is the convergence order and $d$represents the number of functional evaluations required per iteration. Traub (1974) [13] extended this concept by defining the operational efficiency index,

$$IO = p^{1/op}, \tag{59}$$

where op quantifies the number of operations, typically in terms of multiplications or basic arithmetic steps, reflecting the method's overall computational workload.

A more complete analysis of iterative methods can be obtained by merging the efficiency index with the operational efficiency index. This unified quantity is referred to as the Computational Efficiency Index (CEI) and is written as

$$CEI = p^{1/(d+op)}, \tag{60}$$

where d is the number of function evaluations and op is the number of operations per iteration. The CEI was introduced by Cordero (2010).

This refers to a special class of Ostrowski and Chun parametric equations. This is an 4th order optimal method in terms of kung and Traub conjecture [13], requiring 3 function evaluations per iteration.$2^{\mathrm{n}-1}$ where 'n' is the number of function evaluations. If $n = 3$, then, $2^{3-1} = 2^2 = 4$, Which is optimal. Apply as for the proposed method 4 function evaluations,

Then $2^{4-1} = 2^3 = 8$

Which is optimal.

3.2.1 Comparison of Computational Efficiency Index (C.E.I) with Existing Methods

Table 1: Displays the computed efficiency index results for both the new method and earlier methodologies.

Method	**Order**	**Function Evaluation**	**C.E.I**
M1	5	$2n^2 + 3n$	$\frac{1}{5^{(\frac{2}{3}n^3+6n^2+\frac{7}{3}n)}}$
M2	6	$2n^2 + 2n$	$\frac{1}{6^{(\frac{2}{3}n^3+10n^2+\frac{4}{3}n)}}$
M3	7	$3n^2 + n$	$\frac{1}{7^{(\frac{1}{3}n^3+26n^2+\frac{2}{3}n)}}$
M4	8	$2n^2 + 3n$	$\frac{1}{8^{(\frac{2}{3}n^3+12n^2+\frac{7}{3}n)}}$
PM	8	$4n^2$	$\frac{1}{8^{(\frac{1}{3}n^3+8n^2-\frac{1}{3}n)}}$

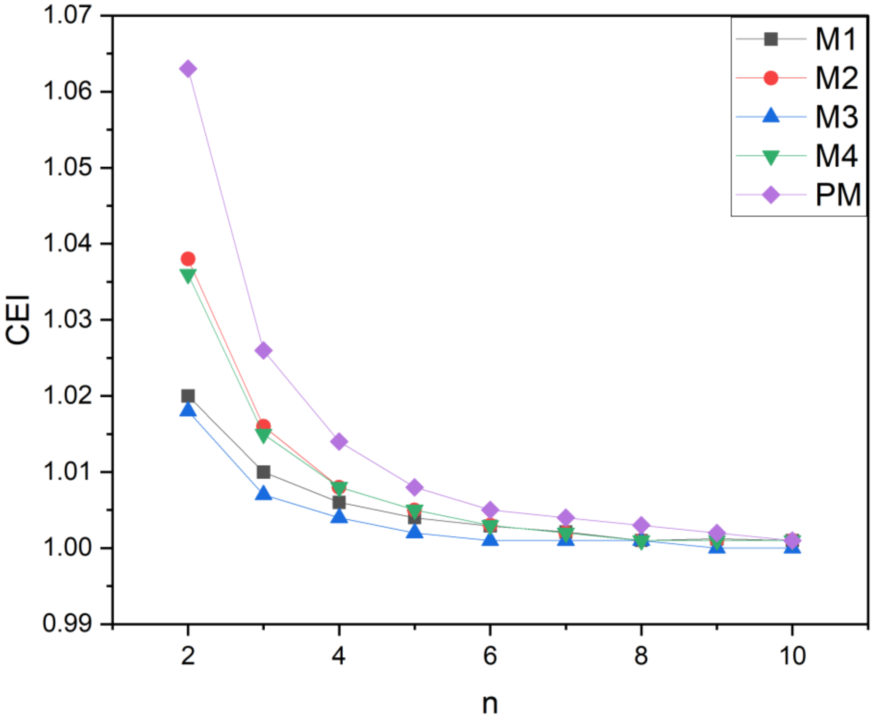
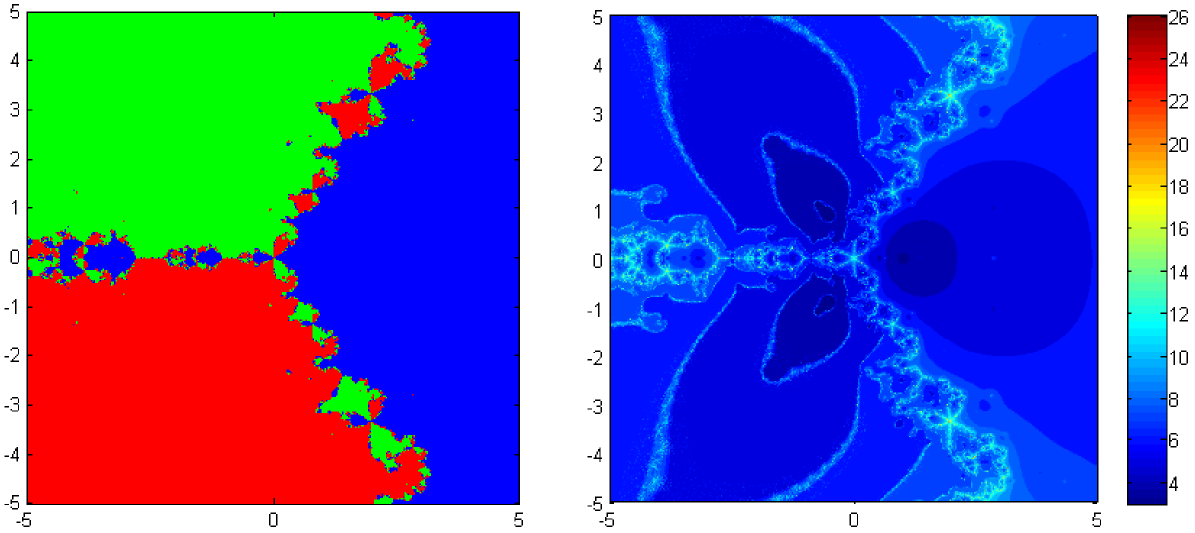


Figure 1: Graphical Representation of Computational Efficiency Index (C.E.I).

The graphical representation of the computational Efficiency Index of the proposed method and comparison methods has been provided in Figure 1. The x-axis is used to indicate the various values of n, and the y-axis is used to indicate the Index of computational efficiency. The highest efficiency of the proposed approach is compared to any other method of comparison.

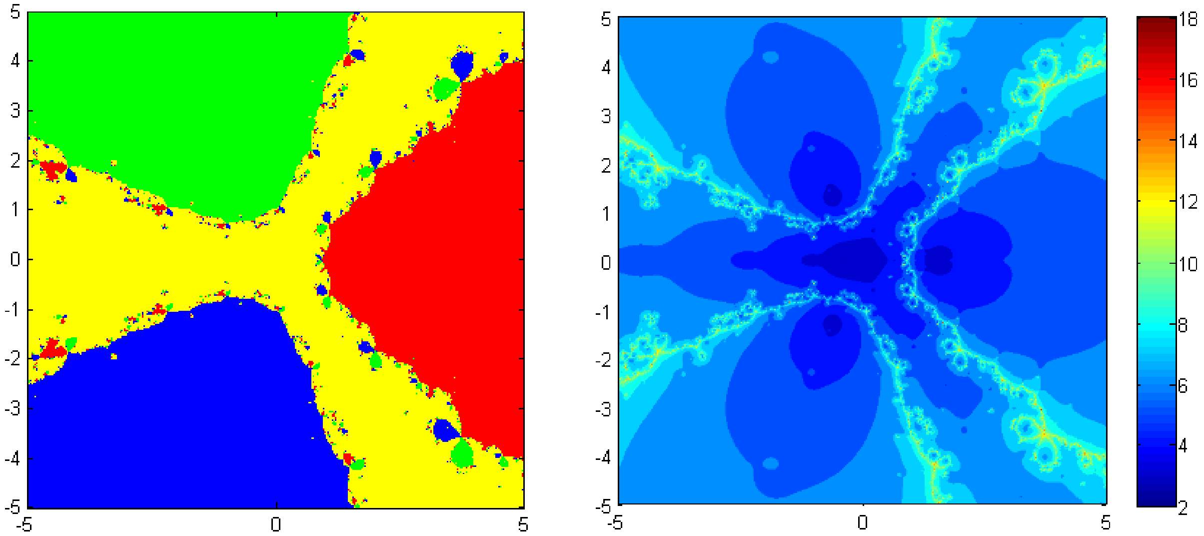
4. STABILITY ANALYSIS

The complex dynamics of the optimal proposed approach (9) is examined in order to analyze the stability of the optimal approach to the one-dimensional case. Prerequisite to this subject may be found in [20], and many works may also be found on stability, at [20]. In these works, and rationale function $R: \mathbb{C} \to \mathbb{C}$ is thought in the Riemann arena $\mathbb{C}$. The resulting expression of $p(z) = (z - a)(z - b) \ldots$, where $a, b, \ldots \in \mathbb{C}$, was used on (9) is the rational operator. The last is based on a 600×600 grid of the square $D = [-5, 5] \times [-5, 5] \subset C$ and give to every point $z_0 \in \mathbb{C}$ the simple root with which the orbit of the iterative method beginning at z_0 converges. We indicate the point black in the case the orbit fails to converge to a root in the sense that upon at most 25 successive iterations the distance to any of the roots is greater than 10^{-9}. By so doing, we differentiate polynomiographs, in relation to color.



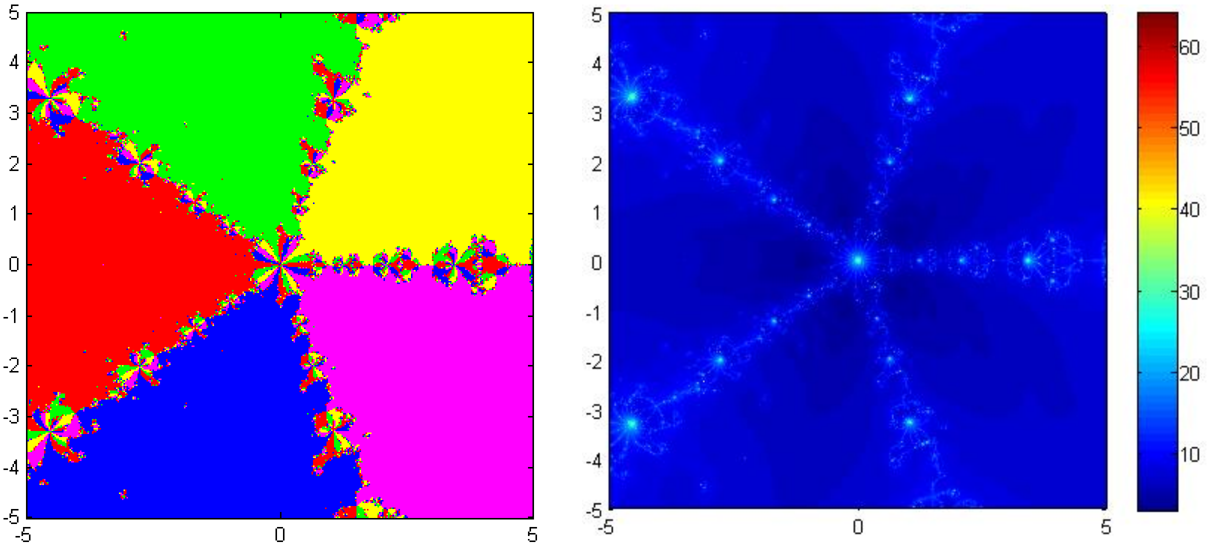
$\mathrm{Root}(p_1(z))$ $\mathrm{Iteration}(p_1(z))$

$p_1(z) = z^3 - 1$



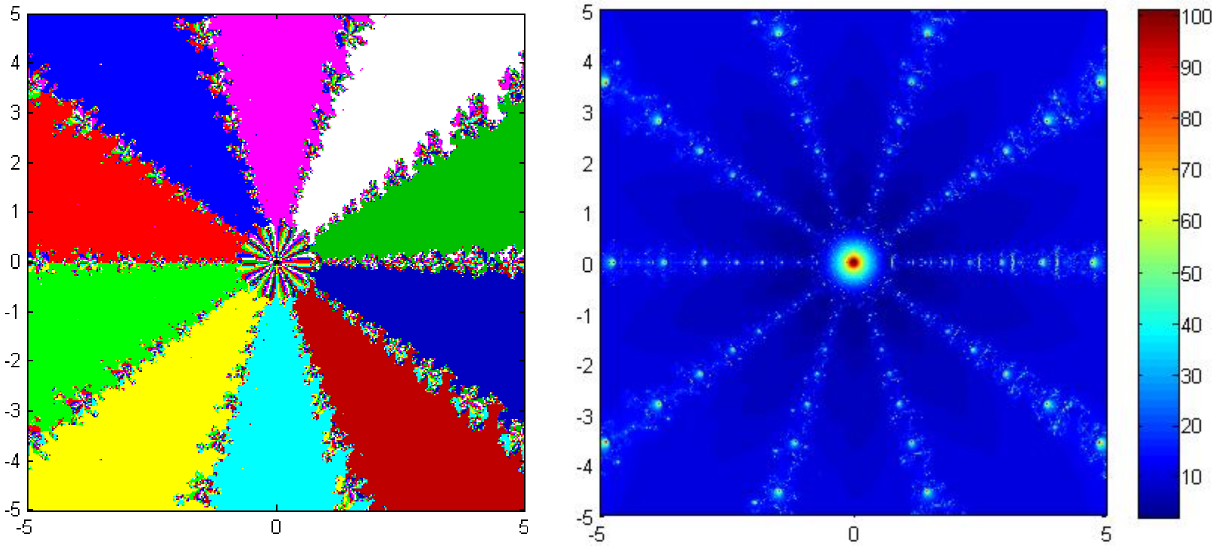
$\mathrm{Root}(p_2(z))$ $\mathrm{Iteration}(p_2(z))$

$p_2(z) = z^4 - 3z - 1$



Root$\left(p_3(z)\right)$ Iteration$\left(p_3(z)\right)$

$$p_3(z) = z^5 + 1$$



Root$\left(p_4(z)\right)$ Iteration$\left(p_4(z)\right)$

$$p_4(z) = z^{10} + 1$$

Figure 2: Polynomiographs obtained by the proposed method

5. NUMERICAL RESULTS

Example 1. Nonlinear System in Mathematical Biology and Neuroscience

Consider a nonlinear system that can arise in models of interacting biological populations or coupled neuronal dynamics:

$$F(x,y)=\begin{cases} x^2-x-y^2-1=0, \\ \sin(x)+y=0. \end{cases} \tag{41}$$

Here, x and y can represent, for instance:

- x: the normalized activity of a neuron or neural population,
- y: a feedback variable, such as a synaptic input or neurotransmitter concentration.

This system captures nonlinear interactions similar to those seen in neuronal firing models or in predator-prey-like population dynamics in biological networks. Solving this system can help determine equilibrium points of the system or the steady-state response of a neural circuit. Iterative methods, such as the proposed three-step eighth-order method, can be used to compute (x,y) with high accuracy and computational efficiency.

Table 2: Numerical performance of the existing technique applied to Example 1 with initial vector $x^0=(-0.15,-0.15)^t$.

Method	**N**	**CPU(s)**	$\|\boldsymbol{F}(\boldsymbol{x^k})\|$	$\boldsymbol{e_k^{tol}}$	$\boldsymbol{\rho}$
M1	6	1.54	8.55×10^{-784}	8.55×10^{-684}	5.0000
M2	5	1.45	5.71×10^{-2747}	5.71×10^{-2647}	6.0000
M3	--	--	--	--	7.0000
M4	5	3.32	3.03×10^{-9360}	3.03×10^{-9260}	8.0000
PM	4	0.76	1.00×10^{-9999}	1.00×10^{-9899}	8.0000

Table 2 presents a comparative analysis of numerical simulations for the nonlinear system described in Example 1, with an initial guess of $x^0=(-0.15,-0.15)^T$. The table includes several iterative methods denoted as M1, M2, M3, M4 and the proposed method (PM), highlighting their computational performance in terms of the number of iterations N, CPU time, the norm of the function $\| F(x^k) \|$, the error tolerance e_k^{tol}, and the order of convergence ρ.

From the table, it is evident that the proposed method (PM) exhibits superior computational efficiency compared to the existing methods. Specifically, PM achieves convergence in only 4 iterations, which is fewer than all other methods except M2 and M4, while also requiring the least CPU time (0.76 seconds). Moreover, the residual $\| F(x^k) \|$for PM reaches an extremely small value of 1.00×10^{-9999}, indicating high numerical accuracy and demonstrating the method's ability to resolve the equilibrium points of the system effectively.

The convergence rate ρ also produces a stronger indication of the effectiveness of the suggested approach. PM attains the desired eighth-order convergence, as expected in the development of the three-step method based on Hermite approximations. Compared to them, M1 and M2 converge at lower orders (5 and 6 respectively), and M4 also converges in eighth order, but with more iterations and greater CPU time. The data of M3 is not available, which may imply convergence failure or a limitation on the computations in the case of this example.

These findings are of particular importance to mathematical biology and neuroscience. The nonlinear system denotes interrelations like neuronal activity as well as coupled biological groups. Precise and effective calculation of equilibrium points is vital to know equilibrium stability in systems, neuron firing, or stable behavior of interacting groups. The proposed PM methodology is not only less costly in terms of computations, but also guarantees the determination of steady states, which is of high accuracy that is necessary in ensuring sound simulations in the biological or neural modeling.

In general, Table 2 shows that the proposed approach is characterized by a high convergence order, low number of iterations, and low-computational effort and is, therefore, very appropriate to address nonlinear systems that are frequently used in the fields of computational neuroscience and mathematical biology.

Example 2. Nonlinear System in Mathematical Biology and Neuroscience:

Consider the following three-dimensional nonlinear system of equations:

$$F(x,y,z)=\begin{cases}12x-3y^2-4z-7.17=0,\\ x^2+10y^3-z-11.54=0,\\ \quad y^2+7z-7.631=0.\end{cases}\tag{42}$$

In the context of mathematical biology or neuroscience, the variables can be interpreted as follows:

- x: the activity of a neural population or signaling molecule concentration,
- y: a secondary interacting variable, such as another neural population, gene expression, or neurotransmitter level,
- z: a feedback or regulatory component influencing the system dynamics.

This system models nonlinear interactions between coupled biological components, such as feedback loops in neural circuits or biochemical pathways. Solving this system allows us to find equilibrium points or steady states of the network, which are essential for understanding the stability and dynamics of the modeled biological or neural system.

Iterative methods, particularly high-order schemes like the proposed three-step eighth-order method, are suitable for efficiently computing solutions to this type of system, providing high accuracy with fewer iterations and computational resources.

Table 3: numerical simulations obtained with the existing method, considering Example 2 with an initial guess of $x^0 = (0.8,1.8,3.0)^t$.Comparison of

Method	**N**	**CPU(s)**	$\|\boldsymbol{F}(\boldsymbol{x}^k)\|$	$\boldsymbol{e}_k^{tol}$	$\boldsymbol{\rho}$
M1	6	0.51	3.99×10^{-124}	3.99×10^{-24}	5.0000
M2	5	0.55	4.53×10^{-446}	4.53×10^{-346}	6.0000
M3	--	--	--	--	7.0000
M4	5	2.21	2.75×10^{-773}	2.75×10^{-673}	8.0000
PM	4	0.34	5.15×10^{-1449}	5.15×10^{-1349}	8.0000

Table 3 presents the numerical performance of different iterative methods for solving the nonlinear system in Example 2, with an initial guess $x^0 = (0.8,1.8,3.0)^T$. The proposed method (PM) outperforms all other approaches, converging in just 4 iterations with the lowest CPU time (0.34 s) and achieving an extremely small residual norm of 5.15×10^{-1449}. While M4 also reaches eighth-order convergence, it requires more iterations and higher computational effort, and lower-order methods (M1, M2) converge more slowly and with larger residuals. From the perspective of biological and neural networks, these results indicate that the PM method can efficiently determine stable equilibrium points of complex, multidimensional systems representing interacting neuronal populations, feedback loops, or coupled biochemical pathways. Fast and accurate convergence ensures reliable identification of steady-state network behaviors, which is critical for understanding neural firing patterns, synaptic regulation, or the dynamic balance in biological systems.

Example 3. Nonlinear System in Medical Modeling:

Let us examine the following nonlinear system of four equations:

$$F(x, y, z, w) = \begin{cases} x^3 + y + z + w - 15 = 0, \\ x + y^3 + z + w - 16 = 0, \\ x + y + z^3 + w - 17 = 0, \\ x + y + z + w^3 - 18 = 0. \end{cases} \qquad 43$$

In a medical or biomedical context, the variables can represent interacting physiological or pharmacological quantities:

- x: concentration of a primary drug or therapeutic agent,
- y: biomarker level associated with the treatment response,
- z: metabolic factors or enzyme activity influencing drug processing,
- w: secondary physiological response, such as immune activity or hormone level.

This system captures nonlinear interactions among multiple components in a medical system, such as drug-drug interactions, feedback mechanisms in metabolic pathways, or coupled responses in multi-organ models. Solving this system provides insights into equilibrium states of the physiological system, such as steady concentrations of drugs and biomarkers, helping in dose optimization, treatment planning, or understanding homeostatic balance. High-order iterative methods, like the proposed eighth-order three-step method, are particularly useful for efficiently computing such steady states in complex medical models with high accuracy.

Table 4: Comparison of the numerical simulations calculated with consideration of the existing method of Example 3 as an initial guess of $x^0 = (1.5, 1.5, 1.5, 1.5)^t$.

Method	**N**	**CPU(s)**	$\|F(x^k)\|$	e_k^{tol}	ρ
M1	--	--	--	--	5.0000
M2	5	1.22	1.02×10^{-721}	1.02×10^{-621}	6.0000
M3	--	--	--	--	7.0000
M4	--	--	--	--	8.0000
PM	4	0.64	2.37×10^{-1966}	2.37×10^{-1866}	8.0000

Table 4 presents the numerical performance of different iterative methods for solving the nonlinear system in Example 3, with an initial guess of $x^0 = (1.5,1.5,1.5,1.5)^T$. The proposed method (PM) demonstrates superior efficiency, converging in only 4 iterations with a CPU time of 0.64 seconds and achieving an extremely small residual norm of 2.37×10^{-1966}, suggesting extremely great accuracy. A number of methods (M1, M3, M4) either did not converge or had no results, whereas M2 converged in 5 iterations, but took nearly twice as much CPU time, and achieved a larger residual. Mathematically, in medical modeling terms, this reflects the fact that the PM method can compute accurately steady states of complex physiological systems, e.g. drug-biomarker interactions, multi-organ feedback loops or metabolic pathway equilibria. Such models require fast and accurate convergence to have reliable predictions to plan the treatment, optimize the dosage, and study the nonlinear dynamics of feedback among the interacting biomedical variables.

Table 2-4 indicates that the proposed method PM is the best in problem 1-3. After four iterations, PM had a minimum weighted error and functional norm, which is much lower than comparative approaches; M1- M4 and diverged in multiple situations. The fastest CPU time was also recorded by PM with respect to M1-M4 methods further demonstrating its high accuracy and efficiency even though all methods demonstrated theoretical optimality based on ACOC values.

6. CONCLUDING REMARKS

This research proposes a new method of iteration that integrates Hermite approximation with a three-step framework to obtain the eighth-order convergence. Unlike any of the existing high-order methods, the method only needs to make four function evaluations and considers both computational efficiency and allows the solving of nonlinear systems. It always converges quicker, the solutions are highly accurate with fewer iterations, and the cost of computation is low. The results affirm that the Hermite based eighth-order method is not only faster than the traditional methods, more precise, and efficient, but the method also has a greater index of efficiency. Above all, the approach shows a high and tolerable convergence in all test problems.

Mathematically speaking, the implications of the proposed methodology are substantial as far as mathematical biology and neuroscience are concerned. Numerous biological systems e.g. coupled neuronal populations, gene regulatory systems, or multicompartment physiological models are nonlinear and multidimensional in nature. The correct identification of their equilibrium points, steady states, or dynamic responses is important to comprehend the stability of the system, neuron firing patterns, synaptic feedback or biomarker interactions. The new iterative approach offers very efficient and accurate computational aid in analyzing such systems which can be very reliable even in complex situations when the traditional methods will not work or consume a lot of computational power. The examples of the nonlinear systems of neuronal and biological interactions given in Section 4 and the numerical findings supporting the approach indicate the effectiveness of the methodology in predicting the biologically relevant behaviors and network dynamics.

To conclude, the presented Hermite-based eighth-order method is highly stable, precise and computationally efficient, and it can be not only an important addition to the numerical analysis of the problem, but also an effective tool of modeling, simulation and analysis of the mathematical biology and neuroscience issues. The fact that it reliably solves complex nonlinear problems implies that it may be widely applicable in scientific and engineering problems involving coupled, nonlinear biological processes.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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