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NUMERICAL SIMULATION OF TWO-DIMENSIONAL POLLUTANT DISTRIBUTION USING THE CRANK-NICOLSON METHOD IN WASTE EQUALIZATION PONDS

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Abstract. Domestic wastewater flowing from residential areas to wastewater stabilization ponds generally contains various pollutants. In this study, research was conducted to investigate the distribution of pollutants in wastewater ponds by observing changes in biochemical oxygen demand (BOD) parameters. The concentration of pollutant tested was same as the concentration of BOD. The phenomenon of waste particle distribution in domestic wastewater stabilization ponds is viewed as an advection-diffusion scheme. The advection-diffusion scheme can be developed into a mathematical model, specifically a two-dimensional partial differential equation. In this study, numerical methods were used to solve these equations. The Crank-Nicolson method was used to discretize partial differential equations in time and space. The purpose of this study was to determine the points of pollutant dispersion in liquid wastewater stabilization ponds. The dispersion of wastewater pollutants is displayed in simulations performed using Python 3.14.0. The stability analysis of the Crank-Nicolson method is investigated using the Von Neumann method. The results of the analysis show that the Crank-Nicolson method is convergent and unconditionally stable.

Keywords: biological oxygen demand; advection-diffusion model; Crank-Nicolson method; mathematical modelling.

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1. INTRODUCTION

The issue of built environment quality in residential areas in almost all major cities in Indonesia is a multidimensional problem. Rapid urban development is marked by an increase in the number of people living in urban residential areas. This increase in population has implications for the volume of solid and liquid waste produced by the community, and consequently, the amount of waste per unit area also increases [1]. If no appropriate action is taken to address this issue, it will cause serious problems. The more densely populated an area is, the more complex the pollution problem will be, especially with household or domestic waste. Uncontrolled domestic waste has caused pollution in almost all rivers in Indonesia, especially in Java [2].

Domestic waste or household waste consists of dirty water discharged from bathrooms, toilets, and kitchens. This waste is a mixture of mineral and organic substances in many forms, including large and small particles, solids, and residues of dissolved substances in a floating state and in colloidal and semi-colloidal forms [3]. Wastewater treatment is one issue that needs to be considered in an ecosystem. Wastewater production is proportional to population growth in an area [4]. Wastewater treatment is carried out to reuse water in accordance with its intended use and environmental quality standards, helping overcome water scarcity in an area. Untreated wastewater can have adverse effects on the environment [5]. To increase water quality, appropriate wastewater treatment procedures are needed so that it can be reused for irrigation. Wastewater treatment is also important to prevent possible environmental pollution and protect public health from related hazards [6].

Centralized settlements lead to high levels of domestic wastewater in drainage systems. This can reduce river water quality if the wastewater is discharged without prior treatment. To solve this problem, wastewater must undergo treatment before being discharged into the water system. One way to control wastewater pollution is to build a centralized wastewater pond to treat domestic wastewater [7]. Biological treatment processes aim to reduce organic content by utilizing or maintaining microorganisms that degrade organic matter in wastewater treatment units, such as Wastewater Treatment Plants (WWTP) [8]. Domestic wastewater entering the IPAL contains organic pollutants. Biological Oxygen Demand (BOD) is a measure of the oxygen demand in water required by microorganisms to degrade organic matter. Therefore, BOD can be used as a parameter in assessing wastewater quality because BOD levels are proportional to the level of water pollution [9], [10].

Changes in water quality can be assessed from the distribution of BOD. In study [11], the BOD distribution model is described as an advection-diffusion mechanism, represented by partial differential equations. The advection process is the transfer of mass by fluid flow, while the diffusion process is the movement of mass due to the random motion of water molecules [11]. Research on advection-diffusion has been conducted in pollution transport modeling by [12], mass and heat transfer [13], tumor growth [14], and population dynamics [15]. Because advection-diffusion is important across many fields, many researchers use it to conduct research.

An advection-diffusion equation can be represented as a partial differential equation. Partial differential equations can be solved using analytical methods [16] and numerical methods [17]–[19]. Analytical solutions to partial differential equations are difficult to obtain, so numerical methods are used to solve these equations [20]. In studies [18], [21], numerical methods such as the finite element method were used to solve partial differential equations. Research by Jingxian et al. [20] used smooth particle hydrodynamics and a one-dimensional finite-difference method to model the distribution of chemical pollutants. High-order time discretization methods for solving stochastic partial differential equations were studied by Yukun et al. [22]. The solution of groundwater flow diffusion equations using a modified central finite difference method with backward finite differences was carried out by [23]. Research on pollutant distribution based on advection-diffusion equations using the forward-time central-space (FTCS) finite-difference method was conducted by Gita et al [24]. Numerical simulations for 1D, 2D, and 3D models using explicit finite-difference methods to describe particle movement were studied by Sunarsih et al. [11]. Guang et al. [25] use the alternating direction implicit (ADI) scheme to solve two-dimensional fractional diffusion equations. The Crank Nicolson–ADI numerical method is unconditionally stable for two-dimensional advection-diffusion equations [26].

One implicit method for solving partial differential equations is the Crank-Nicolson method. In solving partial differential equations, the Crank-Nicolson method involves time- and space-derivative terms [27]. In the Crank-Nicolson method, time and space derivatives are approximated by central differences. Based on wastewater quality, as measured by BOD concentration, this paper examines the distribution of BOD in wastewater stabilization ponds. The concentration of pollutant tested was same as the concentration of BOD measured horizontally and vertically, accounting for temporal and spatial changes. Changes in BOD concentration over time and space can be represented by partial differential equations. The proposed model considers a two-dimensional advection-diffusion process solved using the Crank-Nicolson method. The purpose of this study is to develop a numerical scheme and describe the phenomenon of BOD distribution in wastewater stabilization ponds.

2. ADVECTION-DIFFUSION PHENOMENON

A particle dissolved in water will undergo a dispersion process. The basic mechanisms that influence this process are advection and diffusion. The advection process is caused by a unidirectional flow that does not change the shape of the particle when it is moved. A simple example of the advection process is the flow of water towards the outlet of a pool, while the diffusion process is the movement of mass due to the random motion of water molecules. A simple example of diffusion is sugar spreading and dissolving in water.

The advection and diffusion equations can be represented by partial differential equations that describe the distribution of particles in fluid flow. There are two basic mechanisms used, namely the advection mechanism and the diffusion mechanism. The movement of a particle in a space can be illustrated as a particle passing through a control volume.

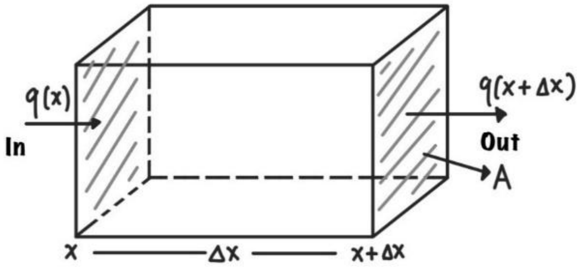

FIGURE 1. Volume Control in Particle Distribution

The total dissolved mass in the advection process that passes through the control volume as shown in Figure 1 can be expressed in the following equation [28].

$$q = u\,C \tag{1}$$

With mass flux (q), flow velocity (u), and particle concentration (C).

Equation (1) is the total dissolved mass of a mass transport due to a one-dimensional advection process. If the equation is developed into two dimensions, equation (1) becomes

$$q = uC + vC \tag{2}$$

Where u is the flow velocity in the x-direction, and v is the flow velocity in the y-direction.

The diffusion process can be represented by Fick's Law. Fick's Law is a statement that relates the flux of a mass to the concentration gradient [28]. Flux is the number of molecules or particles that pass through a certain space at a certain time. Based on Fick's First Law, the flux of particle flow is proportional to the concentration gradient.

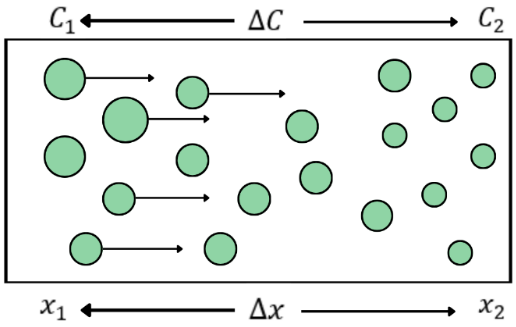

FIGURE 2. Illustration of One-Way Particle Flow

When a mass of particles flows from position x_1 to position x_2 and the mass of particles undergoes a change in concentration from C_1 to C_2 because of Brownian motion. A diffusion process is not only influenced by particle movement, but also by temperature, pressure, and the properties of the solvent. This will produce a diffusion coefficient whose value differs depending on the properties of the solvent.

In Fick's first law, the gradient represents the change in concentration with respect to space. Fick's second law states that the rate of change of particle concentration in a space over time is proportional to the second derivative of the change in particle concentration with respect to space. The second derivative of Fick's second law represents how sharp the change in the gradient is. An illustration of Fick's second law can be seen in Figure 3 and Figure 4

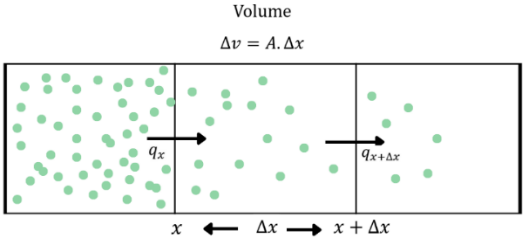

FIGURE 3. Flow of Particle Concentration Changes

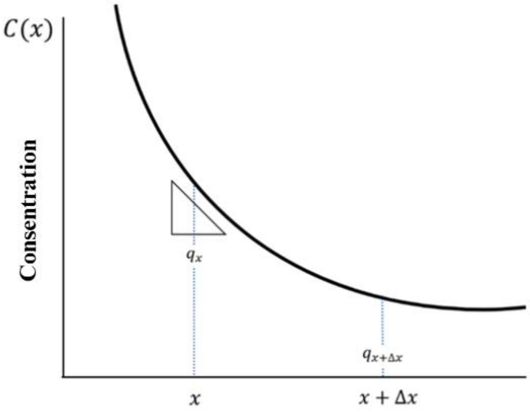

FIGURE 4. Graph of Particle Concentration Change Flow

Figure 3 shows that particles move from a high concentration area to a low concentration area within a control volume with A being its cross-sectional area. Figure 4 shows the concentration gradient (slope) against distance. The slope formed represents the displacement of particles from their position x to $x+\Delta x$. If a particle enters a control volume at position x, then based on Fick's first law, there is a flux q_x. This also occurs with particles moving out of the control volume at position $x+\Delta x$ then the flux is obtained $q_{x+\Delta x}$.

In this paper, to describe the distribution of pollutants along the length and depth of the pond, a two-dimensional diffusion-advection model is used, assuming diffusion-advection flow in the horizontal and vertical directions. The biochemical transformation process, which is a mass conservation process in the pond, is written as a diffusion-advection equation represented in the form of a partial differential equation as follows.

$$\frac{\partial \mathrm{C}}{\partial \mathrm{t}} = -\mathrm{u}\frac{\partial \mathrm{C}}{\partial \mathrm{x}} - \mathrm{v}\frac{\partial \mathrm{C}}{\partial \mathrm{y}} + \mathrm{D}_{\mathrm{mx}}\frac{\partial^2 \mathrm{C}}{\partial \mathrm{x}^2} + \mathrm{D}_{\mathrm{my}}\frac{\partial^2 \mathrm{C}}{\partial \mathrm{y}^2} \tag{3}$$

The descriptions of $\mathrm{C}, \mathrm{u}, \mathrm{v}, \mathrm{D}_{\mathrm{mx}}$, and D_{my} used in equation (3) are given as follows

C : Konsentrasi partikel
u : Advection velocity in the x direction
v : Advection velocity in the y direction
D_{mx} : Diffusion coefficient in the x direction
D_{my} : Diffusion coefficient in the y direction

Equation (3) is applied to each variable that forms the biochemical model. The bottom of the pool is sediment, which is considered inactive and therefore not included in the model. The longitudinal advection process is ignored in the sediment column.

3. CRANK NICOLSON METHOD

The finite difference method is a numerical method commonly used to solve technical and mathematical problems of physical phenomena that have a regular geometric form. The principle of the finite difference method is to replace the derivatives in differential equations with finite difference discretization based on Taylor series. Physically, the Taylor series can be interpreted as a quantity of interest in a space and time [14]. To apply finite difference discretization, the spatial domain is divided into a grid with a step size of Δx in the x direction and Δy in the y direction, and with a time step of Δt. The grid points are defined as follows

$$\begin{aligned} x_i &= i\Delta x, i = 0,1,2,\dots,N_x \\ y_i &= i\Delta y, i = 0,1,2,\dots,N_y \\ t^n &= n\Delta t, n = 0,1,2,\dots \end{aligned} \tag{4}$$

By defining the discrete form as follows

$$C_{i,j}^n = C(x_i\,, y_i\,, t^n)$$

In equation (3), the approach for time derivatives uses a forward finite difference scheme

$$\frac{\partial C}{\partial t} = \frac{C_{i,j}^{n+1} - C_{i,j}^n}{\Delta t} \tag{5}$$

The first derivative with respect to x and y for the advection form using a second-order central finite difference scheme

$$\frac{\partial C}{\partial x} = \frac{C_{i+1,j} - C_{i-1,j}}{2\Delta x} \tag{6}$$

$$\frac{\partial C}{\partial y} = \frac{C_{i,j+1} - C_{i,j-1}}{2\Delta y} \tag{7}$$

The second derivative with respect to x and y for the diffusion form uses a second-order central finite difference scheme, resulting in the equation

$$\frac{\partial^2 C}{\partial x^2} = \frac{C_{i+1,j} - 2C_{i,j} + C_{i-1,j}}{(\Delta x)^2} \tag{8}$$

$$\frac{\partial^2 C}{\partial y^2} = \frac{C_{i,j+1} - 2C_{i,j} + C_{i,j-1}}{(\Delta y)^2} \tag{9}$$

The implicit finite difference method can be applied to the advection-diffusion model in a wastewater pond. The discretization of a pond is shown in Figure 5.

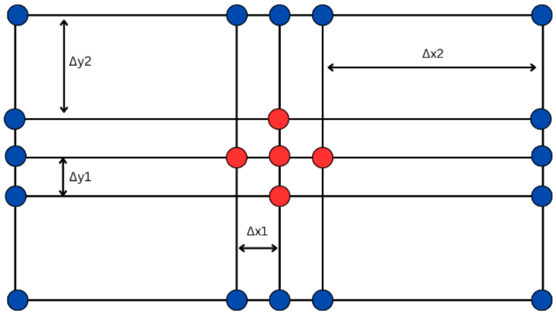

FIGURE 5. The Discretization of a Pond

4. RESULTS AND DISCUSSIONS

The advection-diffusion equation is solved using the implicit finite difference method. The discretization results are obtained by substituting into equation (3).

$$\frac{C_{i,j}^{n+1}-C_{i,j}^{n}}{\Delta t}=-\frac{1}{2}\left[u\left(\frac{C_{i+1,j}^{\mathrm{n}+1}-C_{i-1,j}^{\mathrm{n}+1}}{2\Delta x}\right)+u\left(\frac{C_{i+1,j}^{\mathrm{n}}-C_{i-1,j}^{\mathrm{n}}}{2\Delta x}\right)\right]-\frac{1}{2}\left[v\left(\frac{C_{i,j+1}^{\mathrm{n}+1}-C_{i,j-1}^{\mathrm{n}+1}}{2\Delta y}\right)+v\left(\frac{C_{i,j+1}^{\mathrm{n}}-C_{i,j-1}^{\mathrm{n}}}{2\Delta y}\right)\right]+\frac{1}{2}\left[D_{mx}\left(\frac{C_{i+1,j}^{\mathrm{n}+1}-2C_{i,j}^{\mathrm{n}+1}+C_{i-1,j}^{\mathrm{n}+1}}{(\Delta x)^2}\right)+D_{mx}\left(\frac{C_{i+1,j}^{\mathrm{n}}-2C_{i,j}^{\mathrm{n}}+C_{i-1,j}^{\mathrm{n}}}{(\Delta x)^2}\right)\right]+\frac{1}{2}\left[D_{my}\left(\frac{C_{i,j+1}^{\mathrm{n}+1}-2C_{i,j}^{\mathrm{n}+1}+C_{i,j-1}^{\mathrm{n}+1}}{(\Delta y)^2}\right)+D_{my}\left(\frac{C_{i,j+1}^{\mathrm{n}}-2C_{i,j}^{\mathrm{n}}+C_{i,j-1}^{\mathrm{n}}}{(\Delta y)^2}\right)\right] \tag{10}$$

By rearranging equation (10), the following discretization results are obtained

$$\begin{aligned}&(1+2\lambda_x+2\lambda_y)C_{i,j}^{n+1}+(q_x-\lambda_x)C_{i+1,j}^{\mathrm{n}+1}+(-q_x-\lambda_x)C_{i-1,j}^{\mathrm{n}+1}+(q_y-\lambda_y)C_{i,j+1}^{\mathrm{n}+1}+\\&(-q_y-\lambda_y)C_{i,j-1}^{\mathrm{n}+1}=(1-2\lambda_x-2\lambda_y)C_{i,j}^{n}+(-q_x+\lambda_x)C_{i+1,j}^{\mathrm{n}}+(q_x+\lambda_x)C_{i-1,j}^{\mathrm{n}}+\\&(-q_y+\lambda_y)C_{i,j+1}^{\mathrm{n}}+(q_y+\lambda_y)C_{i,j-1}^{\mathrm{n}}\end{aligned} \tag{11}$$

With $q_x=u\frac{\Delta t}{4\Delta x}; q_y=u\frac{\Delta t}{4\Delta y};\ \lambda_x=D_{mx}\frac{\Delta t}{2(\Delta x)^2};\quad \lambda_y=D_{my}\frac{\Delta t}{2(\Delta y)^2}$.

The consistency test of the Crank-Nicolson finite difference equation is performed by substituting the Taylor series expansion of each discrete form involved

$$\begin{aligned}&\frac{1}{\Delta t}\left[\left(C_{i,j}^{n}+(\Delta t)(C_t)_{i,j}^{n}+\frac{(\Delta t)^2}{2!}(C_{tt})_{i,j}^{n}+\frac{(\Delta t)^3}{3!}(C_{ttt})_{i,j}^{n}+\cdots\right)-C_{i,j}^{n}\right]\\
&=-\frac{1}{2}\left[u\left(\frac{\left(C_{i,j}^{n+1}+(\Delta x)(C_x)_{i,j}^{n+1}+\frac{(\Delta x)^2}{2!}(C_{xx})_{i,j}^{n+1}+\frac{(\Delta x)^3}{3!}(C_{xxx})_{i,j}^{n+1}+\cdots\right)-\left(C_{i,j}^{n+1}-(\Delta x)(C_x)_{i,j}^{n+1}+\frac{(\Delta x)^2}{2!}(C_{xx})_{i,j}^{n+1}-\frac{(\Delta x)^3}{3!}(C_{xxx})_{i,j}^{n+1}+\cdots\right)}{2\Delta x}\right)\right.\\
&\left.+u\left(\frac{\left(C_{i,j}^{n}+(\Delta x)(C_x)_{i,j}^{n}+\frac{(\Delta x)^2}{2!}(C_{xx})_{i,j}^{n}+\frac{(\Delta x)^3}{3!}(C_{xxx})_{i,j}^{n}+\cdots\right)-\left(C_{i,j}^{n}-(\Delta x)(C_x)_{i,j}^{n}+\frac{(\Delta x)^2}{2!}(C_{xx})_{i,j}^{n}-\frac{(\Delta x)^3}{3!}(C_{xxx})_{i,j}^{n}+\cdots\right)}{2\Delta x}\right)\right]\\
&-\frac{1}{2}\left[v\left(\frac{\left(C_{i,j}^{n+1}+(\Delta y)(C_y)_{i,j}^{n+1}+\frac{(\Delta y)^2}{2!}(C_{yy})_{i,j}^{n+1}+\frac{(\Delta y)^3}{3!}(C_{yyy})_{i,j}^{n+1}+\cdots\right)-\left(C_{i,j}^{n+1}-(\Delta y)(C_y)_{i,j}^{n+1}+\frac{(\Delta y)^2}{2!}(C_{yy})_{i,j}^{n+1}-\frac{(\Delta y)^3}{3!}(C_{yyy})_{i,j}^{n+1}+\cdots\right)}{2\Delta y}\right)\right.\\
&\left.+v\left(\frac{\left(C_{i,j}^{n}+(\Delta y)(C_y)_{i,j}^{n}+\frac{(\Delta y)^2}{2!}(C_{yy})_{i,j}^{n}+\frac{(\Delta y)^3}{3!}(C_{yyy})_{i,j}^{n}+\cdots\right)-\left(C_{i,j}^{n}-(\Delta y)(C_y)_{i,j}^{n}+\frac{(\Delta y)^2}{2!}(C_{yy})_{i,j}^{n}-\frac{(\Delta y)^3}{3!}(C_{yyy})_{i,j}^{n}+\cdots\right)}{2\Delta y}\right)\right]\\
&+\frac{1}{2}\left[D_{mx}\left(\frac{\left(C_{i,j}^{n+1}+(\Delta x)(C_x)_{i,j}^{n+1}+\frac{(\Delta x)^2}{2!}(C_{xx})_{i,j}^{n+1}+\frac{(\Delta x)^3}{3!}(C_{xxx})_{i,j}^{n+1}+\cdots\right)-2C_{i,j}^{\mathrm{n}+1}+\left(C_{i,j}^{n+1}-(\Delta x)(C_x)_{i,j}^{n+1}+\frac{(\Delta x)^2}{2!}(C_{xx})_{i,j}^{n+1}-\frac{(\Delta x)^3}{3!}(C_{xxx})_{i,j}^{n+1}+\cdots\right)}{(\Delta x)^2}\right)\right.\\
&\left.+D_{mx}\left(\frac{\left(C_{i,j}^{n}+(\Delta x)(C_x)_{i,j}^{n}+\frac{(\Delta x)^2}{2!}(C_{xx})_{i,j}^{n}+\frac{(\Delta x)^3}{3!}(C_{xxx})_{i,j}^{n}+\cdots\right)-2C_{i,j}^{\mathrm{n}}+\left(C_{i,j}^{n}-(\Delta x)(C_x)_{i,j}^{n}+\frac{(\Delta x)^2}{2!}(C_{xx})_{i,j}^{n}-\frac{(\Delta x)^3}{3!}(C_{xxx})_{i,j}^{n}+\cdots\right)}{(\Delta x)^2}\right)\right]\\
&+\frac{1}{2}\left[D_{my}\left(\frac{\left(C_{i,j}^{n+1}+(\Delta y)(C_y)_{i,j}^{n+1}+\frac{(\Delta y)^2}{2!}(C_{yy})_{i,j}^{n+1}+\frac{(\Delta y)^3}{3!}(C_{yyy})_{i,j}^{n+1}+\cdots\right)-2C_{i,j}^{\mathrm{n}+1}+\left(C_{i,j}^{n+1}-(\Delta y)(C_y)_{i,j}^{n+1}+\frac{(\Delta y)^2}{2!}(C_{yy})_{i,j}^{n+1}-\frac{(\Delta y)^3}{3!}(C_{yyy})_{i,j}^{n+1}+\cdots\right)}{(\Delta y)^2}\right)\right.\\
&\left.+D_{my}\left(\frac{\left(C_{i,j}^{n}+(\Delta y)(C_y)_{i,j}^{n}+\frac{(\Delta y)^2}{2!}(C_{yy})_{i,j}^{n}+\frac{(\Delta y)^3}{3!}(C_{yyy})_{i,j}^{n}+\cdots\right)-2C_{i,j}^{\mathrm{n}}+\left(C_{i,j}^{n}-(\Delta y)(C_y)_{i,j}^{n}+\frac{(\Delta y)^2}{2!}(C_{yy})_{i,j}^{n}-\frac{(\Delta y)^3}{3!}(C_{yyy})_{i,j}^{n}+\cdots\right)}{(\Delta y)^2}\right)\right]\end{aligned} \tag{12}$$

If the grids are made close to zero, that is, with limits $\Delta t\to 0, \Delta x\to 0,$ and $\Delta y\to 0$, the equation returns to the form of the original partial differential equation. In other words, the Crank-Nicolson method is consistent with the two-dimensional advection-diffusion differential equation.

Von Neumann stability analysis is also known as Fourier stability analysis. The method used to check the stability of finite difference schemes involves Fourier series expansion and observing changes in amplitude in Fourier components.

The Von Neumann stability analysis begins by substituting the Fourier series $(C_t)_{i,j}^{n} = \alpha^{n} e^{i\beta_1 ij}$ into equation (11), we get

$$\begin{aligned}&\left(1+2\lambda_x+2\lambda_y\right)\alpha^{n+1}e^{i\beta ij} + (q_x-\lambda_x)\alpha^{n+1}e^{i\beta(i+1)j} + (-q_x-\lambda_x)\alpha^{n+1}e^{i\beta(i-1)j} + \\ &\left(q_y-\lambda_y\right)\alpha^{n+1}e^{i\beta i(j+1)} + \left(-q_y-\lambda_y\right)\alpha^{n+1}e^{i\beta i(j+1)} = \left(1-2\lambda_x-2\lambda_y\right)\alpha^{n}e^{i\beta ij} + \\ &(-q_x+\lambda_x)\alpha^{n}e^{i\beta(i+1)j} + (q_x+\lambda_x)\alpha^{n}e^{i\beta(i-1)j} + \left(-q_y+\lambda_y\right)\alpha^{n}e^{i\beta i(j+1)} + \left(q_y + \right. \\ &\left.\lambda_y\right)\alpha^{n}e^{i\beta i(j-1)}\end{aligned} \tag{13}$$

Using the fact that $e^{i\beta x} = \cos(\beta x) + \sin(\beta x)$ and $e^{-i\beta x} = \cos(\beta x) - \sin(\beta x)$, we have

$$\alpha = \frac{\left(1-2\lambda_x-2\lambda_y\right)-q_x(2\sin(\beta j))+\lambda_x\left(2-4sin^2\left(\frac{\beta j}{2}\right)\right)-q_y(2\sin(\beta i))+\lambda_y\left(\left(2-4sin^2\left(\frac{\beta i}{2}\right)\right)\right)}{\left(1+2\lambda_x+2\lambda_y\right)+q_x(2\sin(\beta j))+\lambda_x\left(-2+4sin^2\left(\frac{\beta j}{2}\right)\right)+q_y(2\sin(\beta i))+\lambda_y\left(\left(-2+4sin^2\left(\frac{\beta i}{2}\right)\right)\right)} \tag{14}$$

For stability, it is required that $|\alpha| \leq 1$. From equation (14), it is known that $|\alpha| \leq 1$ [29]. Thus, the difference equation is said to be unconditionally stable. Therefore, knowing that the Crank-Nicolson difference equation is consistent and stable, the Crank-Nicolson difference equation converges.

The pollutant concentration at time n in equation (11) is used to calculate the pollutant concentration at $n+1$. The equation can be solved using Python software by providing the pool parameter values. The parameters used are flow velocity, diffusion coefficient, and flow time. The parameter values were obtained from field data at the Sewon Wastewater Treatment Plant. The flow velocity parameter value is $u = v = 2{,}526\, m/j$ (assuming that the flow velocity in the x-axis and y-axis directions are the same). The diffusion coefficient parameter value is $Dmx = Dmy = 8{,}3x10^{-5}\, m^2/j$ (assuming that the diffusion coefficients in the x-axis and y-axis directions are equal). The flow time is $n = 24h$. The size of the IPAL pool is $77\, m$ in length, $70\, m$ in width, grid length $\Delta x1 = 11\, m$, $\Delta x2 = 27.5\, m$, grid width $\Delta y1 = 10\, m, \Delta y2 = 25\, m$.

Based on the advection-diffusion mechanism, the distribution of BOD pollutants was calculated using an implicit finite difference numerical method applied in Python software. The calculation results were simulated in a 2D graph, which was also generated using Python software. The simulation of BOD distribution based on the advection-diffusion mechanism is shown in Figure below.

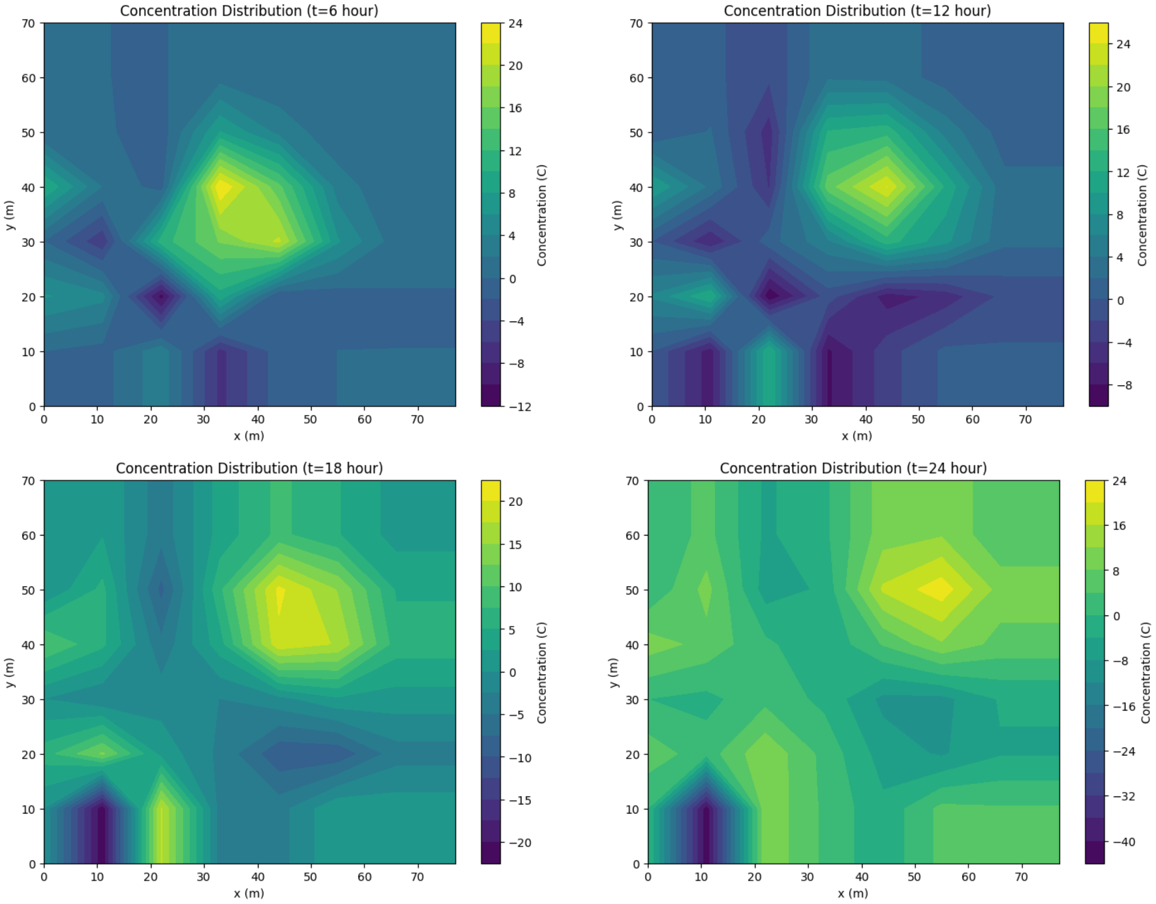

FIGURE 6. Distribution of BOD Concentration

When $t = 6$, the concentration begins to spread from the inlet towards the center of the pond due to advection flow. The BOD concentration is still concentrated at the inlet of the pond, but slowly spreads to the top and bottom of the pond due to diffusion. In general, the BOD concentration around the pond is still low because the flow from the inlet has only reached the center.

When $t = 12$, the concentration from the inlet moves further to the right of the pool. The concentration change becomes more uniform due to the diffusion process. The concentration, which was initially high around the inlet, begins to merge with its surroundings due to particle dispersion. The BOD concentration at the right outlet begins to increase. The concentration distribution is now more uniform along the horizontal axis.

When $t = 18$, the concentration spreads throughout almost the entire pool. The concentration at the right outlet is already high compared to the beginning of the simulation, while the diffusion process obscures the concentration changes at each point. The concentration changes around the outlet show a significant increase, indicating that the flow and diffusion have reached the end of the pool.

When $t = 24$, the BOD concentration in the pond is close to stable, meaning that BOD particles are evenly distributed throughout the pond. This can occur because there is an aerator machine in the middle of the pond.

TABLE 1. BOD Concentration Actual data, Numerical data, and error

	Actual Data	**Numerical Data**	**Error**
$C_{(i-1),j}$	20.41	19.27	0.056
$C_{(i+1)j}$	17.21	18.33	0.506
$C_{i,j}$	19.67	19.79	0.504
$C_{i,(j+1)}$	20.094	21.32	0.556
$C_{i,(j-1)}$	19.16	17.91	0.553

5. CONCLUSIONS

In this study, the two-dimensional diffusion advection equation was solved using a Crank-Nicolson numerical method that considers changes in space and time. The diffusion advection equation represents the movement of BOD particles in a wastewater pond. Based on the analysis conducted, the Crank-Nicolson method showed unconditional stability, consistency, and convergence. The numerical simulation shows that the flow of BOD particles is influenced by the advection process that starts from the inlet of the pond, while the dispersion of BOD particles is influenced by the diffusion process. Therefore, the advection parameter and diffusion coefficient are very influential. The error from data validation shows the accuracy of the prediction data and actual data.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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