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NONLINEAR DYNAMICS OF A PREDATOR–PREY SYSTEM WITH KLEPTOPARASITIC INTERACTIONS

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Abstract: This study explores a predator–prey model featuring one prey and two competing predators linked through kleptoparasitism. Unlike traditional approaches, we introduce a nonlinear formulation where the interaction simultaneously suppresses the host predator's growth while boosting the kleptoparasite. We determine the system's equilibrium points and evaluate their stability through the Routh–Hurwitz criteria and Lyapunov methods. Our findings reveal that the system's dynamics are highly sensitive to parameter shifts. Numerical simulations reveal three main results: full species coexistence, competitive elimination, and bistability. Importantly, the environmental carrying capacity serves as a key factor that initiates changes among these varying dynamic states. Additionally, the presence of bistability highlights that the system's long-term fate can be completely reliant on the numbers of initial populations. These findings imply that our proposed formulation presents a novel viewpoint compared to traditional interference-based models. By addressing these complex responses, this framework delivers a more refined comprehension of how kleptoparasitic behavior influence predator-prey relationships across various ecological environments.

Keywords: kleptoparasitic; predator-prey; nonlinear dynamics.

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1. INTRODUCTION

In the study of ecosystem dynamics, predator–prey models serve as a fundamental framework for analyzing food chain interactions. The classical representation of this interaction was introduced by Lotka and Volterra using a system of nonlinear differential equations [1, 2]. Over time, this model has been extended to better capture of more realistic biological processes. Initially, the Lotka–Volterra model described a two-species interaction between predator and prey. This framework was later extended to a three-dimensional system involving one prey and two predators. In such systems, the dynamics become more complex due to competition between predators for the same prey [3–10]. Several studies have explored additional effects in three-species models, including diffusion processes [11, 12] and seasonal or behavioral interactions among predators [13]. It is well known that increasing resource availability does not always stabilize such systems; in some cases, it can lead to instability, a phenomenon known as the paradox of enrichment [14]. In recent developments of ecological modeling, the incorporation of kleptoparasitism into predator-prey systems has attracted increasing attention. Kleptoparasitism refers to a foraging strategy in which one species obtains food by stealing prey captured by another species. Materassi et al. [15] introduced this concept into multi-trophic network models, showing that the intensity of food theft can influence ecosystem stability. This approach was further developed by Spencer and Broom [16] using a game-theoretical framework, demonstrating that kleptoparasitic success depends on differences in competitive ability. Garay et al. [17] combined evolutionary game theory with optimal foraging theory and showed that time and energy constraints can alter classical prey selection behavior. Empirical evidence of kleptoparasitism has also been observed in marine ecosystems, such as interactions between grey reef sharks and whitetip reef sharks [18, 19]. These findings indicate that kleptoparasitism plays an important role in shaping ecological dynamics [20].

Most mathematical models incorporating kleptoparasitism describe its effect through modified functional responses or behavioral terms. Consequently, different modeling approaches may lead to different mathematical representations of the interaction mechanisms between predator populations. This motivates the consideration of alternative formulations that provide different perspectives on how kleptoparasitic interactions can be incorporated into continuous-time dynamical systems.

The model developed in this study is motivated by the work of Bhattacharjee et al. [21], which incorporates kleptoparasitism into the framework proposed by Hsu et al. [22]. Their model describes the interaction between one prey population (x), a host predator (y_1), and a

kleptoparasitic predator (y_2), which their growths are given by:

$$\begin{aligned}\frac{dx}{dt} &= r_1 x\left(1-\frac{x}{k}\right) - b_2 x y_1 - b_3 x y_2,\\ \frac{dy_1}{dt} &= \frac{b_2 c_4 x y_1}{1+\alpha y_2} - k_1 y_1 y_2 - d_2 y_1,\\ \frac{dy_2}{dt} &= b_3 c_5 x y_2 - k_2 y_1 y_2 - d_3 y_2 + c_6 x y_1 y_2.\end{aligned} \tag{1}$$

In this formulation, the effect of kleptoparasitism is incorporated through a modified functional response, where the term $\frac{1}{1+\alpha\, y_2}$ represents the reduction in the host predator's growth due to kleptoparasitic interference. The term $c_6 x y_1 y_2$ represents the gain obtained by the kleptoparasitic predator through its interaction with both the prey and the host predator. This provides one possible mathematical representation of kleptoparasitic interaction within the system.

In this study, we consider an alternative formulation in which kleptoparasitism is represented through nonlinear interaction terms involving the prey and both predator populations. This formulation offers a different mathematical perspective for describing kleptoparasitic interactions and their influence on population dynamics.

This study is inspired by ecological interactions observed in coral reef ecosystems, where grey reef sharks may act as kleptoparasites by taking food from whitetip reef sharks, while stingrays are treated as the prey [23]. The main objective of this study is to investigate how this alternative representation of kleptoparasitic interaction influences the stability structure and coexistence conditions of a predator–prey system with one prey and two competing predators.

The main contributions of this study can be summarized as follows: (i) proposing a predator–prey model with an alternative formulation of kleptoparasitic interaction, (ii) establishing the well-posedness of the system, (iii) analyzing the local and global stability of equilibrium points, and (iv) supporting the analytical results with numerical simulations. This study provides an additional mathematical perspective for understanding the role of kleptoparasitism in ecological dynamics.

2. MODEL FORMULATION

In this study, we propose a predator–prey model that incorporates both interspecific competition and kleptoparasitic interactions between two predator populations. In contrast to the formulation in system (1), the kleptoparasitic interaction is represented here through an alternative functional form. This approach is intended to provide another mathematical representation for

describing kleptoparasitic mechanisms in population dynamics. The dynamics of the modified model are given by:

$$\begin{aligned}\frac{dx}{dt} &= r_1 x\left(1-\frac{x}{k}\right) - b_2 x y_1 - b_3 x y_2,\\ \frac{dy_1}{dt} &= b_2 c_4 x y_1 - a b_2 c_4 x y_1 y_2 - k_1 y_1 y_2 - d_2 y_1,\\ \frac{dy_2}{dt} &= b_3 c_5 x y_2 - k_2 y_1 y_2 - d_3 y_2 + c_6 x y_1 y_2.\end{aligned} \tag{2}$$

All parameters are assumed to be positive and the initial conditions satisfy $x(0) > 0, y_1(0) > 0$, and $y_2(0) > 0$. A complete description of each parameter and its biological significance can be found in Table 1.

By following a logistic growth pattern, the prey population inherently accounts for the environment's carrying capacity. Both predator populations consume the same prey, resulting in in direct competition through resource exploitation as well as direct interspecific interactions. We model these kleptoparasitic dynamics using nonlinear terms $x\,y_1 y_2$, where the host's growth is suppressed to fuel the kleptoparasite's expansion. This approach offers a distinct mathematical alternative to the model (1).

3. Existence and Uniqueness of Solution

This section establishes the existence and uniqueness of solution of system (2) as stated in the following theorem.

Table 1. Parameter Description

r_1	:	Intrinsic growth rate of the prey population
k	:	Environmental carrying capacity
b_2	:	Predation rate of prey by the first predator
b_3	:	Predation rate of prey by the second predator
c_4	:	Rate of biomass conversion of prey into births of the first predator
c_5	:	Rate of biomass conversion of prey into births of the second predator
c_6	:	Rate of food conversion from kleptoparasitism into biomass of the second predator
k_1	:	Interspecies competition rate for the first predator
k_2	:	Interspecies competition rate for the second predator
d_2	:	Mortality rate of the first predator
d_3	:	Mortality rate of the second predator
a	:	Intensity rate of kleptoparasitism by the second predator

Theorem 1. Let,

$$\Omega = \{Q = (x, y_1, y_2) \in \mathbb{R}^3_+ , 0 \leq x, y_1, y_2 \leq \frac{\rho}{\psi}\}$$

be a bounded region for some positive constant M. Then, for any initial condition $(x(0), y_1(0), y_2(0)) \in \Omega$, system (2) admits a unique solution defined on $\Omega \times [0, T]$, for T > 0.

Proof. Let $Q = (x, y_1, y_2)$ and $\bar{Q} = (\bar{x}, \overline{y_1}, \overline{y_2})$ are two different solutions of system (2) and define the mapping

$$G(Q) = (G_1(Q), G_2(Q), G_3(Q)),$$

where

$$G_1(Q) = r_1 x \left(1 - \frac{x}{k}\right) - b_2 x y_1 - b_3 x y_2,$$
$$G_2(Q) = b_2 c_4 x y_1 - a b_2 c_4 x y_1 y_2 - k_1 y_1 y_2 - d_2 y_1,$$
$$G_3(Q) = b_3 c_5 x y_2 - k_2 y_1 y_2 - d_3 y_2 + c_6 x y_1 y_2.$$

We equip $\mathbb{R}^3_+$ with the L^1 −norm defined by

$$\|(Q) - (\bar{Q})\| = |x - \bar{x}| + |y_1 - \overline{y_1}| + |y_2 - \overline{y_2}|.$$

Since Ω is bounded, there exists a constant $M > 0$ such that

$$0 \leq x, y_1, y_2, \ \bar{x}, \ \overline{y_1}, \overline{y_2} \leq M, \text{ for all } Q, \bar{Q} \in \Omega.$$

For any $Q, \bar{Q} \in \Omega$ we consider

$$\|G(Q) - G(\bar{Q})\| = |G_1(Q) - G_1(\bar{Q})| + |G_2(Q) - G_2(\bar{Q})| + |G_3(Q) - G_3(\bar{Q})|.$$

First, for G_1, we have

$$|G_1(Q) - G_1(\bar{Q})| = \left| r_1 x \left(1 - \frac{x}{k}\right) - b_2 x y_1 - b_3 x y_2 - \left(r_1 \bar{x} \left(1 - \frac{\bar{x}}{k}\right) - b_2 \bar{x}\, \overline{y_1} - b_3 \bar{x}\, \overline{y_2}\right)\right|.$$

Using triangle inequality theorem and the boundedness of variable, we obtain

$$|G_1(Q) - G_1(\bar{Q})| \leq \left(r_1 + \frac{2r_1}{k} M + b_2 M + b_3 M\right) |x - \bar{x}| + [b_2 M]|y_1 - \overline{y_1}| + [b_3 M]|y_2 - \overline{y_2}|.$$

Now, for G_2, we have

$$|G_2(Q) - G_2(\bar{Q})|$$
$$= | b_2 c_4 x y_1 - a\, b_2 c_4 x y_1 y_2 - k_1 y_1 y_2 - d_2 y_1 - (b_2 c_4 \bar{x}\, \overline{y_1} - a b_2 c_4 \bar{x}\, \overline{y_1}\ \overline{y_2} - k_1 \overline{y_1}\, \overline{y_2} - d_2 \overline{y_1})|.$$

By applying the triangle inequality and using the boundedness of the variables, we obtain

$$|G_2(Q) - G_2(\bar{Q})| \leq (b_2 c_4 M + a b_2 c_4 M^2)|x - \bar{x}| + [b_2 c_4 M + a b_2 c_4 M^2 + k_1 M + d_2\ ]|y_1 - \overline{y_1}| + [a b_2 c_4 M^2 + k_1 M]|y_2 - \overline{y_2}|.$$

Similarly, for G_3, we obtain

$|G_3(Q) - G_3(\bar{Q})|$

$= |b_3c_5xy_2 - k_2y_1y_2 - d_3y_2 + c_6xy_1y_2 - (b_3c_5\bar{x}\overline{y_2} - k_2\overline{y_1}\,\overline{y_2} - d_3\overline{y_2} + c_6\bar{x}\overline{y_1}\,\overline{y_2})|.$

Thus,

$$|G_3(Q) - G_3(\bar{Q})| \leq (b_3c_5M + c_6M^2)|x - \bar{x}| + (k_2M + c_6M^2)|y_1 - \overline{y_1}| + (b_3c_5M + k_2M + d_3 + c_6M^2)|y_2 - \overline{y_2}|.$$

Combining the above inequalities, we obtain

$$\|G(Q) - G(\bar{Q})\| \leq S_1(|x - \bar{x}| + S_2|y_1 - \overline{y_1}| + S_3|y_2 - \overline{y_2}|)$$

where

$$S_1 = r_1 + \left(\frac{2r_1}{k} + b_2 + b_3 + b_2c_4 + b_3c_5\right)M + (ab_2c_4 + c_6)M^2,$$
$$S_2 = d_2 + (b_2 + b_2c_4 + k_1 + k_2)M + (ab_2c_4 + c_6)M^2,$$
$$S_3 = d_3 + (b_3 + b_3c_5 + k_1 + k_2)M + (ab_2c_4 + c_6)M^2.$$

Let $S = \max\{S_1, S_2, S_3\}$, Then

$$\|G(Q) - G(\bar{Q})\| \leq S\|Q - \bar{Q}\|.$$

Thus, $G(Q)$ satisfies the Lipschitz condition on Ω. By the existence and uniqueness theorem for ordinary differential equations [24], system (2) admits a unique solution for any initial condition in Ω.

4. POSITIVITY AND BOUNDEDNESS

This section establishes the non-negativity and boundedness of solution of system (2) as shown in the Theorem 2 and Theorem 3 respectively.

Theorem 2. All solution of system (2) with initial value $(x(0), y_1(0), y_2(0)) \in \mathbb{R}^3_+$ remain non-negative for a $t > 0$.

Proof. We first prove that $x(t) > 0$ for all $t > 0$. Suppose $x(0) > 0$. Assume, by contradiction, that there exists $t_1 > 0$ such that

$$x(t) > 0 \text{ for } 0 < t < t_1, \qquad x(t_1) = 0, \qquad x(t) < 0 \text{ for } t > t_1.$$

From the first equation of system (2), we obtain

$$\left.\frac{dx}{dt}\right|_{t=t_1} = 0.$$

Thus, the derivative at $t = t_1$ is non-negative, which inplies that the solution cannot cross the axis $x = 0$. This contradicts the assumption that $x(t)$ become negative for $t > t_1$. Hence, $x(t) \geq 0$ for all $t > 0$.

Using the same argument, it can be shown that $y_1(t) \geq 0$ and $y_2(t) \geq 0$ for all $t \geq 0$.

Theorem 3. All solutions of system (2) are bounded in the region

$\Omega = \left\{(x, y_1, y_2) \in \mathbb{R}^3_+ : 0 \leq x + \frac{y_1}{c_4} + \frac{y_2}{c_5} \leq \frac{\rho}{\psi}\right\}$, for some positive constants ρ and ψ.

Proof. From the first equation in system (2), we have

$$\frac{dx}{dt} = r_1 x \left(1 - \frac{x}{k}\right) - b_2 x y_1 - b_3 x y_2 \leq r_1 x \left(1 - \frac{x}{k}\right),$$

since $-b_2 x y_1 - b_3 x y_2 \leq 0$. Hence, by comparison with the logistic equation, we obtain

$\lim_{t\to\infty} sup\ x(t) \leq k$, so that $x(t)$ is bounded.

Next, define the function $H(t) = x + \frac{y_1}{c_4} + \frac{y_2}{c_5}$.

Differentiating $H(t)$ along the solutions of system (2), we obtain

$$\frac{dH}{dt} = r_1 x \left(1 - \frac{x}{k}\right) - \left(ab_2 - \frac{c_6}{c_5}\right) x y_1 y_2 - \left(\frac{k_1}{c_4} + \frac{k_2}{c_5}\right) y_1 y_2 - \frac{d_2}{c_4} y_1 - \frac{d_3}{c_5} y_2.$$

Assume that $ab_2 > \frac{c_6}{c_5}$ so that the term involving $x\, y_1\, y_2$ is non-positive. Choose a positive constant ψ such that $\psi < min\{d_2, d_3\}$. By grouping the linear terms and using the above assumption, we obtain $\frac{dH}{dt} + \psi H \leq r_1 x \left(1 - \frac{x}{k} + \frac{\psi}{r_1}\right)$,

completing the square yields $\frac{dH}{dt} + \psi H \leq -\frac{r_1}{k}\left(x - \frac{(r_1+\psi)k}{2r_1}\right)^2 + \frac{(r_1+\psi)^2 k}{4r_1}$.

Thus, $\frac{dH}{dt} + \psi H \leq \rho$, where $\rho = \frac{(r_1+\psi)^2 k}{4r_1}$.

Multiplying both sides by $e^{\psi t}$ and integrating, we obtain

$$H(t) \leq \frac{\rho}{\psi}\left(1 - e^{-\psi t}\right) + H(0) e^{-\psi t},$$

Taking the limit as $t \to \infty$, we get $0 < H(t) \leq \frac{\rho}{\psi}$. Therefore, all solutions of system (2) are bounded in Ω .

5. Equilibrium Point and Local Stability

The equilibrium points of system (2) are determined by setting the right-hand sides equal to zero and solving the resulting algebraic system. The corresponding equilibrium points and their existence conditions are presented in the following theorem.

Theorem 4. System (2) admits the following equilibrium points:

1. The trivial equilibrium, denoted by $E_0 = (0,0,0)$, which always exists in $\mathbb{R}^3_+$.

2. The predator-free equilibrium, denoted by $E_1 = (k, 0, 0)$, which always exists for $k > 0$.
3. The host predator extinction equilibrium is given by $E_2 = \left(\frac{d_3}{b_3 c_5}, 0, \frac{r_1(kb_3c_5 - d_3)}{b_3^2 k c_5}\right)$.

   This equilibrium exists in R_+^3 provided that $k > \frac{d_3}{b_3 c_5}$.
4. The kleptoparasite predator extinction equilibrium is given by $E_3 = \left(\frac{d_2}{b_2 c_4}, \frac{r_1(kb_2c_4 - d_2)}{b_2^2 c_4 k}, 0\right)$.

   This equilibrium exists in R_+^3 provided that $k > \frac{d_2}{b_2 c_4}$.
5. The coexistence equilibrium is denoted by $E_4 = (x^*, {y_1}^*, {y_2}^*)$,

   where, $x^* = \frac{k_1 {y_2}^* + d_2}{b_2 c_4 - a b_2 c_4 {y_2}^*}$, ${y_1}^* = \frac{d_3(b_2 c_4 - a b_2 c_4 {y_2}^*) - b_3 c_5 (k_1 {y_2}^* + d_2)}{c_6(k_1 {y_2}^* + d_2) - k_2(b_2 c_4 - a b_2 c_4 {y_2}^*)}$.

   The variable ${y_2}^*$ are the positive real roots of the cubic equation

   $c_3 y_2^3 + c_2 y_2^2 + c_1 y_2 + c_0 = 0$,

where,

$$c_3 = k\, a\, b_2 b_3 c_4 (k_1 c_6 + k_2 a b_2 c_4),$$

$$c_2 = -k k_1 a b_2^2 b_3 c_4 c_5 + k a b_2 b_3 c_4 c_6 d_2 - k k_1 a r_1 b_2 c_4 c_6 - k k_2 a^2 r_1 b_2^2 c_4^2 - k_1^2 r_1 c_6 + k_1 k_2 a r_1 b_2 c_4 - k a^2 d_3 b_2^3 c_4^2 - k k_1 b_2 b_3 c_4 c_6 - 2 k k_2 a b_2^2 b_3 c_4^2,$$

$$c_1 = k k_1 r_1 b_2 c_4 c_6 + 2 k k_2 a r_1 b_2^2 c_4^2 + k_1 k_2 r_1 b_2 c_4 + k_2 a d_2 r_1 b_2 c_4 - k a d_2 r_1 b_2 c_4 c_6 - 2 k_1 d_2 r_1 c_6 + 2 k d_3 a b_2^3 c_4^2 + k k_1 b_2^2 b_3 c_4 c_5 - k a d_2 b_2^2 b_3 c_4 c_5 - k d_2 b_2 b_3 c_4 c_6 + k k_2 b_2^2 b_3 c_4^2,$$

$$c_0 = k d_2 r_1 b_2 c_4 c_6 - k k_2 r_1 b_2^2 c_4^2 - d_2^2 r_1 c_6 + k_2 d_2 r_1 b_2 c_4 - k d_3 b_2^3 c_4^2 + k d_2 b_2^2 b_3 c_4 c_5.$$

The coexistence equilibrium E_4 exists in $\mathbb{R}_+^3$ if $1 - a y_2^* > 0$, $c_6(k_1 y_2^* + d_2) > k_2 b_2 c_4 (1 - a y_2^*)$, and $d_3 b_2 c_4 (1 - a y_2^*) > b_3 c_5 (k_1 y_2^* + d_2)$.

The local stability of the equilibrium points of system (2) is analyzed using linearization. The Jacobian matrix of the system at an arbitrary point (x, y_1, y_2) is given by

$$j = \begin{pmatrix} i_{11} & i_{12} & i_{13} \\ i_{21} & i_{22} & i_{23} \\ i_{31} & i_{32} & i_{33} \end{pmatrix},$$

where,

$$i_{11} = r_1 - \frac{2 r_1 x}{k} - b_2 y_1 - b_3 y_2,\ i_{12} = -b_2 x,\ i_{13} = -b_3 x,\ i_{21} = b_2 c_4 y_1 - a b_2 c_4 y_1 y_2,$$

$$i_{22} = b_2 c_4 x - a b_2 c_4 x y_2 - k_1 y_2 - d_2,\ i_{23} = -a b_2 c_4 x y_1 - k_1 y_1,$$

$$i_{31} = b_3 c_5 y_2 + c_6 y_1 y_2,\ i_{32} = -k_2 y_2 + c_6 x y_2,\ i_{33} = b_3 c_5 x - k_2 y_1 - d_3 + c_6 x y_1.$$

The local stability of each equilibrium point is assessed by evaluating the Jacobian

matrix at the corresponding equilibrium and analyzing its eigenvalues. It is easy to observe that E_0 is always unstable, while E_1 can be asymptotically stable if $k < \min\left\{\frac{d_2}{b_2c_4}, \frac{d_3}{b_3c_5}\right\}$.

Theorem 5. *The equilibrium point* $E_2 = (x^*, 0, {y_2}^*)$, *is locally asymptotically stable if* $b_2c_4x^* < ab_2c_4\, x^*{y_2}^* + k_1{y_2}^* + d_2$, while $E_3 = \left(x^*, {y_1}^*, 0\right)$ is as if $x^* < \frac{k_2{y_1}^* + d_3}{b_3c_5 + c_6{y_1}^*}$.

Proof. The Jacobian matrix at E_2 is

$$J(E_2) = \begin{bmatrix} -\frac{r_1x^*}{k} & -b_2x^* & -b_3x^* \\ 0 & b_2c_4x^* - ab_2c_4\, x^*{y_2}^* - k_1{y_2}^* - d_2 & 0 \\ b_3c_5{y_2}^* & -k_2{y_2}^* + c_6x^*{y_2}^* & 0 \end{bmatrix}.$$

The first eigenvalue $\lambda_1 = b_2c_4x^* - ab_2c_4\, x^*{y_2}^* - k_1{y_2}^* - d_2$ is negatif if

$$b_2c_4x^* < ab_2c_4\, x^*{y_2}^* + k_1{y_2}^* + d_2.$$

The remaining eigenvalues are obtained from the submatrix $\begin{pmatrix} -\frac{r_1x^*}{k} & -b_3x^* \\ b_3c_5{y_2}^* & 0 \end{pmatrix}$,

where its trace is negative and its determinant is positive. Therefore, both eigenvalues have negative real parts. The stability condition for E_3 can be proven analogically.

Theorem 6. *The coexistence equilibrium* $E_4 = (x^*, {y_1}^*, {y_2}^*)$ *is locally asymptotically stable if the Routh-Hurwitz conditions* $\sigma_1 > 0$, $\sigma_3 > 0$, $\sigma_1\sigma_2 - \sigma_3 > 0$, are satisfied.

Proof. The Jacobian matrix at E_4 is given by $J(E_4) = [K_{ij}]$, where the entries are obtained by evaluating the Jacobian at$= (x^*, {y_1}^*, {y_2}^*)$. The characteristic equation is

$$\lambda^3 + \sigma_1\lambda^2 + \sigma_2\lambda + \sigma_3 = 0.$$

By the Routh-Hurwitz criterion, all eigenvalues have negative real parts if the stated conditions hold. Due to the algebraic complexity of the coefficients, the Routh-Hurwitz conditions are verified numerically in the subsequent section.

6. GLOBAL STABILITY ANALYSIS

In this section, we establish the global asymptotic stability of the equilibrium points using Lyapunov functions and LaSalle's invariance principle.

Theorem 7. *The equilibrium point* $E_1 = (k, 0, 0)$ *is globally asymptotically stable if*

$$k < \min\left\{\frac{d_2}{b_2(1 + c_4)}, \frac{d_3}{b_3(1 + c_5)}, \frac{k_1 + k_2}{c_6}\right\}.$$

Proof. Consider the Lyapunov function

$$V_1(x, y_1, y_2) = \left(x - x^* - x^* \ln \frac{x}{x^*}\right) + y_1 + y_2,$$

where $x^* = k$. This function is positive definite for $x > 0, y_1 \geq 0$ and $y_2 \geq 0$.

Taking the time derivative of V_1 along the solutions of system (2), we obtain

$$\frac{dV_1}{dt} = \left(\frac{x - x^*}{x}\right)\frac{dx}{dt} + \frac{dy_1}{dt} + \frac{dy_2}{dt}.$$

Substituting system (2) and using $x^* = k$, we have

$$\frac{dV_1}{dt} = (x - k)\left[r_1\left(1 - \frac{x}{k}\right) - b_2 y_1 - b_3 y_2\right] + [b_2 c_4 x y_1 - a b_2 c_4 x y_1 y_2 - k_1 y_1 y_2 - d_2 y_1]$$
$$+[b_3 c_5 x y_2 - k_2 y_1 y_2 - d_3 y_2 + c_6 x y_1 y_2].$$

Let

$$A_1 = (x - k)\left[r_1\left(1 - \frac{x}{k}\right) - b_2 y_1 - b_3 y_2\right],$$
$$A_2 = b_2 c_4 x y_1 - a b_2 c_4 x y_1 y_2 - k_1 y_1 y_2 - d_2 y_1,$$
$$A_3 = b_3 c_5 x y_2 - k_2 y_1 y_2 - d_3 y_2 + c_6 x y_1 y_2.$$

Then $\frac{dV_1}{dt} = A_1 + A_2 + A_3$.

First, we estimate A_1. Since $x \leq k$ from boundedness, we obtain

$$A_1 = r_1(x - k)\left(1 - \frac{x}{k}\right) - (x - k)(b_2 y_1 + b_3 y_2) \leq k b_2 y_1 + k b_3 y_2.$$

Next, ignoring negative terms in A_2, we have

$$A_2 \leq b_2 c_4 x y_1 - k_1 y_1 y_2 - d_2 y_1.$$

Similarly

$$A_3 \leq b_3 c_5 x y_2 - k_2 y_1 y_2 - d_3 y_2 + c_6 x y_1 y_2.$$

Using the boundedness $x \leq k$, we obtain

$$b_2 c_4 x y_1 \leq b_2 c_4 k y_1, \qquad b_3 c_5 x y_2 \leq b_3 c_5 k y_2, \qquad c_6 x y_1 y_2 \leq c_6 k y_1 y_2.$$

Thus,

$$\frac{dv_1}{dt} \leq [k b_2(1 + c_4) - d_2] y_1 + [k b_3(1 + c_5) - d_3] y_2 + [c_6 k - k_1 - k_2] y_1 y_2.$$

Therefore, $\frac{dv_1}{dt} < 0$ if $k < \frac{d_2}{b_2(1+c_4)}$, $k < \frac{d_3}{b_3(1+c_5)}$, $k < \frac{k_1 + k_2}{c_6}$.

Moreover $\frac{dv_1}{dt} = 0$ if and only if $(x, y_1, y_2) = (k, 0, 0)$. Hence, by LaSalle's invariance principle, the equilibrium point E_1 is globally asymptotically stable under the stated conditions.

Theorem 8. *The equilibrium point* $E_2 = (x^*, 0, y_2^*)$ is globally asymptotically stable if

$$b_2x^* + \frac{k_2}{c_5}\ y_2^* < \frac{d_2}{c_4}\ , \qquad \frac{c_6}{c_5}k < \frac{k_1}{c_4} + \frac{k_2}{c_5}.$$

Proof. Consider the Lyapunov function

$$V_2(x, y_1, y_2) = \left(x - x^* - x^* ln\frac{x}{x^*}\right) + \frac{1}{c_4}y_1 + \frac{1}{c_5}\left(y_2 - \ y_2^* - \ y_2^*\ ln\frac{y_2}{y_2^*}\ \right),$$

where $x^* = \frac{d_3}{b_3c_5}, y_2^* = \frac{r_1(kb_3c_5 - d_3)}{kb_3^2c_5}$.

It is clear that V_2 is positive definite.

Taking the derivative of V_2 along the solutions of system (2), we obtain

$$\frac{dV_2}{dt} = \frac{x - x^*}{x}\frac{dx}{dt} + \frac{1}{c_4}\frac{dy_1}{dt} + \frac{1}{c_5}\left(\frac{y_2 - y_2^*}{y_2}\ \right)\frac{dy_2}{dt}.$$

Substituting system (2), we get

$$\frac{dV_2}{dt} = B_1 + B_2 + B_3,$$

where,

$$B_1 = \ (x\ -\ x^*)\ \left[r_1\left(1 - \frac{x}{k}\right) - b_2y_1 - b_3y_2\right],$$

$$B_2 = \frac{1}{c_4}(b_2c_4xy_1 - ab_2c_4xy_1y_2 - k_1y_1y_2 - d_2y_1),$$

$$B_3 = \frac{1}{c_5}\ (y_2 - y_2^*)(b_3c_5x - k_2y_1 - d_3 + c_6xy_1).$$

Since E_2 is an equilibrium point, we have $b_3c_5x^* - d_3 = 0$. Using this relation, B_3 becomes

$$B_3 = b_3(x - x^*)(y_2 - y_2^*) - \frac{k_2}{c_5}y_1(y_2 - y_2^*) + \frac{c_6}{c_5}xy_1(y_2 - y_2^*).$$

Next, using the identity $1\ - \frac{x}{\mathrm{k}} = 1\ - \frac{x^*}{\mathrm{k}} - \frac{x - x^*}{k}$, and the fact that $1\ - \frac{x^*}{\mathrm{k}} = \frac{b_3y_2^*}{\mathrm{r}_1}$,

we obtain $B_1 = \ b_3y_2^*(x\ -\ x^*) - \frac{\mathrm{r}_1}{\mathrm{k}}(x - x^*)^2\ - (x\ -\ x^*)(b_2y_1 + b_3y_2)$.

Thus

$$B_1 \le -b_2(x - x^*)y_1 - b_3(x - x^*)(y_2 - y_2^*).$$

Combining $B_1, B_2,$ and B_3, we obtain

$$\frac{dV_2}{dt} \le b_2x^*y_1 - \frac{d_2}{c_4}\ y_1 + \frac{k_2}{c_5}\ y_1y_2^* + \frac{c_6}{c_5}xy_1y_2 - \frac{k_1}{c_4}y_1y_2 - \frac{k_2}{c_5}y_1y_2.$$

Since $x \le k$, we have $\frac{c_6}{c_5}xy_1y_2 \le \frac{c_6}{c_5}ky_1y_2$.

Therefore,

$$\frac{dV_2}{dt} \leq \left(b_2x^* + \frac{k_2}{c_5}y_2^* - \frac{d_2}{c_4}\right)y_1 + \left(\frac{c_6}{c_5}k - \frac{k_1}{c_4} - \frac{k_2}{c_5}\right)y_1y_2$$

Hence, $\frac{dV_2}{dt} < 0$ if $b_2x^* + \frac{k_2}{c_5}\ y_2^* < \frac{d_2}{c_4}$, $\frac{c_6}{c_5}k < \frac{k_1}{c_4} + \frac{k_2}{c_5}$.

Moreover, $\frac{dV_2}{dt} = 0$ if and only if $(x, y_1, y_2) = (x^*, 0, y_2^*)$. Thus, by LaSalle's invariance principle, the equilibrium point E_2 is globally asymptotically stable.

Theorem 9. *The equilibrium point* $E_3 = (x^*, y_1^*, 0)$ is globally asymptotically stable if $\frac{c_6k}{c_5} < \frac{k_1}{c_4} + \frac{k_2}{c_5}$, $b_3x^* + \frac{k_1}{c_4}y_1^* + ab_2k\ y_1^* < \frac{d_3}{c_5}$.

Proof. Consider the Lyapunov function

$$V_3(x, y_1, y_2) = \left(x - x^* - x^*\ln\frac{x}{x^*}\right) + \frac{1}{c_4}\left(y_1 - y_1^* - y_1^*\ln\frac{y_1}{y_1^*}\right) + \frac{1}{c_5}y_2,$$

where

$$x^* = \frac{d_2}{b_2c_4}, y_1^* = \frac{r_1(kb_2c_4 - d_2)}{b_2^2c_4k}.$$

Clearly, V_3 is positive definite.

Taking the time derivative along the solutions of system (2), we obtain

$$\frac{dV_3}{dt} = \left(\frac{x - x^*}{x}\right)\frac{dx}{dt} + \frac{1}{c_4}\left(\frac{y_1 - y_1^*}{y_1}\right)\frac{dy_1}{dt} + \frac{1}{c_5}\frac{dy_2}{dt}.$$

Substituting system (2), we write

$$\frac{dV_3}{dt} = C_1 + C_2 + C_3,$$

where

$$C_1 = (x - x^*)\left[r_1\left(1 - \frac{x}{k}\right) - b_2y_1 - b_3y_2\right],$$

$$C_2 = \frac{(y_1 - y_1^*)}{c_4}(b_2c_4x - ab_2c_4xy_2 - k_1y_2 - b_2c_4x^*),$$

$$C_3 = \frac{1}{c_5}(b_3c_5xy_2 - k_2y_1y_2 - d_3y_2 + c_6xy_1y_2).$$

Since E_3 is an equilibrium point, we have $b_2c_4x^* - d_2 = 0$.

Using the identity $1 - \frac{x}{k} = 1 - \frac{x^*}{k} - \frac{x - x^*}{k}$ and the relation $1 - \frac{x^*}{k} = \frac{b_2y_1^*}{r_1}$,

we obtain $C_1 = b_2y_1^*(x - x^*) - \frac{r_1}{k}(x - x^*)^2 - (x - x^*)(b_2y_1 + b_3y_2)$.

Thus, $C_1 \leq -b_2(x - x^*)(y_1 - y_1^*) - b_3y_2(x - x^*)$.

Next, substituting $d_2 = b_2c_4x^*$ into C_2, we obtain

$$C_2 = \frac{(y_1 - {y_1}^*)}{r_4}(b_2c_4x - ab_2c_4xy_2 - k_1y_2 - b_2c_4x^*),$$

$$C_2 = b_2(x - x^*)(y_1 - {y_1}^*) - ab_2xy_2(y_1 - {y_1}^*) - \frac{k_1}{c_4}y_2(y_1 - {y_1}^*).$$

Moreover, $C_3 = b_3xy_2 - \frac{k_2}{c_5}y_1y_2 - \frac{d_3}{c_5}y_2 + \frac{c_6}{c_5}xy_1y_2$. Combining C_1, C_2, and C_3, we obtain

$$\frac{dV_3}{dt} \leq \left(b_3x^* + \frac{k_1}{c_4}y_1^* + ab_2xy_1^* - \frac{d_3}{c_5}\right)y_2 + \left(\frac{c_6x}{c_5} - \frac{k_2}{c_5} - \frac{k_1}{c_4}\right)y_1y_2.$$

Since $x \leq k$, we obtain $ab_2xy_1^* \leq ab_2ky_1^*$, $\frac{c_6}{c_5}x \leq \frac{c_6k}{c_5}$.

Thus,

$$\frac{dV_3}{dt} \leq \left(b_3x^* + \frac{k_1}{c_4}y_1^* + ab_2ky_1^* - \frac{d_3}{c_5}\right)y_2 + \left(\frac{c_6k}{c_5} - \frac{k_2}{c_5} - \frac{k_1}{c_4}\right)y_1y_2.$$

Hence, $\frac{dV_3}{dt} < 0$ if $b_3x^* + \frac{k_1}{c_4}y_1^* + ab_2ky_1^* < \frac{d_3}{c_5}$, $\frac{c_6}{c_5}k < \frac{k_2}{c_5} + \frac{k_1}{c_4}$.

Moreover $\frac{dV_3}{dt} = 0$ if and only if $(x, y_1, y_2) = (x^*, y_1^*, 0)$. Therefore, by LaSalle's invariance principle, the equilibrium point E_3 is globally asymptotically stable.

7. NUMERICAL SIMULATION

To illustrate the analytical results, we performed a series of numerical simulations using a fourth-order Runge-Kutta scheme. We specifically chose four different initial conditions (3,0.01,0.4), (3,0.01,0.8), (0.8,0.3,0.3), and (1,0.8,0.3) to show the sensitivity of the solution to the initial condition which may leads to the bistability. Each set of parameters was carefully tuned to meet the stability criteria we derived.

We first illustrate the stability of coexistence equilibrium E_4 by setting the parameters as follows

$$r_1 = 0.9, \quad b_2 = 0.9, \quad b_3 = 0.7, \quad c_4 = 0.7, \quad c_5 = 0.25, \quad c_6 = 1,$$
$$a = 0.88, \quad k_1 = 1.5, \quad k_2 = 0.01, \quad d_2 = 0.28, \quad d_3 = 0.25, \quad k = 1.$$

By using these parameters the system (2) has 4 equilibrium point, namely E_0, E_1, E_3, and E_4. It is found that $E_4 \approx (0.76705, 0.15292, 0.10557)$, $\sigma_1 = 0.692214$, $\sigma_2 = 0.093257$, $\sigma_3 = 0.007208$ and $\sigma_1\sigma_2 - \sigma_3 = 0.057332$.

Applying the parameter set to the Theorem 6,

$$\sigma_1 > 0, \sigma_2 > 0, \ \sigma_3 > 0, \text{and } \sigma_1\sigma_2 - \sigma_3 > 0.$$

Therefore, all Routh-Hurwitz conditions in Theorem 6 are satisfied, which confirms that the coexistence equilibrium E_4 is locally asymptotically stable.

$\sigma_1 > 0, \sigma_2 > 0,\ \sigma_3 > 0,$ and $\sigma_1\sigma_2 = 0.06454 > \sigma_3 = 0.007208.$

Therefore, all Routh-Hurwitz conditions in Theorem 6 are satisfied, which confirms that the coexistence equilibrium E_4 is locally asymptotically stable.

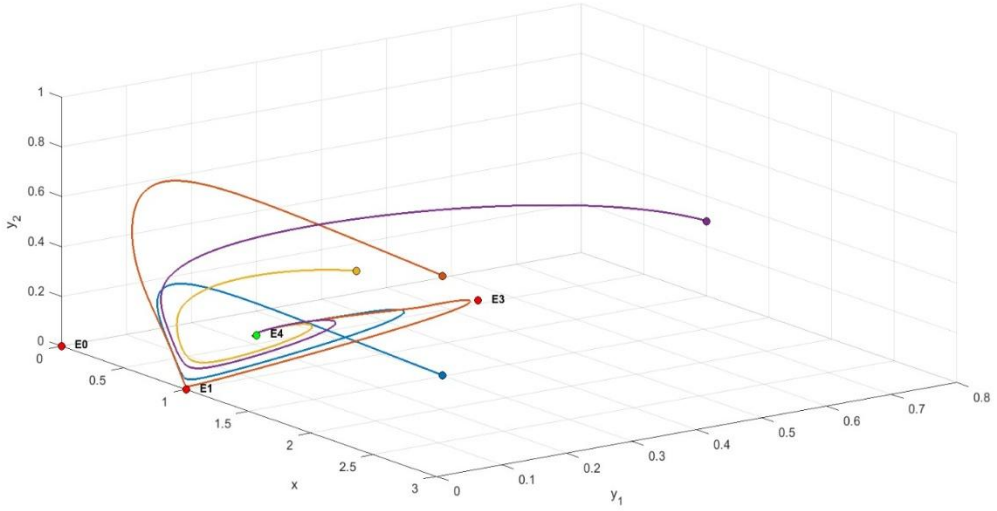

FIGURE 1: Phase portrait illustrating convergence to the coexistence equilibrium E_4 for $k = 1$.

As illustrated in Figure 1, all trajectories converge to E_4 from different initial conditions, indicating that it acts as a strong attractor under this parameter regime. Biologically, this corresponds to stable coexistence between the prey, the host predator, and the kleptoparasitic predator. This indicates that kleptoparasitism is not always a disruptive force; under suitable conditions, it may instead promote a stable ecological equilibrium rather than its disruption.

If we increase the carrying capacity to $k = 1.9$, while all other parameters are kept fixed, we found that, E_4 does not exist, remaining 4 equilibrium point $E_0, E_1, E_2,$ and E_3. Especially, we found that $E_2 = (1.42857,0,0.31901)$. According to Theorem 5, equilibrium point E_2 is locally asymptotically stable because $b_2c_4x^* - (ab_2c_4x^*y_2^* + k_1y_2^* + d_2) = -0.032 < 0.$

As illustrated in Figure 2, all trajectories converge to E_2, marking a clear qualitative shift in the system's dynamics. The coexistence equilibrium observed at $k = 1$ disappears, and the system evolves toward a boundary equilibrium where only one predator population persists. This transition highlights a nontrivial ecological effect: increasing the carrying capacity does not necessarily promote coexistence. Instead, it may drive the system toward competitive exclusion. This change in behavior indicates the presence of parameter-induced transitions in the system dynamics as k varies, leading to a qualitative shift in the long-term outcome.

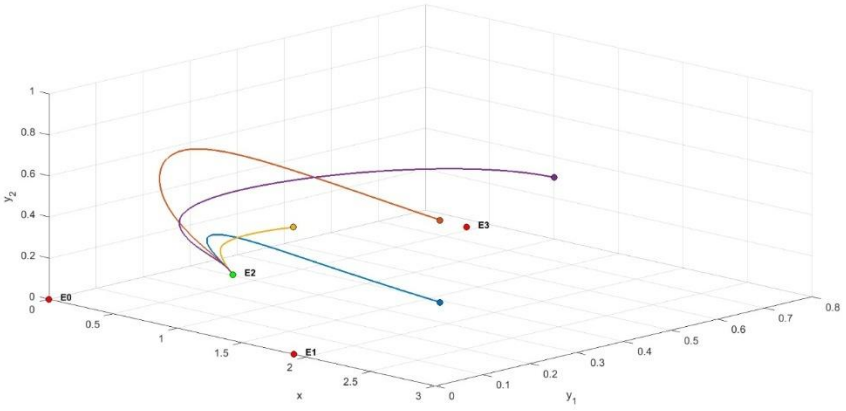

Figure 2: Phase portrait for $k = 1.9$, showing convergence to the equilibrium E_2.

In order to capture the full complexity of the system's dynamics, we shifted our focus to a different parameter set:

$r_1 = 0.4, b_2 = 0.7, b_3 = 0.5, c_4 = 0.6, c_5 = 0.7,\ c_6 = 0.35, a = 0.6, k_1 = 0.3, k_2 = 0.5,$ $d_2 = 0.5, d_3 = 0.75.$

At $k = 6$, we found that both E_2 and E_3 satisfy the stability conditions stated in Theorems 5. For E_2, substituting the parameter values yields

$b_2c_4x^* = 0.9,\ ab_2c_4x^*y_2^* + k_1y_2^* + d_2 = 0.932,$ so that $0.9 < 0.932$, confirming the stability condition. Similarly, for E_3, we obtain $x^* = 1.1905$ and $\frac{k_2y_1^* + d_3}{b_3c_5 + c_6y_1^*} = 1.918$, which gives $1.1905 < 1.918$ verifying the required inequality. These results confirm that both equilibria are locally asymptotically stable under the same parameter set. The phase portrait is shown in Figure 3.

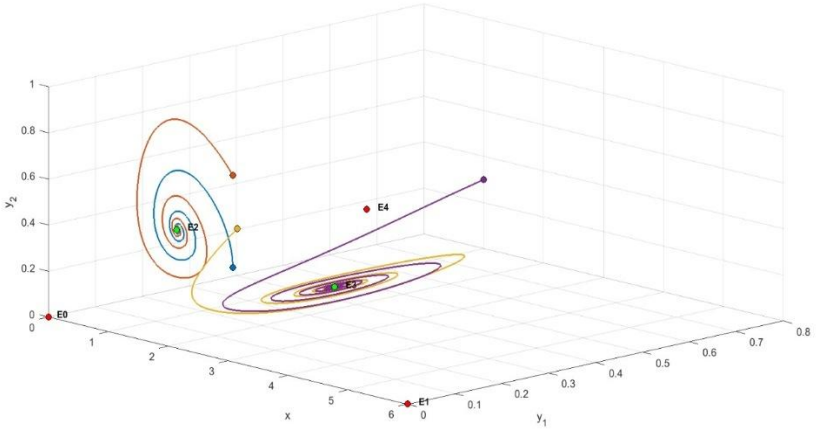

Figure 3. Phase portrait for $k = 6$, illustrating bistability between E_2 and E_3.

As shown in Figure 3, the way trajectories converge to different equilibria depending on where they start clearly points to two distinct basins of attraction. Since the stability conditions for E_2 and E_3 are met simultaneously, their stability domains must overlap, creating a bistable state. In biological terms, this means the system's long-term fate is tied to the initial population distribution; essentially, initial dominance dictates who wins.

These simulations do more than just back up our math; they pull back the curtain on some complex dynamics. We see that the system can shift between coexistence, exclusion, and bistability. One of the more counterintuitive findings is that ramping up the carrying capacity actually kills off coexistence. Instead of facilitating coexistence, increased resources tend to favor a single predator. It is now evident that kleptoparasitism acts as a double-edged sword: it may either stabilize the system or drive competitive exclusion, depending entirely on the parameter regime.

7. Discussion and Conclusion

We explore a predator-prey dynamic involving a single prey and two competing predators linked by kleptoparasitism. Unlike traditional models, we propose a nonlinear formulation where the interaction terms directly suppress the host predator's growth while boosting the kleptoparasite. This shift in perspective offers a more nuanced way to model how predator populations interfere with one another. The analysis reveals that the system admits several equilibrium points, namely E_0, E_1, E_2, E_3, and E_4, whose existence depends on system parameters, particularly the environmental carrying capacity and interaction coefficients. Local stability analysis shows that E_0 is always unstable, while the stability of the remaining equilibria depends on conditions involving parameters such as predation rates, conversion efficiencies, and kleptoparasitism intensity. In addition, global stability conditions are established for the boundary equilibria E_1, E_2, and E_3 using Lyapunov functions and LaSalle's invariance principle. This structure indicates that the system has the potential for multi-equilibrium dynamics, where the existence and stability of equilibria are strongly governed by parameter values. Furthermore, global stability guarantees convergence to a unique equilibrium, whereas the presence of only local stability allows for multiple possible long-term outcomes.

To validate these analytical findings, numerical simulations were conducted to examine the global behavior of the system. Three distinct dynamical scenarios were identified. In the first case, the coexistence equilibrium E_4 is stable, attracting all trajectories and indicating that all populations can persist under the given parameter set. However, as the carrying capacity increases,

a sharp qualitative transition occurs: coexistence disappears and trajectories converge to a boundary equilibrium, illustrating competitive exclusion. In a third scenario, both E_2 and E_3 are locally stable, and the long-term outcome depends on the initial conditions, confirming the presence of bistability and distinct basins of attraction. These simulations demonstrate that the carrying capacity plays a central role in governing regime shifts in the system.

Beyond illustrating the analytical results, the numerical simulations provide further insight into the global dynamics. When global stability conditions are satisfied, all trajectories converge to a single equilibrium, in agreement with the Lyapunov-based analysis. In contrast, when only local stability holds, the system becomes sensitive to initial conditions, giving rise to multiple attractors and bistable behavior. This highlights the limitations of relying solely on local stability analysis to fully capture the system's dynamics.

One important finding of this study is that increasing resource availability does not necessarily promote species coexistence. Paradoxically, increasing the carrying capacity can drive the system from a coexistence state to competitive exclusion. This behavior is consistent with the well-known paradox of enrichment, where increasing resource availability can destabilize ecological systems and even destroy previously stable equilibria [14]. This reveals a non-monotonic relationship, suggesting that resource availability in kleptoparasitic systems is more complex than it appears. Moreover, the presence of bistability indicates the existence of multiple attractors, where long-term outcomes depend strongly on initial conditions.

These findings suggest the occurrence of parameter-induced transitions between different dynamical regimes, as observed in the numerical simulations. In this context, kleptoparasitism acts as a double-edged interaction: it can stabilize the community under certain conditions, while under others it drives competitive dominance. The nonlinear formulation adopted in this study strengthens the coupling between predator populations through simultaneous growth suppression and resource gain, which significantly influences the system's stability structure and contributes to the emergence of diverse dynamical behaviors.

By combining analytical and numerical approaches, this study demonstrates that the system exhibits rich dynamical behavior, including coexistence, competitive exclusion, and bistability. These results provide insight into how kleptoparasitism shapes predator--predator interactions in ecological systems. Future work may extend the model by incorporating time delays, stochastic perturbations, or fractional-order dynamics to capture memory effects. In addition, a more detailed bifurcation analysis would be valuable to systematically characterize the transitions between

different dynamical regimes identified in this study.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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