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NONSTANDARD FINITE DIFFERENCE SCHEMES FOR SOLVING SYSTEMS OF TWO LINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH COMPLEX EIGENVALUES

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Abstract. This paper presents a novel nonstandard finite difference (NSFD) scheme for solving a system of linear Caputo-type fractional differential equations (FDEs) characterized by complex eigenvalues. First, the analytical solutions for a two-dimensional linear FDE system are derived. The proposed numerical method is then constructed by employing complex Mittag-Leffler functions for establishing numerator and denominator functions within the NSFD framework. A rigorous convergence analysis is conducted, establishing the scheme's consistency and stability. It is proven that the method is unconditionally stable provided the real parts of the system's eigenvalues are negative. The applicability of the scheme is extended to fractional harmonic oscillators by reformulating them as equivalent systems of FDEs with complex eigenvalues. The efficacy and accuracy of the method are demonstrated through four numerical examples. The solutions exhibit the expected damped oscillatory behavior. The results show exceptional accuracy, with errors on the order of 10^{-15} or less, even when using relatively large step sizes, demonstrating the robustness and computational efficiency of the proposed scheme.

Keywords: denominator function; Mittag-Leffler function; linear fractional differential equations; nonstandard finite difference methods.

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1. INTRODUCTION

Fractional differential equations (FDEs) are generalizations of classical differential equations. FDEs are particularly useful in modeling processes that exhibit memory and hereditary characteristics, which makes them applicable in various fields such as control theory, viscoelasticity, signal processing, and biological systems [20].

For the system of fractional differential equations [18] and [5] offering valuable insights by illustrate the distinct real eigenvalues, repeated real eigenvalues, and complex eigenvalues.

Complex eigenvalues are fundamental to oscillatory dynamics, purely imaginary eigenvalues yield undamped periodic oscillations while eigenvalues with negative real parts describe damped oscillations, with damping eigenvalues shift left into the complex plane. Several studies [6] - [21] have shown how complex eigenvalues govern frequency, decay, resonance, and bifurcations. [6] analyzed damping in coupled oscillators, while [7] used a matrix approach to reveal how eigenvalues encode oscillatory decay. [10] investigated decoupling in damped systems, [4] demonstrated delay induced Hopf bifurcations, Resonance phenomena were also explained in terms of eigenvalue structure in [3]. Fractional order extensions further generalize these concepts, [12] and [11] showed that fractional oscillators produce Mittag–Leffler type responses under forcing, [22] established bifurcation criteria for fractional order systems, while [21] linked fractional delay systems to Hopf bifurcation and chaos.

Due to the analytical complexity of solving FDEs, many researchers have turned to numerical methods [16] discuss several classical numerical methods such as the Euler method, Runge-Kutta method, and the finite difference method, these techniques provide practical tools for approximating solutions when analytical solutions are difficult or impossible to obtain, [9] introduce Chebyshev polynomial approximations and an improved finite difference scheme, these methods have proven to be reliable and efficient for solving both linear and nonlinear systems of FDEs.

In numerical approaches for differential equations and system of differential equations, the nonstandard finite difference (NSFD) method introduced by Mickens [13, 14, 15], ensures that key qualitative properties of the original systems such as positivity and boundedness are preserved in the numerical solutions.

Recent studies have applied NSFD schemes for systems of FDEs [17, 19, 8, 2], represent significant progress in adapting NSFD to handle complex and nonlinear fractional dynamics.

In [1] we presented the NSFD method for solving systems of two linear fractional differential equations with real and repeated eigenvalues. This work aims to investigating linear fractional systems with complex eigenvalues and extending the NSFD method to higher order cases.

In this paper we consider a Caputo-type system of two fractional differential equations of the form:

$$ {}_0^C D_t^\alpha x(t) = ax(t) + by(t),\ t \in [0,T],\ x(0) = x_0 \tag{1}$$

$$ {}_0^C D_t^\alpha y(t) = cx(t) + dy(t),\ t \in [0,T],\ y(0) = y_0 \tag{2}$$

where a, b, c and d are real constants such that $[trace(A)]^2 < 4 \cdot det(A)$, $T > 0$ is positive real number and ${}_0^C D_t^\alpha$ is the Caputo derivative with respect to t of order $0 < \alpha \leq 1$. Hence the system (1)-(2) has two complex and conjugate eigenvalues λ_1 and λ_2, where $\lambda_2 = \bar{\lambda}_1$.

The objective is to develop a highly accurate and stable nonstandard finite difference method for solving the system.

2. PRELIMINARIES AND DEFINITIONS

In this section, we present some basic definitions

2.1. Caputo fractional derivative.

The Caputo fractional derivative is defined as

$$ {}_0^C D_x^\alpha f(x) = I^{n-\alpha} D^n f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)dt, \qquad n-1 < \alpha \leq n $$

2.2. Properties of the Mittag-Leffler Function $E_\alpha(\lambda t^\alpha)$.

Let $\lambda = \sigma + i\omega = \rho e^{i\theta}$, where $\sigma \in \mathbb{R}$, $\omega \in \mathbb{R}$, $\rho = \sqrt{\sigma^2 + \omega^2} \geq 0$, and $\theta = \tan^{-1}\left(\frac{\omega}{\rho}\right), 0 \leq \theta \leq 2\pi$.

2.2.1. *Series Representation.*

The Mittag-Leffler function $E_\alpha(\lambda t^\alpha)$, where $0 < \alpha \leq 1$, can be expressed in series form as:

$$ E_\alpha(\lambda t^\alpha) = \sum_{k=0}^{\infty} \frac{(\lambda t^\alpha)^k}{\Gamma(\alpha k + 1)}. $$

Using the polar representation of λ, this becomes:

$$ E_\alpha(\rho e^{i\theta} t^\alpha) = \sum_{k=0}^{\infty} \frac{\rho^k e^{ik\theta} t^{\alpha k}}{\Gamma(\alpha k + 1)}. $$

Separating the real and imaginary parts:

$$E_\alpha(\rho e^{i\theta} t^\alpha) = \sum_{k=0}^{\infty} \frac{\rho^k t^{\alpha k} (\cos(k\theta) + i \sin(k\theta))}{\Gamma(\alpha k + 1)} = \sum_{k=0}^{\infty} \frac{\rho^k t^{\alpha k} \cos(k\theta)}{\Gamma(\alpha k + 1)} + i \sum_{k=0}^{\infty} \frac{\rho^k t^{\alpha k} \sin(k\theta)}{\Gamma(\alpha k + 1)}.$$

For $0 < \alpha < 1$, the terms $\cos(k\theta)$ and $\sin(k\theta)$ produce oscillations with a frequency determined by θ, while the amplitude is adjusted by $\rho^k t^{\alpha k}/\Gamma(\alpha k + 1)$. For $\alpha = 1$, $E_1(\lambda t^\alpha) = e^{\lambda t^\alpha}$, which grows or decays exponentially with possible oscillations depending on θ.

2.3. Long-Term Behavior and Boundedness. Let $\lambda = \sigma + i\omega = \rho e^{i\theta}$, where $\rho > 0$ and $\theta \in (-\pi, \pi]$. Consider the Mittag-Leffler function $E_\alpha(\lambda t^\alpha)$ for $t > 0$ and $0 < \alpha \leq 1$. The asymptotic behavior as $t \to \infty$ depends on $\theta = \arg(\lambda)$ and α.

Define the critical angle:

$$\theta_c = \frac{\alpha \pi}{2}.$$

From the asymptotic expansion for the Mittag-Leffler function: For $0 < \alpha < 1$, as $|\rho t^\alpha e^{i\theta}| \to \infty$:

$$E_\alpha(\rho t^\alpha e^{i\theta}) \sim \begin{cases} \frac{1}{\alpha} e^{\left(\rho t^\alpha e^{i\theta}\right)^{1/\alpha}} - \sum_{k=1}^{\infty} \frac{\rho t^\alpha e^{i\theta - k}}{\Gamma(1-\alpha k)}, & |\theta| < \frac{\alpha\pi}{2} \\ \\ -\sum_{k=1}^{\infty} \frac{\rho t^\alpha e^{i\theta - k}}{\Gamma(1-\alpha k)}, & |\theta| > \frac{\alpha\pi}{2} \end{cases}$$

The long-term behavior is characterized as follows:

I. **Growing Oscillations:** Occurs when $|\theta| < \theta_c$ (i.e., λ lies strictly inside the sector of angle θ_c centered around the positive real axis). In this region:

$$E_\alpha(\lambda t^\alpha) \sim \frac{1}{\alpha} \exp\left(\lambda^{1/\alpha} t\right) = \frac{1}{\alpha} e^{\delta t} e^{i\gamma t},$$

where $\lambda^{1/\alpha} = \delta + i\gamma$ with $\delta = \rho^{1/\alpha} \cos(\theta/\alpha) > 0$ and $\gamma = \rho^{1/\alpha} \sin(\theta/\alpha)$. Thus, the function grows exponentially with oscillations.

II. **Periodic Solutions:** Occurs only when $\alpha = 1$ and $|\theta| = \frac{\pi}{2}$ (i.e., λ is purely imaginary). Then:

$$E_1(\lambda t) = e^{i\omega t},$$

which is purely periodic.

III. **Decaying Oscillations:** Occurs when $|\theta| = \theta_c$ and $0 < \alpha < 1$ (i.e., λ lies on the boundary of the sector for $\alpha \neq 1$). On the Stokes lines, the asymptotic expansion yields:

$$E_\alpha(\lambda t^\alpha) \sim \frac{2}{\alpha} e^{ibt} - \frac{t^{-\alpha}}{\Gamma(1-\alpha)},$$

where $b = \rho^{1/\alpha}\sin(\theta/\alpha)$. This represents oscillations with algebraic decay.

IV. **Algebraic Decay:** Occurs when $|\theta| > \theta_c$ (i.e., λ lies outside the sector). The dominant term is:

$$E_\alpha(\lambda t^\alpha) \sim -\frac{\lambda^{-1} t^{-\alpha}}{\Gamma(1-\alpha)},$$

which decays algebraically as $t^{-\alpha}$.

In summary, the long term behaviour of the Mittag-Leffler function $E_\alpha(\lambda t^\alpha) = E_\alpha(\rho t^\alpha e^{i\theta})$ as $t \to \infty$ is as follows:

- **Growing oscillations**: $|\theta| < \theta_c$. In this scenario, $\sigma > \rho\cos(\theta_c) = \rho\cos(\alpha\pi/2)$, indicating that σ must be adequately positive.
- **Periodic solutions**: $\alpha = 1$ and $|\theta| = \pi/2$.
- **Decaying oscillations**: $|\theta| = \theta_c$ and $0 < \alpha < 1$. In this case, $\sigma = \rho\cos(\theta_c)$ (on the boundary).
- **Algebraic decay**: $|\theta| > \theta_c$. In this case, $\sigma < \rho\cos(\theta_c)$ (including negative σ), but only when $|\theta| > \theta_c$.

3. EXACT SOLUTION OF A LINEAR SYSTEM WITH COMPLEX EIGENVALUES

The linear system of FDEs with constant coefficients (1). It can be written in the matrix form as

$$ {}_0^C D_t^\alpha \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{3}$$

The matrix entries a, b, c and d are assumed to fulfill the condition $[trace(A)]^2 < 4det(A)$, that is $(a+d)^2 < 4(ad-bc)$. In this case A has two complex eigenvalues λ_1 and $\lambda_2 = \bar{\lambda}_1$. Hence, if $\lambda_1 = \sigma + i\omega$, then $\lambda_2 = \sigma - i\omega$.

Since $\lambda_1 + \lambda_2 = trace(A)$ and $\lambda_1 \cdot \lambda_2 = det(A)$, then $a + d = \lambda_1 + \lambda_2 = 2\sigma \Rightarrow d = 2\sigma - a$ and $\lambda_1 \cdot \lambda_2 = \sigma^2 + \omega^2 = ad - bc \Rightarrow c = -(\sigma^2 + \omega^2 - 2a\sigma + a^2)/b = -(\lambda_1 - a)(\lambda_2 - a)/b$. The eigenvectors of matrix A are given by:

$$v_1 = \begin{pmatrix} 1 \\ \frac{\lambda_1 - a}{b} \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} 1 \\ \frac{\lambda_2 - a}{b} \end{pmatrix}.$$

Let D be the matrix

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \sigma + \omega i & 0 \\ 0 & \sigma - \omega i \end{pmatrix},$$

and P be the matrix

$$P = \begin{pmatrix} 1 & 1 \\ \frac{\lambda_1 - a}{b} & \frac{\lambda_2 - a}{b} \end{pmatrix}.$$

Then,

$$P^{-1} = \begin{pmatrix} \frac{(\lambda_2 - a) i}{2\omega} & -\frac{b i}{2\omega} \\ -\frac{(\lambda_1 - a) i}{2\omega} & \frac{b i}{2\omega} \end{pmatrix} = \frac{i}{2\omega} \begin{pmatrix} \lambda_2 - a & -b \\ -\lambda_1 + a & b \end{pmatrix}.$$

Now, $A = PDP^{-1}$.

Let

$$z(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

then the linear system (3) can be written as ${}_{0}^{C}D_t^{\alpha} z(t) = Az(t)$,with the initial condition $z_0 = z(t_0) = [x_0, y_0]^T$. By using the transformation $w(t) = P^{-1} z(t)$, the coupled linear system ${}_{0}^{C}D_t^{\alpha} z(t) = Az(t)$ is transformed into the uncoupled linear system ${}_{0}^{C}D_t^{\alpha} w(t) = Dw(t)$, whose solution is given by:

$$w(t) = \begin{pmatrix} C_1 E_\alpha(\lambda_1 t^\alpha) \\ C_2 E_\alpha(\lambda_2 t^\alpha) \end{pmatrix}.$$

Now, by taking the inverse transform $z(t) = Pw(t)$ and solving the equation

$$z(t_0) - [x_0, y_0]^T = [0, 0]^T$$

for C_1 and C_2 we obtain the solution:

$$x(t) = \left[\frac{(\lambda_1 - a) \frac{E_\alpha(\lambda_2 t^\alpha)}{E_\alpha(\lambda_2 t_0^\alpha)} - (\lambda_2 - a) \frac{E_\alpha(\lambda_1 t^\alpha)}{E_\alpha(\lambda_1 t_0^\alpha)}}{\lambda_1 - \lambda_2} \right] x_0 + \left[b \frac{\frac{E_\alpha(\lambda_1 t^\alpha)}{E_\alpha(\lambda_1 t_0^\alpha)} - \frac{E_\alpha(\lambda_2 t^\alpha)}{E_\alpha(\lambda_2 t_0^\alpha)}}{(\lambda_1 - \lambda_2)} \right] y_0 \tag{4}$$

$$y(t) = -\frac{(\lambda_1 - a)(\lambda_2 - a)}{b} \left[\frac{\frac{E_\alpha(\lambda_1 t^\alpha)}{E_\alpha(\lambda_1 t_0^\alpha)} - \frac{E_\alpha(\lambda_2 t^\alpha)}{E_\alpha(\lambda_2 t_0^\alpha)}}{(\lambda_1 - \lambda_2)} \right] x_0 + \left[\frac{(\lambda_1 - a) \frac{E_\alpha(\lambda_1 t^\alpha)}{E_\alpha(\lambda_1 t_0^\alpha)} + (\lambda_2 - a) \frac{E_\alpha(\lambda_2 t^\alpha)}{E_\alpha(\lambda_2 t_0^\alpha)}}{(\lambda_1 - \lambda_2)} \right] y_0 \tag{5}$$

We have $\lambda_1-\lambda_2=2\omega i$, $c=-(\lambda_1-a)(\lambda_2-a)/b$, and $\lambda_1+\lambda_2=a+d$. Then $\lambda_1-a=d-\lambda_2$ and $\lambda_2-a=d-\lambda_1$. Replacing $\lambda_1-\lambda_2$, λ_1-a and λ_2-a by their equivalences in equations (4)-(5) yields:

$$(6)\quad x(t) = \left[a\frac{\left(\frac{E_\alpha(\lambda_2 t^\alpha)}{E_\alpha(\lambda_2 t_0^\alpha)}-\frac{E_\alpha(\lambda_1 t^\alpha)}{E_\alpha(\lambda_1 t_0^\alpha)}\right)i}{2\omega}+\frac{\left(\frac{\lambda_2 E_\alpha(\lambda_1 t^\alpha)}{E_\alpha(\lambda_1 t_0^\alpha)}-\frac{\lambda_1 E_\alpha(\lambda_2 t^\alpha)}{E_\alpha(\lambda_2 t_0^\alpha)}\right)i}{2\omega}\right]x_0+b\left[\frac{\left(\frac{E_\alpha(\lambda_2 t^\alpha)}{E_\alpha(\lambda_2 t_0^\alpha)}-\frac{E_\alpha(\lambda_1 t^\alpha)}{E_\alpha(\lambda_1 t_0^\alpha)}\right)i}{2\omega}\right]y_0$$

$$(7)\quad y(t) = c\left[\frac{\left(\frac{E_\alpha(\lambda_2 t^\alpha)}{E_\alpha(\lambda_2 t_0^\alpha)}-\frac{E_\alpha(\lambda_1 t^\alpha)}{E_\alpha(\lambda_1 t_0^\alpha)}\right)i}{2\omega}\right]x_0+\left[d\frac{\left(\frac{E_\alpha(\lambda_2 t^\alpha)}{E_\alpha(\lambda_2 t_0^\alpha)}-\frac{E_\alpha(\lambda_1 t^\alpha)}{E_\alpha(\lambda_1 t_0^\alpha)}\right)i}{2\omega}+\frac{\left(\frac{\lambda_2 E_\alpha(\lambda_1 t^\alpha)}{E_\alpha(\lambda_1 t_0^\alpha)}-\frac{\lambda_1 E_\alpha(\lambda_2 t^\alpha)}{E_\alpha(\lambda_2 t_0^\alpha)}\right)i}{2\omega}\right]y_0$$

A special case is when $\lambda_1=\omega i$ and $\lambda_2=-\omega i$ are pure imaginary eigenvalues. In this case, $c=-(\lambda_1-a)(\lambda_2-a)/b=-(\omega^2+a^2)/b$ and $d=\lambda_1+\lambda_2-a=-a$. Then, the solution of the linear system (3) is given by:

$$(8)\quad x(t) = \frac{x_0}{2}\left(\frac{E_\alpha\left(i\omega t^\alpha\right)}{E_\alpha\left(i\omega t_0^\alpha\right)}+\frac{E_\alpha\left(-i\omega t^\alpha\right)}{E_\alpha\left(-i\omega t_0^\alpha\right)}\right)+\frac{ax_0+by_0}{2\omega}\left(\frac{E_\alpha\left(-i\omega t^\alpha\right)}{E_\alpha\left(-i\omega t_0^\alpha\right)}-\frac{E_\alpha\left(i\omega t^\alpha\right)}{E_\alpha\left(i\omega t_0^\alpha\right)}\right)i$$

$$(9)\quad y(t) = \frac{y_0}{2}\left(\frac{E_\alpha\left(i\omega t^\alpha\right)}{E_\alpha\left(i\omega t_0^\alpha\right)}+\frac{E_\alpha\left(-i\omega t^\alpha\right)}{E_\alpha\left(-i\omega t_0^\alpha\right)}\right)+\frac{((a^2+\omega^2)x_0+aby_0)}{2b\omega}\left(\frac{E_\alpha\left(i\omega t^\alpha\right)}{E_\alpha\left(i\omega t_0^\alpha\right)}-\frac{E_\alpha\left(-i\omega t^\alpha\right)}{E_\alpha\left(-i\omega t_0^\alpha\right)}\right)\mathrm{i}$$

4. NONSTANDARD FINITE DIFFERENCE SCHEME FOR SYSTEM OF LINEAR FDES

In this section, nonstandard finite difference methods will be developed to solve a system of two fractional differential equations for the case of two conjugate complex eigenvalues.

Let N be a positive integer and $h=T/N$. We discretize the interval $[0,T]$ by points $t_k=h\cdot k, k=0,1,...,N$. Let $x_k\approx x(t_k)$ and $y_k\approx y(t_k)$ for $k=0,1,...,N$.

To obtain a nonstandard finite difference scheme for (1), following Mickens's method and making following substitutions in (6) and (7)

$$(10)\qquad \begin{cases} t_0\to t_k=hk, & t\to t_{k+1}=h(k+1)\\ x_0\to x_k, & x(t)\to x_{k+1}\\ y_0\to y_k, & y(t)\to y_{k+1}\end{cases}$$

then

$$x_{k+1} = \left[a\frac{\left(\frac{E_\alpha(\lambda_1 h^\alpha(k+1)^\alpha)}{E_\alpha(\lambda_1 h^\alpha k^\alpha)}-\frac{E_\alpha(\lambda_2 h^\alpha k^\alpha)}{E_\alpha(\lambda_2 h^\alpha k^\alpha)}\right)i}{2\omega}+\frac{\left(\frac{\lambda_1 E_\alpha(\lambda_2 h^\alpha(k+1)^\alpha)}{E_\alpha(\lambda_2 h^\alpha k^\alpha)}-\frac{\lambda_2 E_\alpha(\lambda_1 h^\alpha(k+1)^\alpha)}{E_\alpha(\lambda_1 h^\alpha k^\alpha)}\right)i}{2\omega}\right]x_k+$$

$$b\left[\frac{\left(\frac{E_\alpha(\lambda_1 h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda_1 h^\alpha k^\alpha)}-\frac{E_\alpha(\lambda_2 h^\alpha k^\alpha)}{E_\alpha(\lambda_2 h^\alpha k^\alpha)}\right)i}{2\omega}\right]y_k \tag{11}$$

$$\begin{aligned} y_{k+1} &= c\left[\frac{\left(\frac{E_\alpha(\lambda_1 h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda_1 h^\alpha k^\alpha)}-\frac{E_\alpha(\lambda_2 h^\alpha k^\alpha)}{E_\alpha(\lambda_2 h^\alpha k^\alpha)}\right)i}{2\omega}\right]x_0+ \\ &\left[d\frac{\left(\frac{E_\alpha(\lambda_1 h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda_1 h^\alpha k^\alpha)}-\frac{E_\alpha(\lambda_2 h^\alpha k^\alpha)}{E_\alpha(\lambda_2 h^\alpha k^\alpha)}\right)i}{2\omega}+\frac{\left(\frac{\lambda_1 E_\alpha(\lambda_2 h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda_2 h^\alpha k^\alpha)}-\frac{\lambda_2 E_\alpha(\lambda_1 h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda_1 h^\alpha k^\alpha)}\right)i}{2\omega}\right]y_0 \end{aligned} \tag{12}$$

Let

$$\phi_k=\frac{\left(\frac{E_\alpha(\lambda_1 h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda_1 h^\alpha k^\alpha)}-\frac{E_\alpha(\lambda_2 h^\alpha k^\alpha)}{E_\alpha(\lambda_2 h^\alpha k^\alpha)}\right)i}{2\omega},$$

and

$$\psi_k=\frac{\left(\frac{\lambda_1 E_\alpha(\lambda_2 h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda_2 h^\alpha k^\alpha)}-\frac{\lambda_2 E_\alpha(\lambda_1 h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda_1 h^\alpha k^\alpha)}\right)i}{2\omega}$$

The NSFD scheme for the linear system (1) with distinct eigenvalues as

$$\frac{x_{k+1}-\psi_k x_k}{\phi_k} = ax_k+by_k, \tag{13}$$

$$\frac{y_{k+1}-\psi_k y_k}{\phi_k} = cx_k+dy_k, \tag{14}$$

5. CONVERGENCE AND STABILITY FOR NONSTANDARD FINITE DIFFERENCE SCHEME FOR SYSTEM OF LINEAR FDES

This section presents proof for the convergence of the constructed nonstandard finite difference scheme by demonstrating its consistency and numerical stability.

5.1. Consistency of the numerical schemes.

We prove the convergence of the NSFD scheme for (13) and (14), from fractional Taylor series expansion

$$x(t_{k+1})=x(t_k)+{}_0^C D_t^\alpha x(t_k)\frac{h^\alpha}{\Gamma(\alpha+1)}+O(h^{2\alpha})$$

similarly

$$y(t_{k+1})=y(t_k)+{}_0^C D_t^\alpha y(t_k)\frac{h^\alpha}{\Gamma(\alpha+1)}+O(h^{2\alpha})$$

Therefore

$${}_0^C D_t^\alpha x(t_k)=\frac{x(t_{k+1})-x(t_k)}{\frac{h^\alpha}{\Gamma(\alpha+1)}}+O(h^\alpha)$$

and

$${}_{0}^{C}D_t^{\alpha} y(t_k) = \frac{y(t_{k+1}) - y(t_k)}{\frac{h^\alpha}{\Gamma(\alpha+1)}} + O(h^\alpha)$$

The local truncation error (LTE) of (13) is given by

$$LTE_x = \frac{x(t_{k+1}) - \varphi_k(h,\lambda,\phi)x(t_k)}{\phi_k(h,\lambda,\phi)} - ax(t_k) - by(t_k)$$

It can be expressed as

$$LTE_x = \frac{x(t_{k+1}) - \varphi_k(h,\lambda,\phi)x(t_k)}{\phi_k(h,\lambda,\phi)} - \frac{x(t_{k+1}) - x(t_k)}{\frac{h^\alpha}{\Gamma(\alpha+1)}} + \frac{x(t_{k+1}) - x(t_k)}{\frac{h^\alpha}{\Gamma(\alpha+1)}} - ax(t_k) - by(t_k)$$

$$LTE_x = \frac{x(t_{k+1})}{\phi_k} - \frac{x(t_k)}{\frac{\phi_k}{\varphi_k}} - \frac{x(t_{k+1}) - x(t_k)}{\frac{h^\alpha}{\Gamma(\alpha+1)}} + \frac{x(t_{k+1}) - x(t_k)}{\frac{h^\alpha}{\Gamma(\alpha+1)}} - ax(t_k) - by(t_k)$$

$$|LTE_x| = \left|\left(\frac{1}{\phi_k} - \frac{1}{\frac{h^\alpha}{\Gamma(\alpha+1)}}\right)x(t_{k+1}) - \left(\frac{1}{\frac{\phi_k}{\varphi_k}} - \frac{1}{\frac{h^\alpha}{\Gamma(\alpha+1)}}\right)x(t_k) + \frac{x(t_{k+1}) - x(t_k)}{\frac{h^\alpha}{\Gamma(\alpha+1)}} - ax(t_k) - by(t_k) + O(h)\right|$$

$$|LTE_x| \le \left|\left(\frac{1}{\phi_k} - \frac{1}{\frac{h^\alpha}{\Gamma(\alpha+1)}}\right)x(t_{k+1}) - \left(\frac{1}{\frac{\phi_k}{\varphi_k}} - \frac{1}{\frac{h^\alpha}{\Gamma(\alpha+1)}}\right)x(t_k)\right| + \left|\frac{x(t_{k+1}) - x(t_k)}{\frac{h^\alpha}{\Gamma(\alpha+1)}} - ax(t_k) - by(t_k) + O(h^\alpha)\right|$$

$$|LTE_x| \le \left|\left(\frac{\frac{h^\alpha}{\Gamma(\alpha+1)} - \phi_k}{\phi_k \frac{h^\alpha}{\Gamma(\alpha+1)}}\right)x(t_{k+1}) - \left(\frac{\frac{h^\alpha}{\Gamma(\alpha+1)} - \frac{\phi_k}{\varphi_k}}{\frac{\phi_k}{\varphi_k}\frac{h^\alpha}{\Gamma(\alpha+1)}}\right)x(t_k)\right| + \left|\frac{x(t_{k+1}) - x(t_k)}{\frac{h^\alpha}{\Gamma(\alpha+1)}} - ax(t_k) - by(t_k)\right| + |O(h^\alpha)|$$

$$|LTE_x| \le \left|\left(\frac{\frac{h^\alpha}{\Gamma(\alpha+1)} - \phi_k}{\phi_k}\right)\frac{x(t_{k+1})}{\frac{h^\alpha}{\Gamma(\alpha+1)}} - \left(\frac{\frac{h^\alpha}{\Gamma(\alpha+1)} - \frac{\phi_k}{\varphi_k}}{\frac{\phi_k}{\varphi_k}}\right)\frac{x(t_k)}{\frac{h^\alpha}{\Gamma(\alpha+1)}}\right| + |O(h^\alpha)| + |O(h^\alpha)| \tag{15}$$

from (15) we realize that $|LTE_x| \longrightarrow 0$ as $h \longrightarrow 0$, and similarly $|LTE_y| \longrightarrow 0$ as $h \longrightarrow 0$.

5.2. Stability of the numerical schemes. To prove the stability of the NSFD scheme for (13) and (14)

Let $e_k^x = x(t_k) - x_k$ and $e_k^y = y(t_k) - y_k$, the NSFD scheme (13) and (14) can be written as

$$\begin{aligned} x_{k+1} &= (\varphi + a\phi)x_k + b\phi y_k \\ y_{k+1} &= c\phi x_k + (\varphi + d\phi)y_k \end{aligned} \tag{16}$$

substituting $x(t_k)$ and $y(t_k)$ instead of x_k and y_k in (16), we obtain

$$\begin{aligned} x(t_{k+1}) &= (\varphi + a\phi)x(t_k) + b\phi y(t_k) \\ y(t_{k+1}) &= c\phi x(t_k) + (\varphi + d\phi)y(t_k) \end{aligned} \tag{17}$$

subtracting (17) from (16), substituting $e_k^x = x(t_k) - x_k$ and $e_k^y = y(t_k) - y_k$

$$\begin{aligned} e_{k+1}^x &= (\psi_k + a\phi_k)e_k^x + b\phi_k e_k^y \\ e_{k+1}^y &= c\phi_k e_k^x + (\psi_k + d\phi_k)e_k^y \end{aligned} \tag{18}$$

we can write (18) as

$$\begin{pmatrix} e_{k+1}^x \\ e_{k+1}^y \end{pmatrix} = \begin{pmatrix} \varphi + a\phi_k & b\phi_k \\ c\phi_k & \varphi + d\phi_k \end{pmatrix} \begin{pmatrix} e_k^x \\ e_k^y \end{pmatrix}$$

Generally

$$\begin{pmatrix} e_k^x \\ e_k^y \end{pmatrix} = \begin{pmatrix} (\psi_k + a\phi_k)^k & (b\phi_k)^k \\ (c\phi_k)^k & (\psi_k + d\phi_k)^k \end{pmatrix} \begin{pmatrix} e_0^x \\ e_0^y \end{pmatrix}$$

or

$$\begin{pmatrix} e_k^x \\ e_k^y \end{pmatrix} = \begin{pmatrix} \psi_k + a\phi_k & b\phi_k \\ c\phi_k & \psi_k + d\phi_k \end{pmatrix}^k \begin{pmatrix} e_0^x \\ e_0^y \end{pmatrix}$$

Let

$$B^{(k)} = \begin{pmatrix} \psi_k + a\phi_k & b\phi_k \\ c\phi_k & \psi_k + d\phi_k \end{pmatrix} = \psi_k \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \phi_k \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The eigenvalues of $B^{(k)}$ are $\psi_k + \lambda_1\phi_k$ and $\psi_k + \lambda_2\phi_k$. Hence, $e_k^x \longrightarrow 0$ and $e_k^y \longrightarrow 0$ as $k \longrightarrow \infty$ if and only if $P(B^{(k)}) < 1$.

we have

$$\sigma_1^k = \psi_k + \lambda_1\phi_k = \frac{E_\alpha(\lambda_1(hk+h)^\alpha)}{E_\alpha(\lambda_1(hk)^\alpha)}$$

where

$$\psi_k = \frac{\left[\lambda_1 \dfrac{E_\alpha(\lambda_2(hk+h)^\alpha)}{E_\alpha(\lambda_2(hk)^\alpha)} - \lambda_2 \dfrac{E_\alpha(\lambda_1(hk+h)^\alpha)}{E_\alpha(\lambda_1(hk)^\alpha)}\right] i}{2\omega}$$

and

$$\phi_k = \frac{\left[\dfrac{E_\alpha(\lambda_1(hk+h)^\alpha)}{E_\alpha(\lambda_1(hk)^\alpha)} - \dfrac{E_\alpha(\lambda_2(hk+h)^\alpha)}{E_\alpha(\lambda_2(hk)^\alpha)}\right] i}{2\omega}$$

then,

$$\left|\sigma_1^k\right| < 1 \quad \Rightarrow \quad \left|\frac{E_\alpha(\lambda_1(hk+h)^\alpha)}{E_\alpha(\lambda_1(hk)^\alpha)}\right| < 1 \Rightarrow |E_\alpha(\lambda_1(hk+h)^\alpha)| < |E_\alpha(\lambda_1(hk)^\alpha)|$$

Since, $(hk+h)^\alpha) > (hk)^\alpha)$, then $|E_\alpha(\lambda_1(hk+h)^\alpha)| < |E_\alpha(\lambda_1(hk)^\alpha)|$ implies that the Mittag-Leffler function is monotonically decreasing function in t. This can occur if and only if $arg(\lambda_1) > \frac{\alpha\pi}{2}$ or equivalently $\sigma = Re(\lambda_1) < 0$. similarly

$$\sigma_2^k = \psi_k + \lambda_2\phi_k \quad \text{and} \quad |\sigma_2^k| < 1 \quad \Rightarrow \quad arg(\lambda_2) > \frac{\alpha\pi}{2} \text{ or } \sigma < 0$$

We end up with the following theorem.

Theorem 5.1. *If the eigenvalues of the fractional linear system* (1) *have negative real parts, the nonstandard finite difference scheme described by equations (13) and (14) is unconditionally stable.*

Theorem 5.1 asserts that if the two eigenvalues of the linear system have negative real parts, the NSFDM will exhibit unconditional stability for any step size and fractional order. A modification to the coefficients a, b, c, or d of the linear system may impact the stability of the numerical methods if it alters the sign of the real part of the eigenvalues of the system. Any perturbation of the linear system parameters that does not change the sign of eigenvalue real part from negative to positive will guarantee the unconditional stability of the NSFDM.

6. APPLICATION TO THE FRACTIONAL HARMONIC OSCILLATOR MODEL

We consider the fractional harmonic oscillator equation

$$ {}_0^C D_t^\alpha u + \omega^2 u = 0,\ u(0) = u_0 \text{and } {}_0^C D_t^\alpha u(0) = v_0, \quad 0 < \alpha < 1, t > 0. \tag{19}$$

Let

$$ {}_0^C D_t^\alpha u = v,$$

then

$$ {}_0^C D_t^\alpha \left({}_0^C D_t^\alpha \right) u = {}_0^C D_t^\alpha v$$

Now, we have a system of two linear fractional differential equations of the form

$$ {}_0^C D_t^\alpha u \quad = \quad v, \quad u(0) = u_0 \tag{20}$$

$$ {}_{0}^{C}D_{t}^{\alpha} v = -\omega^2 u, \quad v(0) = {}_{0}^{C}D_{t}^{\alpha} u(0) = v_0 \tag{21} $$

In the matrix form, it can be expressed as

$$ {}_{0}^{C}D_{t}^{\alpha} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} $$

with eigenvalues

$$ \lambda_{1,2} = \pm \omega i $$

and corresponding eigenvectors,

$$ w_1 = \begin{pmatrix} \frac{i}{\omega} \\ 1 \end{pmatrix} \text{ and } w_1 = \begin{pmatrix} -\frac{i}{\omega} \\ 1 \end{pmatrix} $$

The analytical solution of system (20)-(21) is obtained by substituting $a = d = 0$, $b = 1$, $c = -\omega^2$ and $t_0 = 0$ in equations (8)-(9) to obtain:

$$ u(t) = \left(\frac{E_\alpha(i\omega t^\alpha) + E_\alpha(-i\omega t^\alpha)}{2} \right) u_0 + i \left(\frac{E_\alpha(-i\omega t^\alpha) - E_\alpha(i\omega t^\alpha) i}{2\,\omega} \right) v_0 \tag{22} $$

$$ v(t) = \left(\frac{E_\alpha(i\omega t^\alpha) + E_\alpha(-i\omega t^\alpha)}{2} \right) v_0 + i\,\omega \left(\frac{E_\alpha(i\omega t^\alpha) - E_\alpha(-i\omega t^\alpha)}{2} \right) u_0 \tag{23} $$

We can rewrite (20)-(21) to obtain a NSFD scheme for fractional harmonic oscillater (19) we have

$$ v_k = \frac{u_{k+1} - \psi u_k}{\phi} \text{ and } v_{k+1} = \frac{u_{k+2} - \psi u_{k+1}}{\phi} $$

substitute value of v_k and v_{k+1} in $\frac{v_{k+1} - \psi v_k}{\phi} = -\omega^2 u_k$

$$ \frac{\left[\frac{u_{k+2} - \psi u_{k+1}}{\phi} - \frac{\psi u_{k+1} + \psi^2 u_k}{\phi} \right]}{\phi} = -\omega^2 u_k, $$

which can be simplified as

$$ \frac{u_{k+2} - 2\psi u_{k+1} + \psi^2 u_k}{\phi^2} = -\omega^2 u_k, $$

or

$$ \frac{u_{k+1} - 2\psi u_k + \psi^2 u_{k-1}}{\phi^2} + \omega^2 u_k = 0. \tag{24} $$

7. NUMERICAL RESULTS

This section will present three examples to demonstrate the performance of the nonstandard finite difference methods established in this research.

We ran the numerical simulations in Python 3.12 on a computer with an Intel core i7 CPU. We used Khensin's implementation of the Mittag-Leffler function which can be found on GitHub.

Example 7.1. *Complex eigenvalues with negative real parts* We consider the linear system of FDEs

$$\begin{aligned} {}_0^C D_t^\alpha x(t) &= 2x(t)-5y(t), x(0)=2.0 \\ {}_0^C D_t^\alpha y(t) &= 5x(t)-4y(t), y(0)=-2.0 \end{aligned}$$

Let

$$A=\begin{pmatrix} 2 & -5 \\ 5 & -4 \end{pmatrix}$$

The eigenvalues of A are

$$\lambda_1=-1+4\,i and \lambda_2=-1-4\,i$$

The analytical solution is given by

$$\begin{aligned} x(t) &= (1-2\,i)\left(E_\alpha\left((-1+4\,i)t^\alpha\right)+(1+2\,i)E_\alpha\left((-1-4\,i)t^\alpha\right)\right) \\ y(t) &= \left(-1-\frac{i}{2}\right)E_\alpha\left((-1+4\,i)t^\alpha\right)+\left(-1+\frac{i}{2}\right)E_\alpha\left((-1-4\,i)t^\alpha\right) \end{aligned}$$

The numerator and denominator functions are given by

$$\psi_k(h,\alpha)=\frac{\left[(-1+4i)\dfrac{E_\alpha((-1-4i)h^\alpha(k+1)^\alpha)}{E_\alpha((-1+4i)h^\alpha k^\alpha)}-(-1-4i)\dfrac{E_\alpha((-1+4i)h^\alpha(k+1)^\alpha)}{E_\alpha((-1+4i)h^\alpha k^\alpha)}\right]i}{8}$$

and

$$\phi_k(h,\alpha)=\frac{\left[\dfrac{E_\alpha((-1+4i)h^\alpha(k+1)^\alpha)}{E_\alpha((-1+4i)h^\alpha(k)^\alpha)}-\dfrac{E_\alpha((-1-4i)h^\alpha(k+1)^\alpha)}{E_\alpha((-1-4i)h^\alpha k^\alpha)}\right]i}{8}.$$

The NSFD scheme from ((13)) and (14) as

$$\frac{x_{k+1}-\varphi x_k}{\phi}=2x_k-5y_k$$

$$\frac{y_{k+1} - \varphi y_k}{\phi} = 5x_k - 4y_k$$

Example 7.1 has been solved for fractional orders $\alpha = 0.05, 0.2, 0.35, 0.5, 0.65, 0.8, 0.95$ and 1.0, utilizing step sizes of $h = 0.5, 1.0, 1.25, 2.0, 2.5, 4.0, 5.0$, and 10.0. The infinity norm error for each pair of h and α is presented in Table 1.

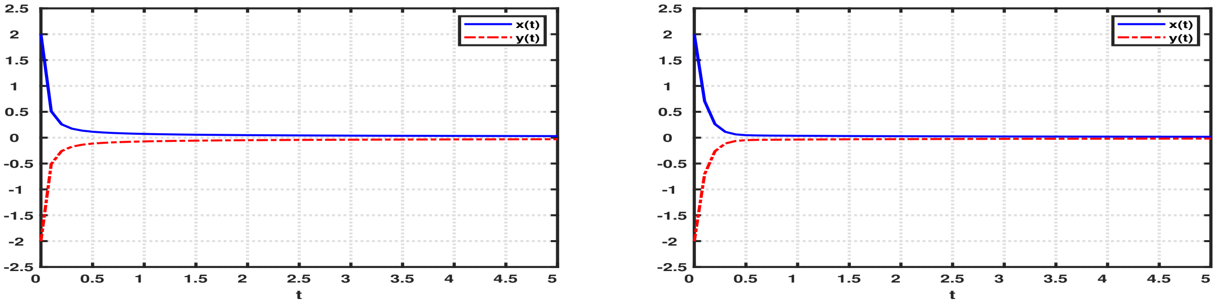
TABLE 1. The infinity norm errors obtained by the NSFD scheme (13)-(14) for Example (7.1), for different step sizes h and orders α.

$h \setminus \alpha$	0.05	0.20	0.35	0.50	0.65	0.80	0.95	1.00
0.50	1.41E-16	1.40E-16	4.18E-17	5.56E-17	2.00E-17	8.67E-18	1.83E-18	9.31E-18
1.00	1.39E-16	5.56E-17	6.94E-17	1.74E-17	8.71E-18	5.27E-18	1.34E-18	3.47E-18
1.25	8.42E-17	1.11E-16	7.19E-17	2.09E-17	1.06E-17	3.47E-18	8.77E-19	4.34E-19
2.00	1.11E-16	4.67E-17	4.17E-17	2.09E-17	7.10E-18	1.81E-18	1.73E-18	3.47E-18
2.50	9.78E-17	4.30E-17	2.08E-17	1.01E-17	3.50E-18	3.47E-18	4.42E-19	4.34E-19
4.00	2.98E-17	4.19E-17	2.78E-17	6.99E-18	3.61E-18	1.81E-18	2.27E-19	5.42E-20
5.00	4.15E-17	2.78E-17	2.83E-17	4.01E-18	3.51E-18	1.73E-18	1.10E-19	1.09E-25
10.00	0.00E+00	1.39E-17	1.40E-17	3.71E-18	1.77E-18	4.78E-19	2.17E-19	7.74E-34

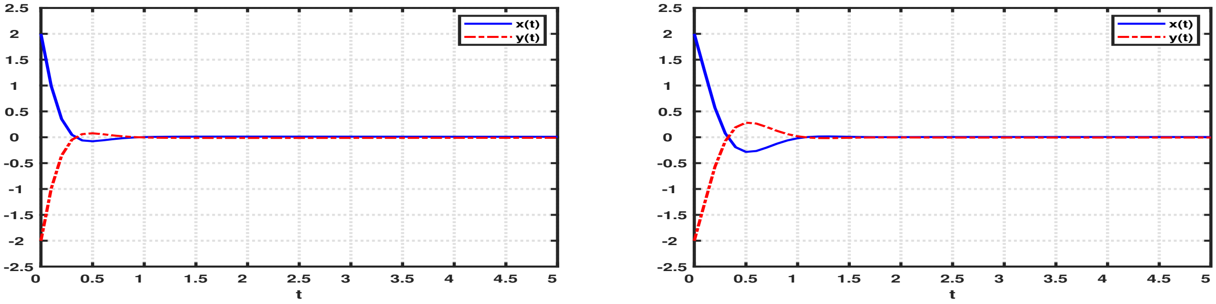
Table 1 demonstrates a steady performance of the numerical scheme, where the infinity norm errors computed by the numerical scheme (13)-(14) are of order 10^{-16} or less, including large step sizes. This indicates that the nonstandard finite difference scheme is exact and is highly stable.

The solutions of Example 7.1, obtained from the NSFDM (13)-(14) for fractional orders $\alpha = 0.25, 0.50, 0.75, 0.95$, and 1.00, with $t \in [0, 20]$, are shown in Figure 1.

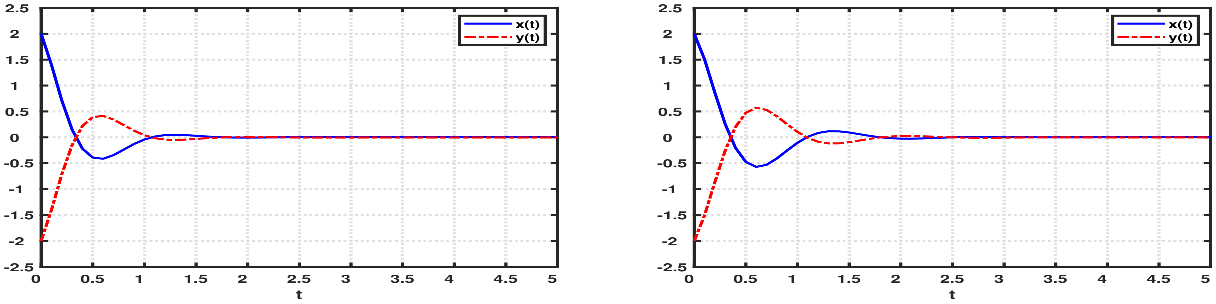
Figure 1 illustrates two distinct types of long-term behavior. For small values of α, the solution exhibits exponential decay. As α increases towards the value 1, the solution exhibits dominant oscillatory behavior, characterized by decaying oscillations that intensify with larger values of alpha.



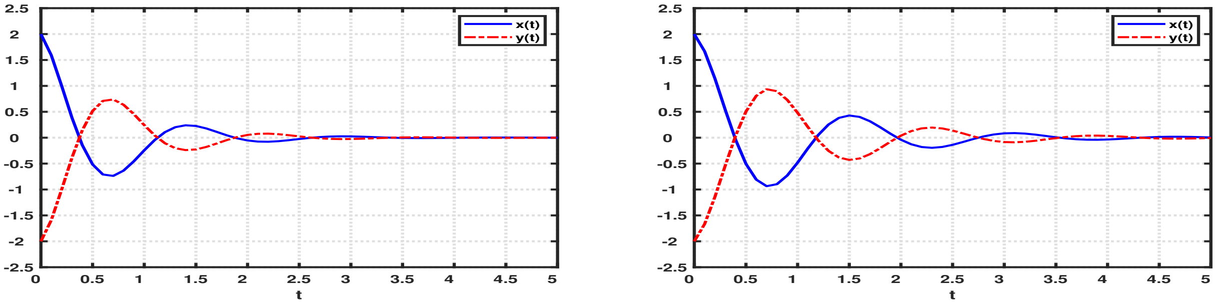
$\alpha = 0.50$ $\alpha = 0.60$



$\alpha = 0.70$ $\alpha = 0.80$



$\alpha = 0.85$ $\alpha = 0.90$



$\alpha = 0.95$ $\alpha = 1.00$

FIGURE 1. The solution of Example 7.1 for $\alpha = 0.50, 0.60, 0.70, 0.80, 0.85, 0.95$ and 1.00

Example 7.2. *Pure imaginary eigenvalues*

$$\begin{aligned} {}_0^CD_t^\alpha x(t) &= 2x(t) - 4y(t), x(0) = 1.0 \\ {}_0^CD_t^\alpha y(t) &= -2x(t) - 2y(t), y(0) = -1.0 \end{aligned}$$

Let

$$A = \begin{pmatrix} 2 & -4 \\ 2 & -2 \end{pmatrix}$$

The eigenvalues of A are

$$\lambda_1 = 2iand\lambda_2 = -2i$$

The analytical solution is given by

$$\begin{aligned} x(t) &= \left(\frac{1}{2} - \frac{3}{2}i\right) E_\alpha\left(2t^\alpha i\right) + \left(\frac{1}{2} + \frac{3}{2}i\right) E_\alpha\left(-2t^\alpha i\right) \\ y(t) &= -\frac{1}{2}\left(E_\alpha\left(2\,it^\alpha\right) + E_\alpha\left(-2\,it^\alpha\right)\right) \end{aligned}$$

The numerator and denominator functions are given by

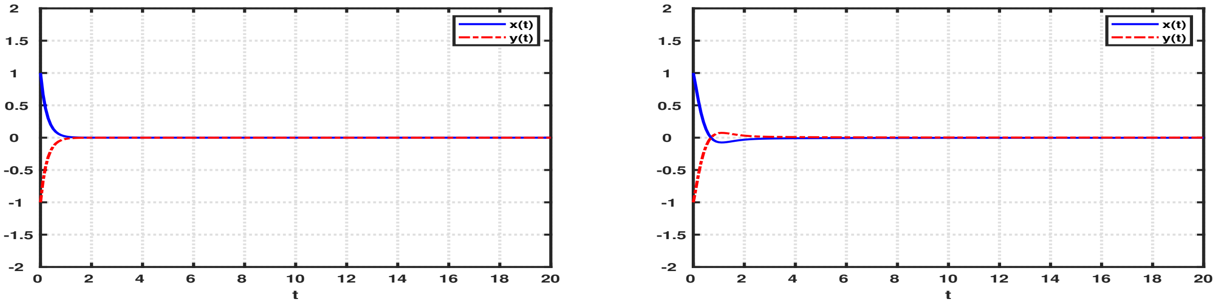
$$\begin{aligned} \psi_k(h,\alpha) &= \frac{-\left[\dfrac{E_\alpha(2h^\alpha(k+1)^\alpha i)}{E_\alpha(2h^\alpha k^\alpha i)} + \dfrac{E_\alpha(-2h^\alpha(k+1)^\alpha i)}{E_\alpha(-2h^\alpha k^\alpha i)}\right]}{2} \\ \phi_k(h,\alpha) &= \frac{\left[\dfrac{E_\alpha(2h^\alpha(k+1)^\alpha i)}{E_\alpha(2h^\alpha(k)^\alpha i)} - \dfrac{E_\alpha(-2h^\alpha(k+1)^\alpha i)}{E_\alpha(-2h^\alpha k^\alpha i)}\right] i}{4}. \end{aligned}$$

We let the fractional order α takes the values $0.05, 0.2, 0.35, 0.5, 0.65, 0.8, 0.95$ and 1.0, and the step size $h = 0.5, 1.0, 1.25, 2.0, 2.5, 4.0, 5.0$ and 10.0. By applying the proposed nonstandard finite difference scheme, we computed the infinity norm errors, defined as

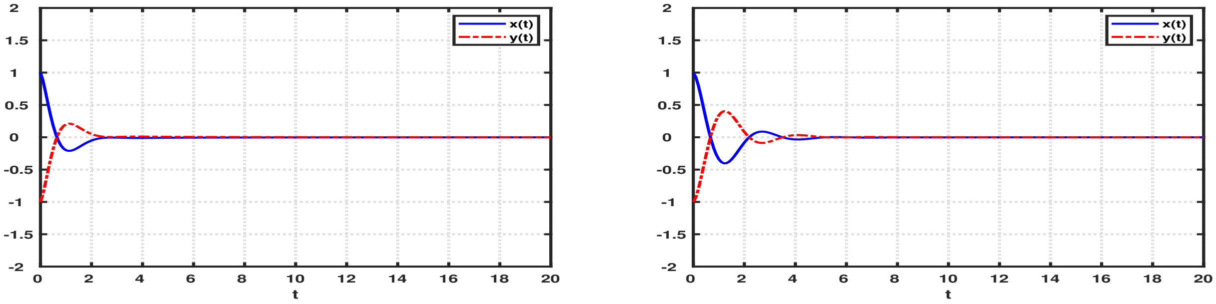
$$||Error||_\infty^{\alpha,h} = \max \max_{k=0,\ldots,N} \{|x_k - x(t_k)|, |y_k - y(t_k)|\}.$$

These errors are illustrated in Table 2.

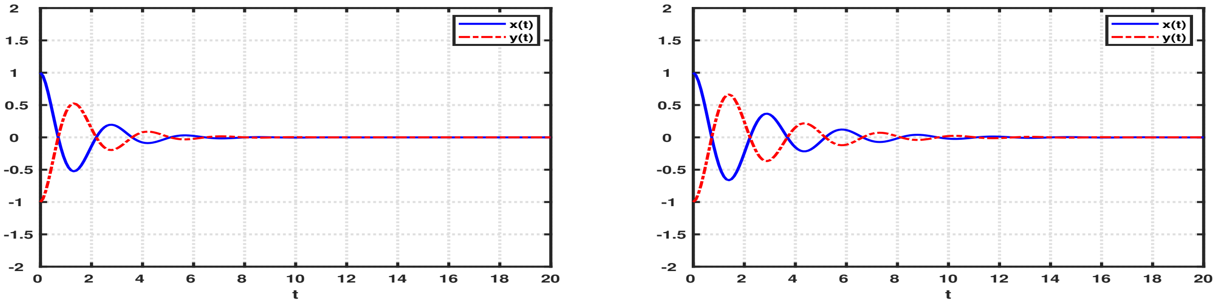
The solutions of Example 7.2, obtained from the NSFDM (13)-(14) for $\alpha = 0.25, 0.50, 0.75, 0.95$, and 1.00, with $t \in [0, 20]$, are demonstrated in Figure 2.



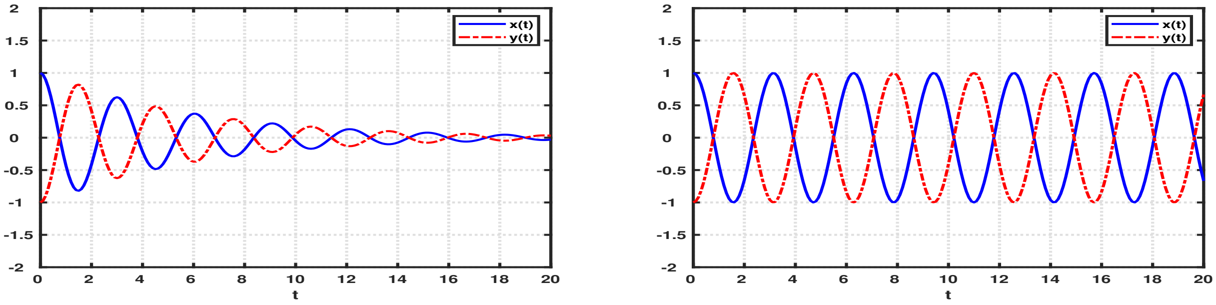
$\alpha = 0.50$ $\alpha = 0.60$



$\alpha = 0.70$ $\alpha = 0.80$



$\alpha = 0.85$ $\alpha = 0.90$



$\alpha = 0.95$ $\alpha = 1.00$

FIGURE 2. The solution of Example 7.2 for $\alpha = 0.50, 0.60, 0.70, 0.80, 0.85, 0.95$ and 1.00

TABLE 2. Norm infinity errors for Example 7.2, for $h \in \{0.5, 1.0, 1.25, 2.00, 2.50, 4.0, 5.0, 10\}$ and $\alpha \in \{0.05, 0.20, 0.35, 0.5, 0.65, 0.80, 0.95, 1.0\}$

$h \setminus \alpha$	0.05	0.20	0.35	0.50	0.65	0.80	0.95	1.00
0.50	1.58E-16	6.24E-17	3.97E-17	1.72E-18	3.19E-17	1.01E-16	1.44E-16	3.79E-16
1.00	1.14E-16	5.91E-17	1.39E-17	3.47E-18	1.39E-17	1.68E-17	1.58E-16	0.00E+00
1.25	9.74E-17	4.22E-17	7.05E-18	4.34E-19	2.79E-18	6.76E-18	7.94E-17	0.00E+00
2.00	2.81E-17	4.66E-17	6.94E-18	2.71E-20	6.94E-18	7.67E-18	1.93E-16	0.00E+00
2.50	8.68E-17	4.34E-17	5.01E-18	3.39E-21	1.73E-18	9.17E-19	7.69E-18	0.00E+00
4.00	1.93E-18	3.18E-17	3.90E-18	1.32E-23	6.40E-19	4.39E-19	5.46E-17	0.00E+00
5.00	4.89E-17	2.58E-17	7.81E-18	4.14E-25	5.84E-19	4.34E-19	7.88E-19	0.00E+00
10.00	0.00E+00	0.00E+00	2.93E-18	2.07E-33	5.62E-20	1.42E-19	1.82E-18	0.00E+00

Figure 2 illustrates three distinct types of long-term behavior. An exponential decay is achieved for low values of the fractional order α. For large values of α, the system exhibits decaying oscillatory behavior. For $\alpha = 1$, the solutions exhibit periodic behavior.

Example 7.3. *Complex eigenvalues with positive real parts* We consider the linear system of FDEs

$$\begin{aligned} {}_{0}^{C}D_t^{\alpha}x(t) &= 5x(t)+5y(t), x(0)=0.5,\ t\in[0,4] \\ {}_{0}^{C}D_t^{\alpha}y(t) &= -5x(t)-3y(t), y(0)=-1.0,\ t\in[0,4] \end{aligned}$$

Let

$$A = \begin{pmatrix} 5 & 5 \\ -5 & -3 \end{pmatrix}$$

The eigenvalues of A are

$$\lambda_1 = 1+3\,i and \lambda_2 = 1-3\,i$$

The analytical solution is given by

$$\begin{aligned} x(t) &= \left(\frac{1}{4}+\frac{1}{2}\,i\right)E_{\alpha}\left((1+3\,i)t^{\alpha}\right)+\left(\frac{1}{4}-\frac{1}{2}\,i\right)E_{\alpha}\left((1-3\,i)t^{\alpha}\right) \\ y(t) &= -\left(\frac{1}{2}-\frac{13}{12}i\right)E_{\alpha}\left((-1+3\,i)t^{\alpha}\right)-\left(\frac{1}{2}+\frac{13}{12}i\right)E_{\alpha}\left((-1-3\,i)t^{\alpha}\right) \end{aligned}$$

The numerator and denominator functions are given by

$$\psi_k(h,\alpha) = \frac{\left[(1+3i)\dfrac{E_\alpha((1-3i)h^\alpha(k+1)^\alpha)}{E_\alpha((1-3i)h^\alpha k^\alpha)} - (1-3i)\dfrac{E_\alpha((1+3i)h^\alpha(k+1)^\alpha)}{E_\alpha((1+3i)h^\alpha k^\alpha)}\right] i}{6}$$

and

$$\phi_k(h,\alpha) = \frac{\left[\dfrac{E_\alpha((1+3i)h^\alpha(k+1)^\alpha)}{E_\alpha((1+3i)h^\alpha(k)^\alpha)} - \dfrac{E_\alpha((1-3i)h^\alpha(k+1)^\alpha)}{E_\alpha((1-3i)h^\alpha k^\alpha)}\right] i}{6}.$$

The NSFD scheme from ((13)) and (14) as

$$\frac{x_{k+1} - \varphi x_k}{\phi} = 5x_k + 5y_k$$

$$\frac{y_{k+1} - \varphi y_k}{\phi} = -5x_k - 3y_k$$

We solved the problem for $\alpha = 0.05, 0.2, 0.35, 0.5, 0.65, 0.8, 0.95$ and 1.0 with step sizes $h = 0.03125, 0.06250, 0.125, 0.250, 0.50, 1.0$ and 2.0. The infinity norm error corresponding to each couple h and α is illustrated in Table 3.

TABLE 3. The infinity norm errors obtained by the NSFD scheme (13)-(14) for Example (7.3), for different step sizes h and orders α.

$h \setminus \alpha$	0.05	0.20	0.35	0.50	0.65	0.80	0.95	1.00
0.03125	7.08E-18	1.58E-17	2.23E-17	2.09E-17	1.29E-16	7.04E-16	1.27E-14	2.85E-14
0.06250	3.73E-18	1.87E-17	1.58E-17	1.86E-17	6.04E-17	3.42E-16	1.16E-15	7.11E-15
0.12500	1.75E-18	7.68E-18	1.75E-17	1.44E-17	7.39E-17	8.37E-17	3.55E-15	7.56E-15
0.25000	1.76E-18	4.23E-18	9.00E-18	7.21E-18	9.05E-18	1.30E-16	0.00E+00	2.14E-14
0.50000	1.06E-18	7.30E-18	3.58E-18	6.97E-18	3.54E-18	3.69E-18	1.51E-15	0.00E+00
1.00000	9.02E-19	4.29E-18	2.00E-18	4.11E-18	3.94E-18	4.80E-19	0.00E+00	0.00E+00
2.00000	8.71E-19	1.48E-18	1.13E-18	9.87E-19	4.57E-19	1.02E-18	0.00E+00	0.00E+00

Despite the eigenvalues of the linear system in Example 7.3 possessing positive real parts, Table 3 illustrates the consistent efficacy of the numerical scheme, wherein the infinity norm errors calculated by the numerical scheme (13)-(14) are of the order of 10^{-14} or lower, even with large step sizes. This signifies that the nonstandard finite difference technique has high accuracy and stability.

The solutions for Example 7.3, derived from the NSFDM (13)-(14) at values of $\alpha = 0.35$, 0.50, 0.65, 0.796, 0.85, 0.90, and 1.00, over the interval $t \in [0,4]$, are presented in Figure 3.

Figure 3 depicts four different kinds of long-term behavior of the solution. For minimal values of the fractional order α, an algebraic decay is achieved. Subsequently, fading oscillations manifest for a secondary range of the fractional order α. A periodic solution is achieved at a critical value of α. Oscillatory growth is achieved when α exceeds the threshold value.

Example 7.4. *Harmonic Oscillator*Consider the fractional harmonic oscillator equation

$$ {}_0^C D_t^{2\alpha} u + u = 0, u(0) = 1.5, {}_0^C D_t^{\alpha} u(0) = 1, \quad 0 < \alpha < 1 $$

In this example, $c = -\omega^2 = -1 \Rightarrow \omega = i$. Therefore the analytical solution of Example 7.4 is given by:

$$ \begin{aligned} x(t) &= \left(\frac{3}{4} - \frac{1}{2} i\right) E_\alpha(it^\alpha) + \left(\frac{3}{4} + \frac{1}{2} i\right) E_\alpha(-it^\alpha) \\ y(t) &= \left(\frac{1}{2} + \frac{3}{4} i\right) E_\alpha(it^\alpha) + \left(\frac{1}{2} - \frac{3}{4} i\right) E_\alpha(-it^\alpha) \end{aligned} $$

The numerator and denominator functions are given by

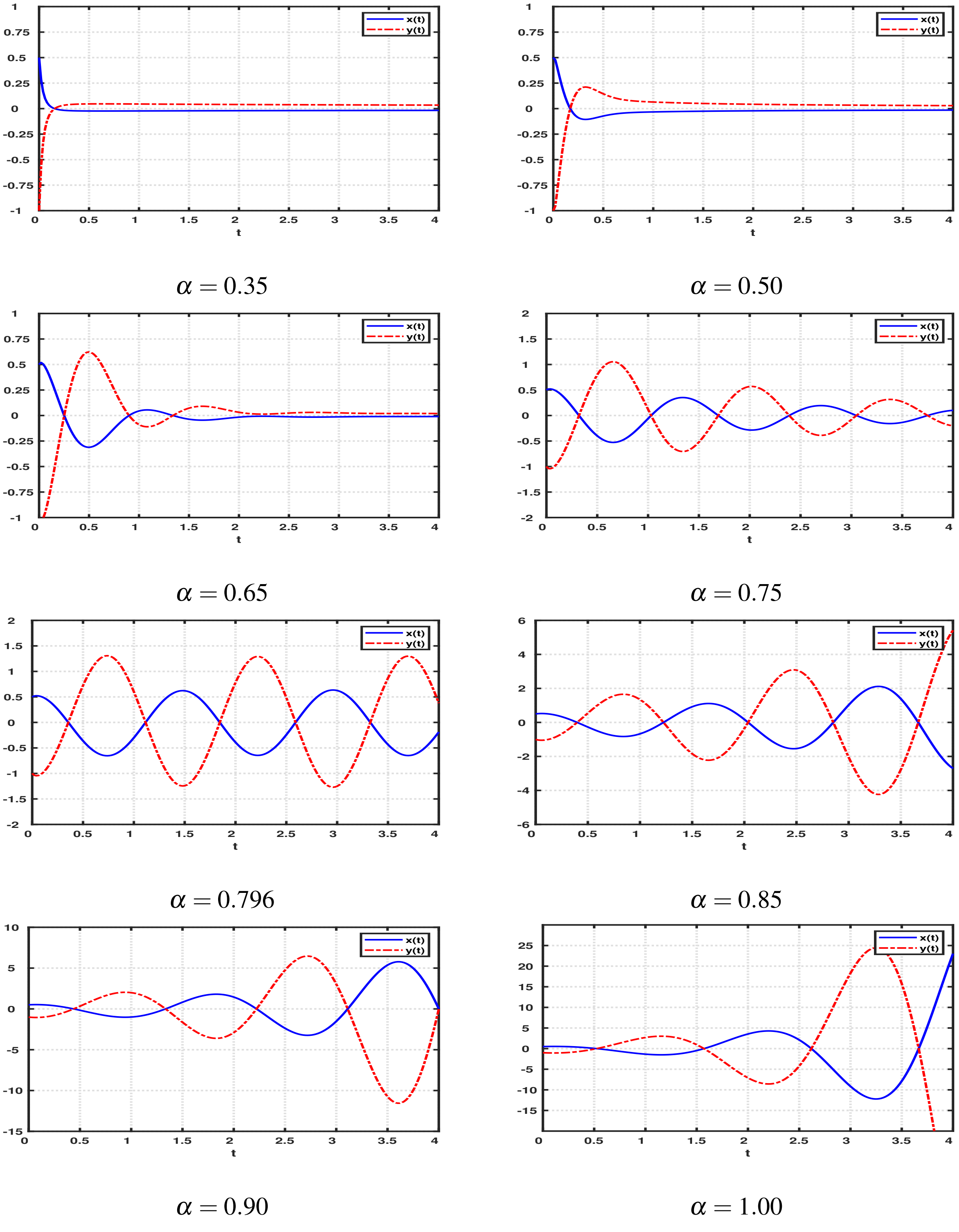
$$ \begin{aligned} \psi_k(h,\alpha) &= \frac{\left[\dfrac{E_\alpha(-ih^\alpha(k+1)^\alpha)}{E_\alpha(-ih^\alpha k^\alpha)} + \dfrac{E_\alpha(ih^\alpha(k+1)^\alpha)}{E_\alpha(ih^\alpha k^\alpha)}\right] i}{2} \\ \phi_k(h,\alpha) &= \frac{\left[\dfrac{E_\alpha(ih^\alpha(k+1)^\alpha)}{E_\alpha(ih^\alpha(k)^\alpha)} - \dfrac{E_\alpha(-ih^\alpha(k+1)^\alpha)}{E_\alpha(-ih^\alpha k^\alpha)}\right] i}{8}. \end{aligned} $$

We let the fractional order α takes the values $0.05, 0.2, 0.35, 0.5, 0.65, 0.8, 0.95$ and 1.0, and the step size h takes the values $h = 0.5, 1.0, 1.25, 2.0, 2.5, 4.0, 5.0$ and 10.0. By applying the proposed nonstandard finite difference scheme, we computed the infinity norm errors, defined as

$$ ||Error||_\infty^{\alpha,h} = \max \max_{k=0,\ldots,N} \{|x_k - x(t_k)|, |y_k - y(t_k)|\}. $$

These errors are illustrated in Table 4.

The solution of Example 7.4, obtained from the NSFDM (13)-(14) for $\alpha = 0.25, 0.50, 0.75, 0.95$, and 1.00, with $t \in [0,20]$, is shown in Figure 4.



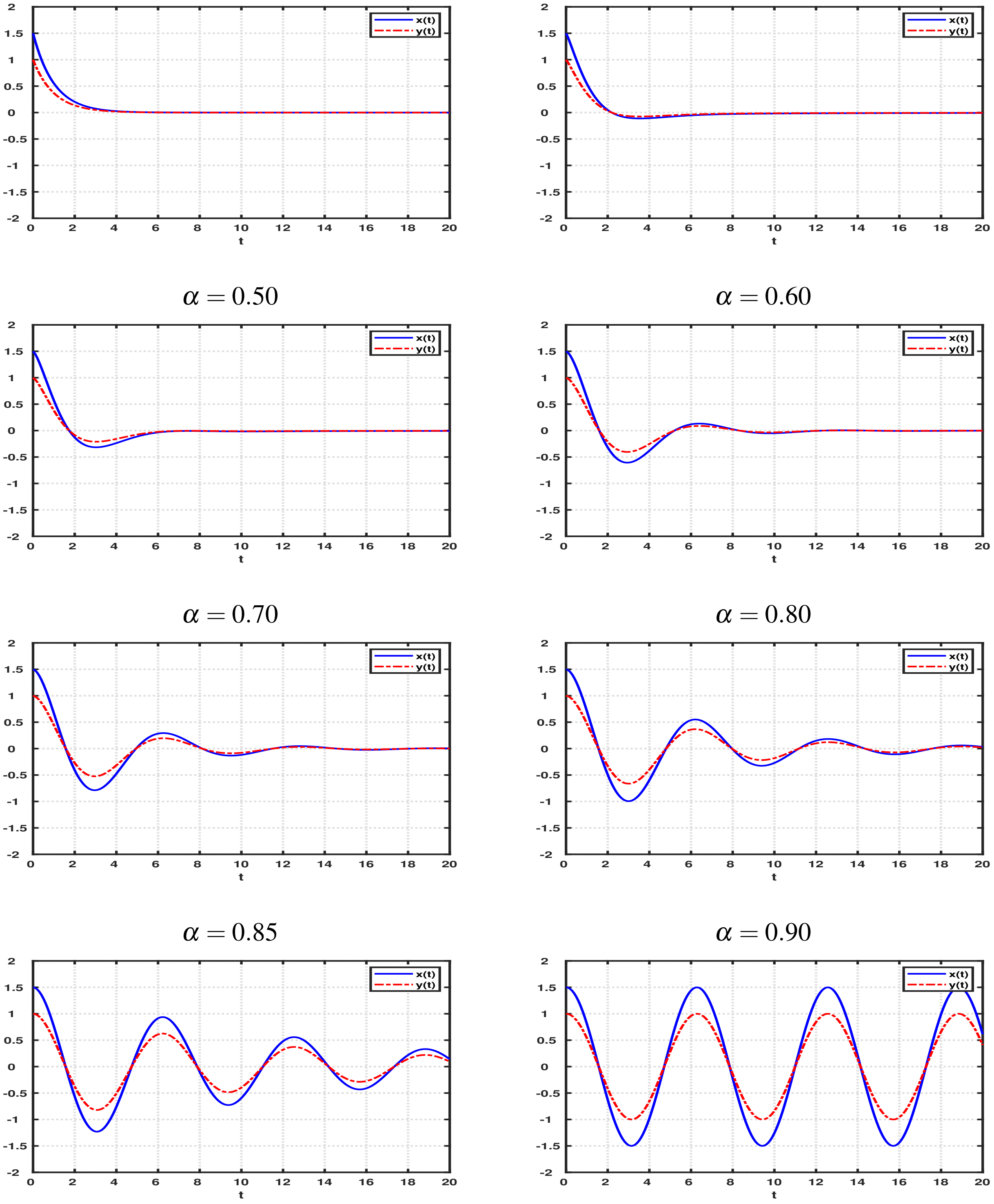
$\alpha = 0.35$ $\alpha = 0.50$

$\alpha = 0.65$ $\alpha = 0.75$

$\alpha = 0.796$ $\alpha = 0.85$

$\alpha = 0.90$ $\alpha = 1.00$

FIGURE 3. The solution of Example 7.1 for $\alpha = 0.35, 0.50, 0.65, 0.785, 0.85, 0.90$ and 1.00



$\alpha = 0.50$ $\alpha = 0.60$

$\alpha = 0.70$ $\alpha = 0.80$

$\alpha = 0.85$ $\alpha = 0.90$

$\alpha = 0.95$ $\alpha = 1.00$

FIGURE 4. The solution of Example 7.4 for $\alpha = 0.50, 0.60, 0.70, 0.80, 0.85, 0.95$ and 1.00

TABLE 4. Norm infinity errors for Example 4, for $h \in \{0.5, 1.0, 1.25, 2.00, 2.50, 4.0, 5.0, 10\}$ and $\alpha \in \{0.05, 0.20, 0.35, 0.5, 0.65, 0.80, 0.95, 1.0\}$

$h \setminus \alpha$	0.05	0.20	0.35	0.50	0.65	0.80	0.95	1.00
0.50	5.57E-16	1.69E-16	8.93E-17	1.12E-16	5.56E-17	9.35E-17	3.38E-16	6.66E-16
1.00	2.36E-16	6.66E-17	6.04E-17	2.78E-17	3.19E-17	6.82E-17	1.68E-16	4.46E-16
1.25	1.24E-16	2.25E-16	5.74E-17	1.75E-18	8.37E-17	1.13E-16	3.33E-16	0.00E+00
2.00	3.34E-16	8.11E-17	2.86E-17	8.61E-19	2.21E-17	3.47E-17	7.18E-17	0.00E+00
2.50	2.24E-16	6.44E-17	8.38E-17	1.76E-18	2.83E-17	4.16E-17	1.36E-16	0.00E+00
4.00	1.23E-16	5.56E-17	2.82E-17	1.73E-18	6.94E-18	8.68E-19	0.00E+00	0.00E+00
5.00	7.49E-17	2.65E-17	1.55E-17	2.17E-19	1.75E-18	4.37E-19	0.00E+00	0.00E+00
10.00	1.48E-17	0.00E+00	1.65E-18	1.69E-21	1.13E-18	1.44E-19	0.00E+00	0.00E+00

Figure 4 illustrates three distinct types of long-term behavior exhibited by the harmonic oscillator. For small to medium values of the fractional order α, the system exhibits exponential decay. For large values of α, a decaying oscillatory behavior is observed. The system exhibits periodic behavior when $\alpha = 1$. The harmonic oscillator, as a particular case in point of linear systems characterized by pure imaginary eigenvalues, exhibits long-term behavior akin to that of systems with imaginary eigenvalues.

8. CONCLUSIONS

In this work, we have developed accurate nonstandard finite difference methods (NSFDMs) for solving systems of linear fractional differential equations (FDEs) with complex eigenvalues. The studied cases were categorized into four types: complex eigenvalues with negative real parts, purely imaginary eigenvalues, complex eigenvalues with positive real parts, and finally, the classical model of the fractional harmonic oscillator, the stability and accuracy properties of all cases were thoroughly analyzed.

Example 7.1 complex eigenvalues with negative real parts was solved using fractional orders $\alpha = 0.05, 0.2, \ldots, 1.0$ and time step sizes up to $h = 10.0$. As shown in Table 1, the errors produced by the NSFDM method were on the order of 10^{-16} or less, indicating remarkable accuracy and stability, even with large step sizes.

Figure 1 illustrates a distinctive long-term behavior, rapid exponential decay at small values of α, while near $\alpha = 1.0$.

Example 7.2 using purely imaginary eigenvalues with the same range of fractional orders α and step sizes h shown in Table 2, the results exhibited high accuracy and strong stability of the method.

Figure 2 reveals three distinct behaviors of the solution as α varies: exponential decay at small values, damped oscillations at medium to high values, and full periodic behavior when $\alpha = 1.0$.

In Example 7.3 the complex eigenvalues with positive real parts typically leads to numerical instability and the results in Table 3 showed that the proposed method maintained a low error level down to the order of 10^{-14}, even with large step sizes, demonstrating its effectiveness under these challenging conditions.

In Figure 3, the solution displays four distinct long-term behaviors depending on α: starting with algebraic decay, followed by damped oscillations, then periodic behavior at a critical value of α, and finally persistent oscillatory growth at higher values.

As a special case of systems 7.4 is fractional harmonic oscillator with purely imaginary eigenvalues, Table 4 shows that the ∞-norm error.

Figure 4 exhibits three characteristic behaviors: exponential decay for small to moderate values of α, damped oscillations at higher values, and purely periodic behavior when $\alpha = 1.0$.

The results demonstrate that the proposed nonstandard finite difference methods are highly effective in solving systems of fractional differential equations with complex eigenvalues, the methods exhibit near-unconditional stability, except for systems with eigenvalues having positive real parts, which may require more refined analysis.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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