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# EXPONENTIATED WEIBULL DISTRIBUTION BASED ON THE MARSHALL-OLKIN METHOD

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**Abstract.** This paper proposes the Marshall-Olkin Exponentiated Weibull distribution (MOEW), a new distribution by adding an additional shape parameter to the current Exponentiated Weibull distribution. The goal is to increase the flexibility of the existing Exponentiated Weibull distribution by including an extra shape parameter, resulting in a more flexible distribution that can provide a better fit to various data sets than the baseline distribution. A generator method introduced by Marshall and Olkin is used to develop the new distribution. Some properties of the new distribution such as hazard rate function, survival function, reversed hazard rate function, cumulative hazard function, odds function, quantile function, moments and order statistics are derived. The method of maximum likelihood estimation is used to derive the parameters of the specified model. Monte Carlo simulation is used to evaluate the behavior of the estimators through the average bias and the root mean squared error. The new distribution is fitted and compared with some existing distributions such as the Marshall-Olkin Weibull (MOW), Exponentiated Weibull (EW) and Weibull distributions, on two data sets, namely Breaking stress of carbon fibers (in GPa) and analgesic data sets. Based on the goodness-of-fit statistics and information criteria values,

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it is demonstrated that the new distribution provides a better fit for the two data sets than the other distributions considered in the study.

**Keywords:** exponentiated Weibull distribution; maximum likelihood estimation; Marshall-Olkin family.

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## 1. Introduction

The Weibull distribution, introduced by [1], is one of the most recognized and widely used probability distributions for analyzing lifetime data. It has found extensive application across various fields, including engineering, hydrology, biology, and economics, to model phenomena such as material strength, product reliability, and failure times [2]. Numerous studies have investigated the properties and applications of the Weibull distribution. [3] focused on parameter estimation techniques relevant to electric breakdown phenomena, emphasizing both graphical methods and Maximum Likelihood Estimation (MLE). Despite its popularity, the Weibull distribution has limitations in accurately modeling certain types of data, particularly in contexts like wind speed analysis [4]. To address these short-comings, researchers have proposed new distributions that offer greater flexibility in modeling hazard rate shapes. [5] proposed the exponentiated Weibull family, which incorporates a shape parameter to improve model flexibility. A particular variation of this family, known as the Exponentiated Weibull (EW) distribution, was later presented by [6]. This version improved the model by adding another parameter, which helped it fit many different types of data even better. It became useful in many real-world applications where the original Weibull model was not flexible enough. The Exponentiated Inverted Weibull distribution was applied by [7] to the Modi family to formulate a new distribution called Modi Exponentiated Inverted Weibull (MEIW) distribution which envisaged to offer greater flexibility than the Exponentiated Inverted Weibull distribution. [4] proposed a method known as the Marshall-Olkin generator to obtain more flexible distributions by adding a new parameter to an existing one, called the Marshall-Olkin family of distributions. [8] applied this approach to the Exponentiated Fréchet distribution, leading to the formulation of a new model called the Marshall-Olkin Exponentiated Fréchet (MOEFr) distribution.

The Marshall-Olkin family of distributions was created by Marshall and Olkin using a technique known as the Marshall-Olkin generator to create more adaptable distributions by adding a new parameter to an already-existing one. Niyoyunguruza developed a new model known as the Marshall-Olkin Exponentiated Fréchet (MOEFr) distribution by applying this method to the Exponentiated Fréchet distribution.

This model performed well because it is to increase the flexibility of the existing Exponentiated Fréchet (EFr) distribution by including an extra shape parameter, resulting in a more flexible distribution that can provide a better fit to various data sets than the baseline distribution.

Many researchers have studied the flexibility of various distributions over time by introducing an additional parameter using the method introduced in [4]. Among these distributions are the generalized exponential Marshall-Olkin distribution presented in [9], the Marshall-Olkin generalized Pareto distribution proposed in [10], the extension of the alpha power transform class called the Marshall-Olkin alpha power family of distributions [11], Inverse Weibull Marshall-Olkin distribution [12], Extended Exponentiated Weibull distribution [13], the Marshall-Olkin exponential Weibull distribution [14], the Marshall-Olkin Fréchet distribution [15]. These authors investigated some of the properties of these distributions, such as moments, moment-generating functions, and order statistics.

A key focus in the literature has been on developing more flexible distributions that can be applied to a broader range of data. Many of these models achieve this flexibility by introducing additional parameters, allowing them to capture more complex failure rate behaviors. However, many remain limited to specific hazard rate patterns, such as unimodal or decreasing rates, which can be restrictive when dealing with more complex datasets, particularly those exhibiting increasing or bathtub-shaped hazard rates.

Thus, this paper proposes a new distribution based on the Marshall-Olkin family of distributions, envisaged to offer greater flexibility than the Exponentiated Weibull distribution.

This paper is structured as follows: Section 2 discusses the Marshall-Olkin family and the baseline distribution, while Section 3 introduces the Marshall-Olkin Exponentiated Weibull distribution and its cumulative distribution function (CDF), probability density function (PDF), hazard rate function, survival function, cumulative hazard function, reversed hazard rate function, and odds function. Statistical and mathematical properties include quantile function, Skewness and kurtosis, Moments and order statistics of the proposed distribution are also derived as shown in section 3. The parameter estimation process for the Marshall-Olkin Exponentiated Weibull distribution is detailed in Section 4. Section 5 exhibits a Monte Carlo simulation to evaluate the performance of the proposed distribution vis-a-vis the traditional models. Section 6 demonstrates the model's goodness-of-fit using two real-world data sets.

## 2. Marshall-Olkin Generator Method and Baseline Distribution

**2.1. Marshall-Olkin Generator Method Theory.** The Marshall-Olkin (MO) family of distributions, presented by [4],serves us a technique for creating adaptable probability distribution by adding a shape parameter ($\theta > 0$) to an existing base distribution $G(x)$. It increases flexibility in modeling data with non-monotonic hazard rates, often used in reliability and survival analysis.

The cumulative distribution function is described as:

$$F(x) = \frac{G(x)}{1-(1-\theta)(1-G(x))} = \frac{G(x)}{\theta+(1-\theta)G(x)}, \theta > 0 \tag{1}$$

while the associated probability density function is given by:

$$f(x,\theta) = \frac{\theta g(x)}{(\theta+(1-\theta)G(x))^2}, x > 0, \theta > 0 \tag{2}$$

**2.2. Exponentiated Weibull Distribution.** [6] introduced the Exponentiated Weibull (EW) distribution as an extension of the traditional Weibull distribution.

The distribution's cumulative function of the distribution is expressed as:

$$G(x,\alpha,\beta,\lambda) = \left[1-e^{-(\lambda x)^\beta}\right]^\alpha, x > 0; \lambda,\beta,\alpha > 0 \tag{3}$$

The probability function distribution is given by:

$$g(x,\alpha,\beta,\lambda) = \alpha\beta\lambda^\beta x^{\beta-1}\left[1-e^{-(\lambda x)^\beta}\right]^{\alpha-1} e^{-(\lambda x)^\beta}, x > 0 \tag{4}$$

where:

The scale parameter is $\lambda$, where $\lambda > 0$ ,

The shape parameters are $\beta > 0$ and $\alpha > 0$ .

The following additional functions are associated with the Exponentiated Weibull distribution:

survival function :

$$S(x,\alpha,\beta,\lambda) = 1 - G(x,\alpha,\beta,\lambda) \tag{5}$$
$$= 1 - \left[1 - e^{-(\lambda x)^{\beta}}\right]^{\alpha}, x > 0; \lambda, \beta, \alpha > 0$$

Hazard Rate Function:

$$h(x,\alpha,\beta,\lambda) = \frac{g(x,\alpha,\beta,\lambda)}{S(x,\alpha,\beta,\lambda)} \tag{6}$$
$$= \frac{\alpha\beta\lambda^{\beta}x^{\beta-1}\left[1 - e^{-(\lambda x)^{\beta}}\right]^{\alpha-1} e^{-(\lambda x)^{\beta}}}{1 - \left[1 - e^{-(\lambda x)^{\beta}}\right]^{\alpha}}$$

Cumulative hazard function:

$$H(x,\alpha,\beta,\lambda) = -logS(x,\alpha,\beta,\lambda) \tag{7}$$
$$= -log\left[1 - \left[1 - e^{-(\lambda x)^{\beta}}\right]^{\alpha}\right]$$

Reversed Hazard Rate Function:

$$r(x,\alpha,\beta,\lambda) = \frac{f(x,,\alpha,\beta,\lambda)}{F(x,\alpha,,\beta,\lambda)} \tag{8}$$
$$= \frac{\alpha\beta\lambda^{\beta}x^{\beta-1}\left[1 - e^{-(\lambda x)^{\beta}}\right]^{\alpha-1} e^{-(\lambda x)^{\beta}}}{\left[1 - e^{-(\lambda x)^{\beta}}\right]^{\alpha}}$$

Odds function:

$$O(x,\alpha,\beta,\lambda) = \frac{F(x,\alpha,\beta,\lambda)}{S(x,\alpha,\beta,\lambda)} \tag{9}$$
$$= \frac{\left[1 - e^{-(\lambda x)^{\beta}}\right]^{\alpha}}{1 - \left[1 - e^{-(\lambda x)^{\beta}}\right]^{\alpha}}$$

## 3. MALSHALL-OLKIN EXPONENTIATED WEIBULL DISTRIBUTION AND ITS PROPERTIES

A typical practice of statistical distribution theory is to add a new parameter to an existing family of distribution functions. Adding an extra shape parameter often adds flexibility to a class of distribution functions, which can be highly beneficial for data analysis.

**3.1. Cumulative Distribution Function and Probability Density Function.** By substituting (2.3) into (2.1), the cumulative distribution function for the Marshall-Olkin Exponentiated Weibull distribution is defined as:

$$F(x,\alpha,\beta,\lambda,\theta)=\frac{\left[1-e^{-(\lambda x)^{\beta}}\right]^{\alpha}}{\theta+(1-\theta)\left[1-e^{-(\lambda x)^{\beta}}\right]^{\alpha}} \tag{10}$$

and its probability density function(PDF) is derived from equation (3.1) as follows:

$$f(x,\alpha,\beta,\lambda,\theta)=\frac{\theta\alpha\beta\lambda^{\beta}x^{\beta-1}e^{-(\lambda x)^{\beta}}\left(1-e^{-(\lambda x)^{\beta}}\right)^{\alpha-1}}{\left[\theta+(1-\theta)\left(1-e^{-(\lambda x)^{\beta}}\right)^{\alpha}\right]^{2}} \tag{11}$$

for all $x>0,\alpha>0,\alpha>0,\lambda>0$, and $\theta>0$.

**3.2. Survival Function and Hazard Rate Function.** According to equation (3.1), the survival function for the Marshall-Olkin exponentiated Weibull distribution is expressed as:

$$\begin{aligned}S(x,\alpha,\beta,\lambda,\theta)&=1-F(x,\alpha,\beta,\lambda,\theta)\\&=1-\frac{\left(1-e^{-(\lambda x)^{\beta}}\right)^{\alpha}}{\theta+(1-\theta)\left(1-e^{-(\lambda x)^{\beta}}\right)^{\alpha}}\\&=\frac{\theta\left[1-\left(1-e^{-(\lambda x)^{\beta}}\right)^{\alpha}\right]}{\theta+(1-\theta)\left(1-e^{-(\lambda x)^{\beta}}\right)^{\alpha}}\end{aligned} \tag{12}$$

The hazard rate function is obtained from equations (3.2) and (3.3)

$$\begin{aligned}h(x,\alpha,\beta,\lambda,\theta)&=\frac{f(x,\alpha,\beta,\lambda,\theta))}{S(x,\alpha,\beta,\lambda,\theta)}\\&=\frac{\frac{\theta\alpha\beta\lambda^{\beta}x^{\beta-1}e^{-(\lambda x)^{\beta}}\left(1-e^{-(\lambda x)^{\beta}}\right)^{\alpha-1}}{\left[\theta+(1-\theta)\left(1-e^{-(\lambda x)^{\beta}}\right)^{\alpha}\right]^{2}}}{\frac{\theta\left[1-\left(1-e^{-(\lambda x)^{\beta}}\right)^{\alpha}\right]}{\theta+(1-\theta)\left(1-e^{-(\lambda x)^{\beta}}\right)^{\alpha}}}\\&=\frac{\alpha\beta\lambda(\lambda x)^{\beta-1}e^{-(\lambda x)^{\beta}}\left(1-e^{-(\lambda x)^{\beta}}\right)^{\alpha-1}}{\left[\theta+(1-\theta)\left(1-e^{-(\lambda x)^{\beta}}\right)^{\alpha}\right]\left[1-\left(1-e^{-(\lambda x)^{\beta}}\right)^{\alpha}\right]}\end{aligned} \tag{13}$$

**3.3. Some Useful Functions of MOEW distribution.** Additional functions related to the MOEW distribution, such as the cumulative hazard function, the reversed hazard rate function, and the odds function are provided below:

The Cumulative hazard function is obtained from equation (3.3):

$$\begin{aligned} H(x,\alpha,\beta,\lambda,\theta) &= -\ln S(x,\alpha,\beta,\lambda,\theta) \\ &= -\ln\left(\frac{\theta\left(1-\left(1-e^{-(\lambda x)^{\beta}}\right)^{\alpha}\right)}{\theta+(1-\theta)\left(1-e^{-(\lambda x)^{\beta}}\right)^{\alpha}}\right) \\ &= \ln\left(\theta+(1-\theta)\left(1-e^{-(\lambda x)^{\beta}}\right)^{\alpha}\right)-\ln\theta-\ln\left(1-\left(1-e^{-(\lambda x)^{\beta}}\right)^{\alpha}\right) \end{aligned} \tag{14}$$

The reversed hazard rate function is calculated as the ratio of the PDF to the CDF:

$$\begin{aligned} r(x,\alpha,\beta,\lambda,\theta) &= \frac{f(x,\alpha,\beta,\lambda,\theta)}{F(x,\alpha,\beta,\lambda,\theta)} \\ &= \frac{\frac{\theta\alpha\beta\lambda^{\beta}x^{\beta-1}e^{-(\lambda x)^{\beta}}\left(1-e^{-(\lambda x)^{\beta}}\right)^{\alpha-1}}{\left(\theta+(1-\theta)\left(1-e^{-(\lambda x)^{\beta}}\right)^{\alpha}\right)^{2}}}{\frac{\left(1-e^{-(\lambda x)^{\beta}}\right)^{\alpha}}{\theta+(1-\theta)\left(1-e^{-(\lambda x)^{\beta}}\right)^{\alpha}}} \\ &= \frac{\theta\alpha\beta\lambda^{\beta}x^{\beta-1}e^{-(\lambda x)^{\beta}}}{(\theta+(1-\theta)\left(1-e^{-(\lambda x)^{\beta}}\right)^{\alpha})\left(1-e^{-(\lambda x)^{\beta}}\right)} \end{aligned} \tag{15}$$

Odds function is defined as the ratio of the CDF to the survival function:

$$\begin{aligned} O(x,\alpha,\beta,\lambda,\theta) &= \frac{F(x,\alpha,\beta,\lambda,\theta)}{S(x,\alpha,\beta,\lambda,\theta)} \\ &= \frac{\left(1-e^{-(\lambda x)^{\beta}}\right)^{\alpha}}{\theta\left(1-\left(1-e^{-(\lambda x)^{\beta}}\right)^{\alpha}\right)} \end{aligned} \tag{16}$$

Figures 1 and 2 illustrate some possible shapes of the MOEW probability density function and the hazard rate function for various parameter values. The MOEW probability density function may show right skewness or symmetry, while the MOEW hazard density function can exhibit a decreasing, increasing, or reversed J-shaped.

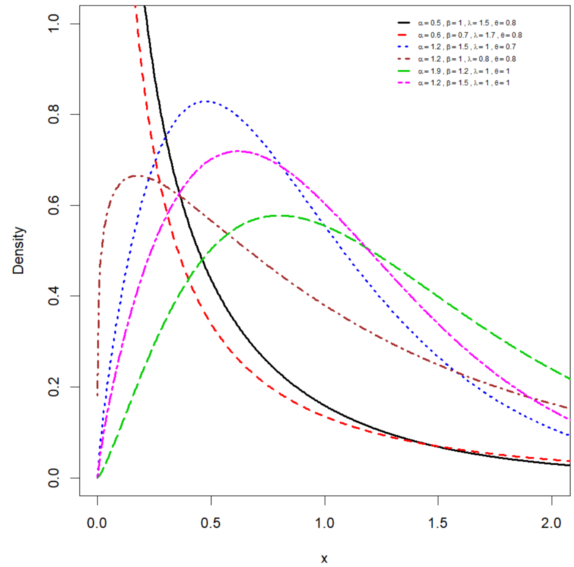


FIGURE 1. Plot of PDF of the MOEW for various values of $\alpha$, $\beta$ $\lambda$, $\theta$

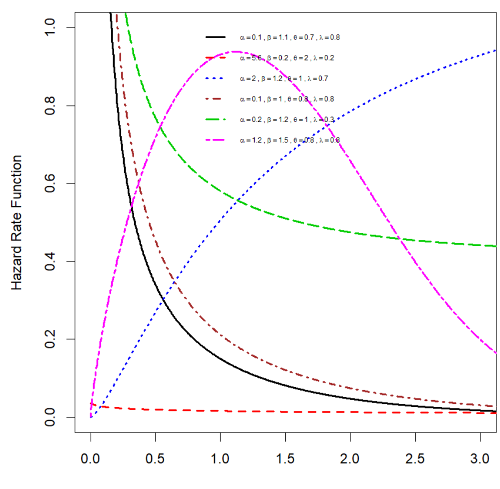


FIGURE 2. Plot of the Hazard rate function of MOEW Distribution for Different various Values of $\alpha$, $\beta$ $\lambda$,$\theta$

### 3.4. Statistical and Mathematical Properties of Marshall-Olkin Exponentiated Weibull Distribution.

**3.4.1.** *Quantile function.* The quantile function is crucial for simulating random samples from a specified distribution. In addition, it serves to describe key characteristics of the distribution, such as median, skewness, and kurtosis.

Let X be a random variable such that $X \sim$ MOEW($\alpha$, $\beta$ , $\lambda$ , $\theta$). Then the quantile function $Q(u)$ is defined as:

$$Q(u) = F^{-1}(u,\alpha,\beta,\lambda,\theta) = \frac{1}{\lambda}\left[-\ln\left(1-\left(\frac{u\theta}{1-u(1-\theta)}\right)^{\frac{1}{\alpha}}\right)\right]^{\frac{1}{\beta}} \tag{17}$$

where $F^{-1}(.)$ denotes the inverse of the cumulative distribution function (CDF) and $u \sim U(0,1)$ is a uniformly distributed random variable in the interval $(0,1)$.

Proof: The quantile function is derived from the inverse of the Cumulative Distribution Function (CDF) of the Marshall-Olkin Exponentiated Weibull distribution, defined in equation (3.1).

Let $F(x;\alpha,\beta,\lambda,\theta) = u$

$$\frac{\left(1-e^{-(\lambda x)^\beta}\right)^\alpha}{\theta+(1-\theta)\left(1-e^{-(\lambda x)^\beta}\right)^\alpha} = u$$

$$\left(1-e^{-(\lambda x)^\beta}\right)^\alpha = u\left(\theta+(1-\theta)\left(1-e^{-(\lambda x)^\beta}\right)^\alpha\right)$$

$$\left(1-e^{-(\lambda x)^\beta}\right)^\alpha = u\theta+u(1-\theta)\left(1-e^{-(\lambda x)^\beta}\right)^\alpha$$

$$\left(1-e^{-(\lambda x)^\beta}\right)^\alpha - u(1-\theta)\left(1-e^{-(\lambda x)^\beta}\right)^\alpha = u\theta$$

$$\left(1-e^{-(\lambda x)^\beta}\right)^\alpha (1-u(1-\theta)) = u\theta$$

$$\left(1-e^{-(\lambda x)^\beta}\right)^\alpha = \frac{u\theta}{1-u(1-\theta)}$$

$$1-e^{-(\lambda x)^\beta} = \left(\frac{u\theta}{1-u(1-\theta)}\right)^{\frac{1}{\alpha}}$$

$$e^{-(\lambda x)^\beta} = 1-\left(\frac{u\theta}{1-u(1-\theta)}\right)^{\frac{1}{\alpha}}$$

$$-(\lambda x)^\beta = \ln\left(1-\left(\frac{u\theta}{1-u(1-\theta)}\right)^{\frac{1}{\alpha}}\right)$$

$$(\lambda x)^\beta = -\ln\left(1-\left(\frac{u\theta}{1-u(1-\theta)}\right)^{\frac{1}{\alpha}}\right)$$

$$\lambda x = \left(-\ln\left(1-\left(\frac{u\theta}{1-u(1-\theta)}\right)^{\frac{1}{\alpha}}\right)\right)^{\frac{1}{\beta}}$$

$$Q(u) = x_u = F^{-1}(u, \alpha, \beta, \lambda, \theta) = \frac{1}{\lambda}\left(-\ln\left(1 - \left(\frac{u\theta}{1-u(1-\theta)}\right)^{\frac{1}{\alpha}}\right)\right)^{\frac{1}{\beta}}$$

By applying the quantile function and substituting u by 1/4, 1/2, and 3/4, one can obtain the values of the lower quartile, median, and upper quartile, respectively.

The lower quartile:

$$Q_1 = \frac{1}{\lambda}\left[-\ln\left(1 - \left(\frac{\theta}{3+\theta}\right)^{\frac{1}{\alpha}}\right)\right]^{\frac{1}{\beta}} \tag{18}$$

The median quartile:

$$Q_2 = \frac{1}{\lambda}\left[-\ln\left(1 - \left(\frac{\theta}{1+\theta}\right)^{\frac{1}{\alpha}}\right)\right]^{\frac{1}{\beta}} \tag{19}$$

The upper quartile:

$$Q_3 = \frac{1}{\lambda}\left[-\ln\left(1 - \left(\frac{3\theta}{1+3\theta}\right)^{\frac{1}{\alpha}}\right)\right]^{\frac{1}{\beta}} \tag{20}$$

**3.4.2.** *Skewness and Kurtosis.* Galton Skewness and Moors Kurtosis for the MOEW distribution are characterized as follows as:

$$S_G = \frac{Q(\frac{3}{4}) + Q(\frac{1}{4}) - 2Q(\frac{1}{2})}{Q(\frac{3}{4}) - Q(\frac{1}{4})} \tag{21}$$

$$K_M = \frac{Q(\frac{7}{8}) - Q(\frac{5}{8}) + Q(\frac{3}{8}) - Q(\frac{1}{8})}{Q(\frac{3}{4}) - Q(\frac{1}{4})} \tag{22}$$

where the value of the quartile is indicated by $Q(.)$

**3.4.3.** *The $r^{th}$ Moments of Marshall-Olkin Exponentiated Weibull Distribution.* Moments play a crucial role as they allow to calculate key characteristics and properties of a probability distribution, including mean, variance, skewness, and kurtosis. The random variable where $x \sim MOEW$ distribution, the $r^{th}$ moment is given by:

$$\mu_r' = E[X^r] = \frac{\alpha}{\lambda^r}\Gamma\left(1 + \frac{r}{\beta}\right)\sum_{j=0}^{\infty}(j+1)(1-\theta)^j \sum_{k=0}^{\infty}(-1)^k \binom{\alpha(j+1)-1}{k}(k+1)^{-\left(1+\frac{r}{\beta}\right)} \tag{23}$$

Proof: The mathematical expression for the $r^{th}$ moment of the MOEW distribution is expressed as:

$$\mu_r' = E[X^r] = \int_0^\infty x^r f(x,\alpha,\beta,\lambda,\theta)\,dx \tag{24}$$

where the probability density function of the distribution is $f(x,\alpha,\beta,\lambda,\theta)$

By substituting (3.2) into (3.15), we obtain

$$\mu_r' = E[X^r] = \int_0^\infty x^r \frac{\theta\alpha\beta\lambda^\beta x^{\beta-1} e^{-(\lambda x)^\beta}\left(1-e^{-(\lambda x)^\beta}\right)^{\alpha-1}}{\left[\theta+(1-\theta)\left(1-e^{-(\lambda x)^\beta}\right)^{\alpha}\right]^2}\,dx$$

$$= \theta\alpha\beta\lambda^\beta \int_0^\infty \frac{x^{r+\beta-1} e^{-(\lambda x)^\beta}\left(1-e^{-(\lambda x)^\beta}\right)^{\alpha-1}}{\left[\theta+(1-\theta)\left(1-e^{-(\lambda x)^\beta}\right)^{\alpha}\right]^2}\,dx$$

To simplify the denominator, write

$$[\theta+(1-\theta)G]^2 = \theta^2\left[1+\frac{1-\theta}{\theta}G\right]^2,$$

where $G = (1-e^{-(\lambda x)^\beta})^\alpha$

Using the expansion

$$\frac{1}{(1+z)^2} = \sum_{j=0}^{\infty}(j+1)(-z)^j,$$

and letting

$$z = \frac{1-\theta}{\theta}G.$$

we obtain

$$\frac{1}{[\theta+(1-\theta)G]^2} = \frac{1}{\theta^2}\sum_{j=0}^{\infty}(j+1)\left(\frac{1-\theta}{\theta}\right)^j G^j \tag{25}$$

Substituting (3.16) into $\mu_r'$, we obtain

$$\mu_r' = \theta\alpha\beta\lambda^\beta \int_0^\infty x^{r+\beta-1} e^{-(\lambda x)^\beta}(1-e^{-(\lambda x)^\beta})^{\alpha-1}$$
$$\times \frac{1}{\theta^2}\sum_{j=0}^{\infty}(j+1)\left(\frac{1-\theta}{\theta}\right)^j (1-e^{-(\lambda x)^\beta})^{\alpha j}dx$$

$$= \frac{\alpha\beta\lambda^\beta}{\theta}\sum_{j=0}^{\infty}(j+1)(1-\theta)^j \int_0^\infty x^{r+\beta-1} e^{-(\lambda x)^\beta}(1-e^{-(\lambda x)^\beta})^{\alpha(j+1)-1}dx$$

Using the binomial expansion

$$(1-z)^m = \sum_{k=0}^{\infty}(-1)^k\binom{m}{k}z^k,$$

with $z = e^{-(\lambda x)^\beta}$, we obtain

$$(1-e^{-(\lambda x)^\beta})^{\alpha(j+1)-1} = \sum_{k=0}^{\infty}(-1)^k \binom{\alpha(j+1)-1}{k} e^{-k(\lambda x)^\beta}$$

Thus,

$$(26)\quad \mu_r' = \frac{\alpha\beta\lambda^\beta}{\theta}\sum_{j=0}^{\infty}(j+1)(1-\theta)^j \sum_{k=0}^{\infty}(-1)^k \binom{\alpha(j+1)-1}{k} \quad \times \int_0^\infty x^{r+\beta-1} e^{-(k+1)(\lambda x)^\beta} dx$$

Using the identity

$$\int_0^\infty x^{m-1} e^{-ax^b} dx = \frac{1}{b} a^{-m/b} \Gamma\left(\frac{m}{b}\right),$$

and letting $m = r+\beta$ and $a = (k+1)\lambda^\beta$, we get

$$(27)\qquad \int_0^\infty x^{r+\beta-1} e^{-(k+1)(\lambda x)^\beta} dx = \frac{1}{\beta}\lambda^{-r-\beta}(k+1)^{-\left(1+\frac{r}{\beta}\right)}\Gamma\left(1+\frac{r}{\beta}\right)$$

Substituting (3.18) back into (3.17) and simplifying, we obtain

$$\mu_r' = \frac{\alpha}{\lambda^r}\Gamma\left(1+\frac{r}{\beta}\right)\sum_{j=0}^{\infty}(j+1)(1-\theta)^j \sum_{k=0}^{\infty}(-1)^k \binom{\alpha(j+1)-1}{k}(k+1)^{-\left(1+\frac{r}{\beta}\right)}$$

Table 1 presents the value of the quantile function and Table 2 displays values of the statistical properties of the MOEW distribution for various parameter sets. It reveals how these parameters influence the central tendency (mean), spread (standard deviation and coefficient of variation), and shape (skewness and kurtosis) of the MOEW distribution. The values of coefficient of skewness (CS) values show that distribution may display right skewness, left skewness, and nearly symmetrical while the coefficient of Kurtosis(CK) suggests that the distribution can be leptokurtic, mesokurtic, and platykurtic.

TABLE 1. Quantiles of the MOEW distribution for some parameter values.

| | $(\alpha,\beta,\lambda,\theta)$ | | | | |
|---|---|---|---|---|---|
| | *I* | *II* | *III* | *IV* | *V* |
| $u$ | $(0.4,0.7,0.8,1.2)$ | $(0.6,0.7,0.8,1.2)$ | $(0.4,0.9,0.8,1.2)$ | $(0.4,0.7,1.5,1,2)$ | $(0.4,0.7,0.8,2.0)$ |
| 0.1 | 0.0015 | 0.0143 | 0.0065 | 0.0008 | 0.0224 |
| 0.2 | 0.0065 | 0.0377 | 0.0276 | 0.0035 | 0.0532 |
| 0.3 | 0.0151 | 0.0666 | 0.0647 | 0.0081 | 0.0884 |
| 0.4 | 0.0276 | 0.1001 | 0.1193 | 0.0147 | 0.1268 |
| 0.5 | 0.0441 | 0.1377 | 0.1936 | 0.0235 | 0.1679 |
| 0.6 | 0.0647 | 0.1793 | 0.2910 | 0.0345 | 0.2115 |
| 0.7 | 0.0897 | 0.2249 | 0.4165 | 0.0478 | 0.2573 |
| 0.8 | 0.1193 | 0.2745 | 0.5789 | 0.0636 | 0.3053 |
| 0.9 | 0.1538 | 0.3284 | 0.7934 | 0.0821 | 0.3556 |

TABLE 2. The moments, standard deviation, variation coefficient, skewness, and kurtosis of the MOEW distribution for different parameter values.

| | $(\alpha,\beta,\lambda,\theta)$ | | | | |
|---|---|---|---|---|---|
| | (1.3, 0.5,2.5,1.8) | (2.0, 0.5,2.5,1.8) | (1.3, 1.0,2.5,1.8) | (1.3, 0.5,4.8,1.8) | (1.3, 0.5,2.5,5.5) |
| $\mu_1$ | 0.32432 | 0.39663 | 0.39880 | 0.16892 | 0.35959 |
| $\mu_2$ | 0.13972 | 0.19042 | 0.19999 | 0.03790 | 0.13438 |
| $\mu_3$ | 0.07420 | 0.10718 | 0.11823 | 0.01048 | 0.05194 |
| $\mu_4$ | 0.04633 | 0.06900 | 0.07949 | 0.00341 | 0.02069 |
| SD | 0.18583 | 0.18195 | 0.20234 | 0.09679 | 0.07122 |
| CV | 0.57297 | 0.45874 | 0.50736 | 0.57297 | 0.19805 |
| CS | 1.01049 | 0.89430 | 0.70313 | 1.01050 | -0.07271 |
| CK | 4.24359 | 4.06995 | 3.48295 | 4.24354 | 2.89424 |

**3.5. Order Statistics.** Let's consider a finite random sample $X_1, X_2, \ldots, X_n$ drawn from a probability density function. From this sample, we define the $n^{th}$ order statistic, $X(r)$, where $X(1)$ represents the smallest value in the sample, $X(2)$ represents the second smallest value, and so on, up to $X(n)$ representing the largest value.

The Probability Density Function (PDF) of the $n^{th}$ order statistic, with $1 \leq r \leq n$, can be defined as:

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} f(x)[F(x)]^{r-1}[1-F(x)]^{n-r} \tag{28}$$

By substituting Equations (3.1) and (3.2) into (3.16), the PDF of the $r^{th}$ order statistic for the MOEW can be expressed as follows:

$$f_{r:n}(x) = \frac{n!\theta^{n-r+1}\alpha\beta\lambda^{\beta}}{(r-1)!(n-r)!} x^{\beta-1} e^{-(\lambda x)^{\beta}} \frac{\left(1-e^{-(\lambda x)^{\beta}}\right)^{\alpha r-1}\left[1-\left(1-e^{-(\lambda x)^{\beta}}\right)^{\alpha}\right]^{n-r}}{\left[\theta+(1-\theta)\left(1-e^{-(\lambda x)^{\beta}}\right)^{\alpha}\right]^{n+1}}, x>0 \tag{29}$$

The probability density function (PDF) attains its minimum for the smallest order statistic in the MOEW distribution when r equals 1. This can be represented as:

$$f_{1:n}(x) = \frac{n\theta^{n}\alpha\beta\lambda^{\beta}x^{\beta-1}e^{-(\lambda x)^{\beta}}\left(1-e^{-(\lambda x)^{\beta}}\right)^{\alpha-1}\left[1-\left(1-e^{-(\lambda x)^{\beta}}\right)^{\alpha}\right]^{n-1}}{\left[\theta+(1-\theta)\left(1-e^{-(\lambda x)^{\beta}}\right)^{\alpha}\right]^{n+1}}, \quad x>0. \tag{30}$$

The probability density function (PDF) attains its maximum for the largest order statistic in the MOEW distribution when r equals n. This can be represented as:

$$f_{n:n}(x)=\frac{n\theta\alpha\beta\lambda^{\beta}x^{\beta-1}e^{-(\lambda x)^{\beta}}\left(1-e^{-(\lambda x)^{\beta}}\right)^{\alpha n-1}}{\left[\theta+(1-\theta)\left(1-e^{-(\lambda x)^{\beta}}\right)^{\alpha}\right]^{n+1}},\quad x>0 \tag{31}$$

## 4. Maximum Likelihood Estimation (MLE)

This section discusses estimating the parameters of MOEW distribution by using the approach called the maximum likelihood method.

Consider a random sample $x_1, x_2, ...x_n$ drawn from the MOEW distribution. The expression for the likelihood function is given as:

$$L(\theta)=\prod_{i=1}^{n}\left(\frac{\theta\alpha\beta\lambda^{\beta}x_i^{\beta-1}e^{-(\lambda x_i)^{\beta}}\left(1-e^{-(\lambda x_i)^{\beta}}\right)^{\alpha-1}}{\left[\theta+(1-\theta)\left(1-e^{-(\lambda x_i)^{\beta}}\right)^{\alpha}\right]^{2}}\right) \tag{32}$$

The related log-likelihood function is derived from equation (4.1)

$$l=\log L=\sum_{i=1}^{n}\log f(x_i;\alpha,\theta,\lambda,\beta)=\sum_{i=1}^{n}\log\prod_{i=1}^{n}\left(\frac{\theta\alpha\beta\lambda^{\beta}x_i^{\beta-1}e^{-(\lambda x_i)^{\beta}}\left(1-e^{-(\lambda x_i)^{\beta}}\right)^{\alpha-1}}{\left[\theta+(1-\theta)\left(1-e^{-(\lambda x_i)^{\beta}}\right)^{\alpha}\right]^{2}}\right)$$

$$\begin{aligned} l=\log L=&n\log\theta+n\log\alpha+n\log\beta+n\beta\log\lambda\quad+(\beta-1)\sum_{i=1}^{n}\log x_i-\sum_{i=1}^{n}(\lambda x_i)^{\beta}\\ &+(\alpha-1)\sum_{i=1}^{n}\log\left(1-e^{-(\lambda x_i)^{\beta}}\right)\quad-2\sum_{i=1}^{n}\log\left[\theta+(1-\theta)\left(1-e^{-(\lambda x_i)^{\beta}}\right)^{\alpha}\right]\end{aligned} \tag{33}$$

The nonlinear equations derived from differentiating the log-likelihood function with respect to $\alpha$, $\beta$, $\lambda$ and $\theta$ are the following:

$$\frac{\partial\ell}{\partial\theta}=\frac{n}{\theta}-2\sum_{i=1}^{n}\frac{1-\left(1-e^{-(\lambda x_i)^{\beta}}\right)^{\alpha}}{\theta+(1-\theta)\left(1-e^{-(\lambda x_i)^{\beta}}\right)^{\alpha}}=0 \tag{34}$$

$$\begin{aligned}\frac{\partial\ell}{\partial\lambda}=&\frac{n\beta}{\lambda}-\beta\sum_{i=1}^{n}x_i^{\beta}\lambda^{\beta-1}\quad+(\alpha-1)\sum_{i=1}^{n}\frac{\beta x_i^{\beta}\lambda^{\beta-1}e^{-(\lambda x_i)^{\beta}}}{1-e^{-(\lambda x_i)^{\beta}}}\\ &-2\sum_{i=1}^{n}\frac{(1-\theta)\alpha\left(1-e^{-(\lambda x_i)^{\beta}}\right)^{\alpha-1}\beta x_i^{\beta}\lambda^{\beta-1}e^{-(\lambda x_i)^{\beta}}}{\theta+(1-\theta)\left(1-e^{-(\lambda x_i)^{\beta}}\right)^{\alpha}}=0\end{aligned} \tag{35}$$

$$\frac{\partial\ell}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\log\left(1-e^{-(\lambda x_i)^{\beta}}\right)\quad-2\sum_{i=1}^{n}\frac{(1-\theta)\left(1-e^{-(\lambda x_i)^{\beta}}\right)^{\alpha}\log\left(1-e^{-(\lambda x_i)^{\beta}}\right)}{\theta+(1-\theta)\left(1-e^{-(\lambda x_i)^{\beta}}\right)^{\alpha}}=0 \tag{36}$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} + n\log\lambda + \sum_{i=1}^{n}\log x_i \quad - \sum_{i=1}^{n}(\lambda x_i)^{\beta}\log(\lambda x_i) \quad + (\alpha-1)\sum_{i=1}^{n}\frac{e^{-(\lambda x_i)^{\beta}}(\lambda x_i)^{\beta}\log(\lambda x_i)}{1-e^{-(\lambda x_i)^{\beta}}}$$

$$-2\sum_{i=1}^{n}\frac{(1-\theta)\alpha\left(1-e^{-(\lambda x_i)^{\beta}}\right)^{\alpha-1}e^{-(\lambda x_i)^{\beta}}(\lambda x_i)^{\beta}\log(\lambda x_i)}{\theta+(1-\theta)\left(1-e^{-(\lambda x_i)^{\beta}}\right)^{\alpha}} = 0 \tag{37}$$

Equations (4.3) - (4.6) are nonlinear and cannot be solved directly, so an iterative quasi-Newton approach, BFGS algorithm, was used to solve them.

## 5. Simulation Study

This section presents the results of a Monte Carlo simulation study conducted to assess the accuracy and precision of the Maximum Likelihood Estimates (MLEs). This study evaluates the accuracy of parameter estimation in the Marshal-Olkin Exponentiated Weibull (MOEW) distribution by examining the average biases (ABs) and root mean square errors (RMSEs).

From the Marshal-Olkin Exponentiated Weibull (MOEW) distribution, random samples of sizes $n = 100, 250, 500, 750$ and 1000 with 5000 iterations for each $n$ value were created using the quantile function provided in equation (4.9), and some selected values of parameters set I:$(\alpha, \beta, \theta, \lambda)$=$(1.8, 1.0, 0.5, 0.7)$ and set II: $(1.5, 1.2, 0.5, 1.5)$ .

The average bias

$$AB_{(\mu)} = \frac{1}{N}\sum_{i=1}^{N}(\hat{\mu}_i - \mu) \tag{38}$$

And the root mean square errors

$$RMSE_{(\mu)} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{\mu}_i - \mu)^2} \tag{39}$$

Where:

- $\mu$ is the parameter to be estimated.
- $\mu_i$ is the estimate at the $i^{th}$ replication for each sample size .
- N is the number of replications.

The following tables show the MLE, AB, and RMSE values of the parameters $\alpha, \beta, \theta$ , and $\lambda$ for various sample sizes.

TABLE 3. Simulation results for Marshal-Olkin Exponentiated Weibull distribution (Set I) with parameters:$(\alpha,\beta,\theta,\lambda)$=$(1.8,1.0,0.5,0.7)$

| **Parameter** | $n$ | **MLE** | **AB** | **RMSE** |
|---|---|---|---|---|
| $\alpha$ | 100 | 2.2236 | 0.4236 | 2.3836 |
| | 250 | 1.9139 | 0.1139 | 0.7283 |
| | 500 | 1.8575 | 0.0575 | 0.4523 |
| | 750 | 1.8297 | 0.0297 | 0.3532 |
| | 1000 | 1.8287 | 0.0287 | 0.3050 |
| $\beta$ | 100 | 1.1012 | 0.1012 | 0.4247 |
| | 250 | 1.0357 | 0.0357 | 0.2093 |
| | 500 | 1.0141 | 0.0141 | 0.1349 |
| | 750 | 1.0113 | 0.0113 | 0.1097 |
| | 1000 | 1.0066 | 0.0066 | 0.0945 |
| $\theta$ | 100 | 0.5055 | 0.0055 | 0.0104 |
| | 250 | 0.5029 | 0.0029 | 0.0057 |
| | 500 | 0.5016 | 0.0016 | 0.0036 |
| | 750 | 0.5013 | 0.0013 | 0.0028 |
| | 1000 | 0.5010 | 0.0010 | 0.0024 |
| $\lambda$ | 100 | 1.0320 | 0.3320 | 4.3486 |
| | 250 | 0.7578 | 0.0578 | 0.2841 |
| | 500 | 0.7263 | 0.0263 | 0.1625 |
| | 750 | 0.7149 | 0.0149 | 0.1251 |
| | 1000 | 0.7134 | 0.0134 | 0.1065 |

TABLE 4. Simulation results for Marshal-Olkin Exponentiated Weibull distribution (Set II) with parameters:$(\alpha, \beta, \theta, \lambda)$=$(1.5, 1.2, 0.5, 1.5)$

| **Parameter** | $n$ | **MLE** | **AB** | **RMSE** |
|---|---|---|---|---|
| $\alpha$ | 100 | 1.7589 | 0.2589 | 1.3634 |
| | 250 | 1.5885 | 0.0885 | 0.5643 |
| | 500 | 1.5361 | 0.0361 | 0.3533 |
| | 750 | 1.5266 | 0.0266 | 0.2756 |
| | 1000 | 1.5162 | 0.0162 | 0.2353 |
| $\beta$ | 100 | 1.3414 | 0.1414 | 0.5495 |
| | 250 | 1.2405 | 0.0405 | 0.2532 |
| | 500 | 1.2211 | 0.0211 | 0.1664 |
| | 750 | 1.2111 | 0.0111 | 0.1302 |
| | 1000 | 1.2096 | 0.0096 | 0.1128 |
| $\theta$ | 100 | 0.5052 | 0.0052 | 0.0101 |
| | 250 | 0.5028 | 0.0028 | 0.0056 |
| | 500 | 0.5017 | 0.0017 | 0.0036 |
| | 750 | 0.5011 | 0.0011 | 0.0027 |
| | 1000 | 0.5011 | 0.0011 | 0.0023 |
| $\lambda$ | 100 | 1.7955 | 0.2955 | 1.5286 |
| | 250 | 1.5863 | 0.0863 | 0.4536 |
| | 500 | 1.5358 | 0.0358 | 0.2721 |
| | 750 | 1.5259 | 0.0259 | 0.2091 |
| | 1000 | 1.5167 | 0.0167 | 0.1767 |

The simulation results for Maximum Likelihood Estimates (MLEs), Absolute Biases (ABs) and Root Mean Squared Errors (RMSEs) are presented in Table 4.3 for the parameter values given as set I and in Table 4.4 for the parameter values given as set II. The simulation results for the Marshall-Olkin Exponentiated Weibull (MOEW) distribution show that as the sample size (n) increases, the Maximum Likelihood Estimates (MLEs) converge toward the true parameter values. Both the Absolute Biases (ABs) and Root Mean Squared Errors (RMSEs) of the parameter estimators decrease with larger sample sizes. This suggests that the MLEs are asymptotically unbiased and consistent.

## 6. APPLICATIONS

In this section, the MOEW distribution was fitted to two real data sets. Its goodness-of-fit was compared with existing distributions such as Weibul, Exponentiated Weibull (EW) and Marshall-Olkin Wibull (MOW) distributions using some measures such as the Kolmogorov-Smirnov (K-S) test, the Cramér-von Mises test, and the Anderson-Darling (AD) test. Information criteria such as Akaike Information Criterion (AIC), Consistent AIC (CAIC), Bayesian (Schwarz) Information Criterion (BIC), and Hannan-Quinn Information Criterion (HQIC) were also used to select the best fitting model.

The distributions to which the MOEW distribution was compared in this section are the Weibull distribution(We) [1], Exponentiated Weibull distribution (EW) [6], Marshall-Olkin Weibull distribution (MOW) [16] with the respective pdfs:

$$We : f(x,\alpha,\beta) = \frac{\beta x^{\beta-1}}{\alpha^{\beta}} \exp\left(-\frac{x}{\alpha}\right)^{\beta} \tag{40}$$

$$EW : g(x,\alpha,\beta,\lambda) = \alpha\beta\lambda^{\beta} x^{\beta-1} \left[1-e^{-(\lambda x)^{\beta}}\right]^{\alpha-1} e^{-(\lambda x)^{\beta}} \tag{41}$$

$$MOW : f(x,\alpha,\beta,\lambda,\theta) = \frac{\alpha\theta\lambda\beta x^{\beta-1} e^{-\lambda x^{\beta}}}{\left(\alpha\theta + (\alpha\bar{\theta}+\bar{\alpha})(1-e^{-\lambda x^{\beta}})\right)^{2}} \tag{42}$$

### 6.1. Dataset I : Breaking stress of carbon fibers (in GPa).

The first data set considered in this study Consists of 100 readings on the breaking stress of carbon fibers (in GPa), as reported by Mahmoud and Mandouh, and list below:

0.920, 0.928, 0.997, 0.997, 1.061, 1.117, 1.162, 1.183, 1.187, 1.192, 1.196, 1.213, 1.215, 1.220, 1.220, 1.224, 1.225, 1.228, 1.237, 1.240, 1.244, 1.259, 1.261, 1.263, 1.276, 1.310, 1.321, 1.329, 1.331, 1.337, 1.351, 1.359, 1.388, 1.408, 1.449, 1.450, 1.450,1.459, 1.471, 1.475, 1.477, 1.480, 1.489, 1.501, 1.507, 1.515, 1.530, 1.530, 1.533, 1.544, 1.544, 1.552, 1.556, 1.562, 1.566, 1.585,1.586, 1.599, 1.602, 1.614, 1.616, 1.617, 1.628, 1.684, 1.711, 1.718, 1.733, 1.738, 1.743, 1.759, 1.777, 1.794, 1.799, 1.806, 1.814,1.816, 1.828, 1.830, 1.884, 1.892, 1.944, 1.972, 1.984, 1.987, 2.020, 2.030, 2.029, 2.035, 2.037, 2.043, 2.046, 2.059, 2.111, 2.165,2.686, 2.778, 2.972, 3.504, 3.863, 5.306.

This data set is transformed on (0,1) interval by dividing it by 6.306.

**Figures 3-6** show The TTT, histogram, violin, and box plots for the dataset consist of 100 readings on the breaking stress of carbon fibers (in GPa). The concave-up shape of the TTT plot suggests an increasing hazard rate in the data. The histogram suggests a right-skewed distribution with a concentration of data around the peak and a longer tail to the right. The box plot indicates some outliers and the violin plot illustrates that values are concentrated around the median.

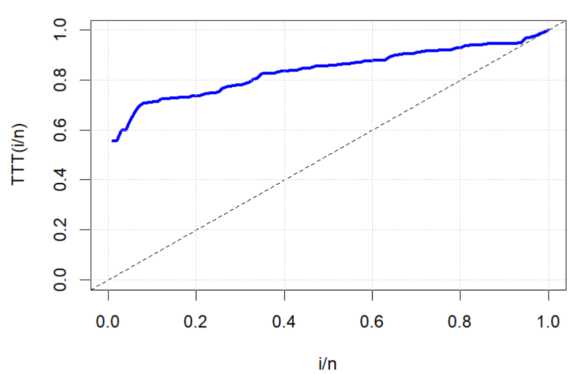


FIGURE 3. TTT Plot for 100 readings on the breaking stress of carbon fibers (in GPa)

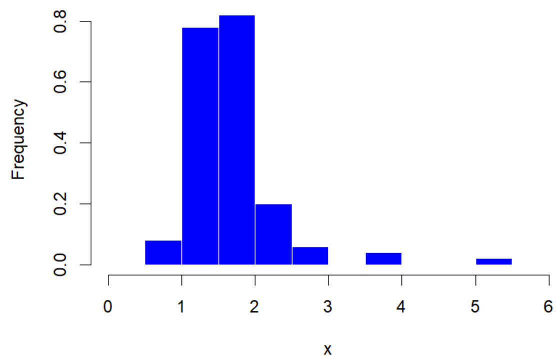


FIGURE 4. Histogram for 100 readings on the breaking stress of carbon fibers (in GPa)

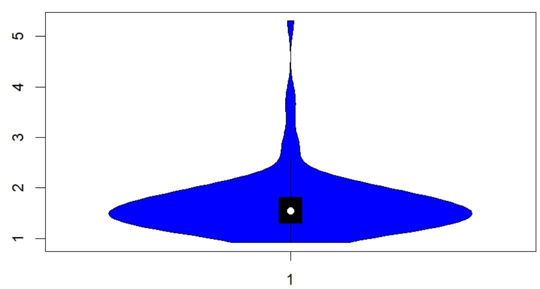


FIGURE 5. Violin plot for 100 readings on the breaking stress of carbon fibers (in GPa)

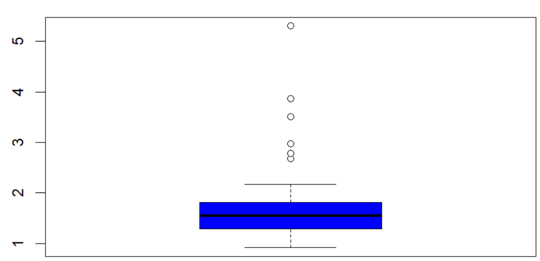


FIGURE 6. Box Plot for 100 readings on the breaking stress of carbon fibers (in GPa)

**Figures 7 and 8** present the estimated probability density function (PDF) and cumulative distribution function (CDF) of the MOEW distribution or 100 readings on the breaking stress of carbon fibers (in GPa). The estimated PDF plot suggests a right-skewed distribution, where most fatigue life values cluster around the peak, with a longer tail that extends to the right. The close alignment between the empirical and estimated CDF indicates a strong fit to the underlying data distribution.

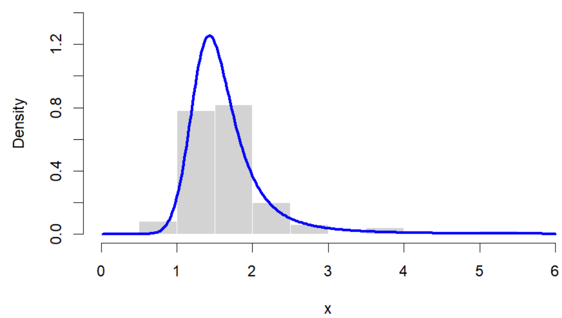


FIGURE 7. Plot of Estimated PDF of 100 readings on the breaking stress of carbon fibers (in GPa)

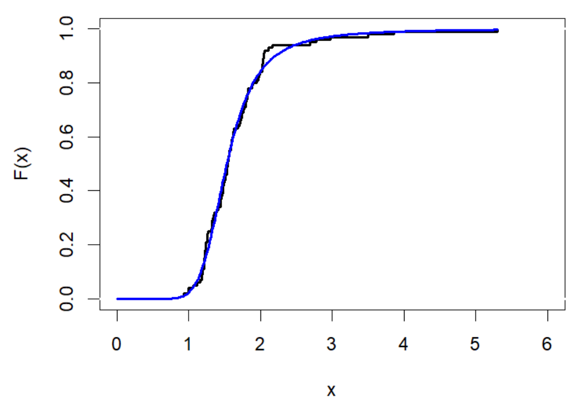


FIGURE 8. Plot of estimated CDF of 100 readings on the breaking stress of carbon fibers (in GPa)

**Figures 9 and 10** display the Kaplan-Meier survival curve and the PP plot, respectively. The Kaplan-Meier curve closely follows the estimated survival function, demonstrating its effectiveness in modeling the material's durability. The close alignment between the empirical and estimated survival curves suggests a reliable model fit. Meanwhile, the PP plot evaluates the goodness-of-fit between the empirical fatigue life data and the theoretical distribution. A closer alignment of points along the diagonal reference line indicates a stronger model fit, confirming that the MOEW distribution provides a better representation of the fatigue life data.

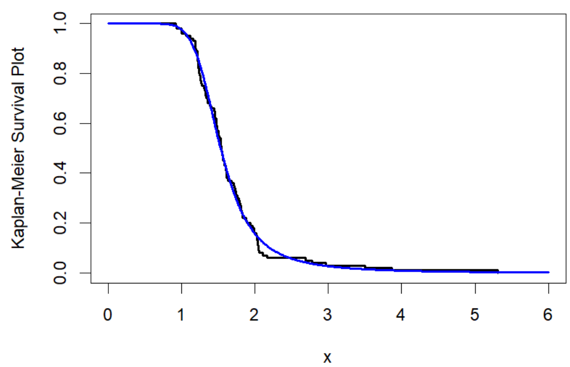


FIGURE 9. Plot of Kaplan-Meier of 100 readings on the breaking stress of carbon fibers (in GPa)

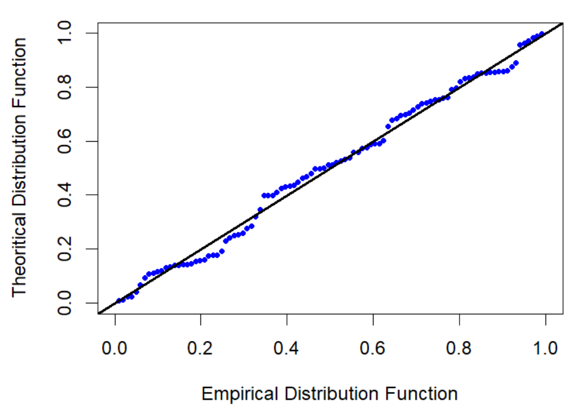


FIGURE 10. PP Plot of 100 readings on the breaking stress of carbon fibers (in GPa)

Lastly, **Figure 11** compares different probability distributions fitted to the fatigue life data. The histogram represents the empirical distribution, while the overlaid curves correspond to different theoretical models (MOEW, MOW, EW and Weibull). Among these, the MOEW distribution aligns most closely with the histogram, indicating that it provides the best fit. This suggests that the MOEW distribution is the most suitable model for accurately characterizing the breaking stress of carbon fibers (in GPa)

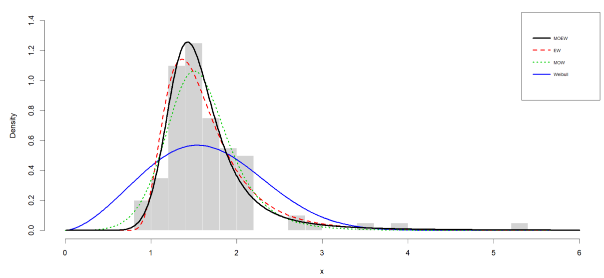


FIGURE 11. Estimated fitted densities of Fatigue Life Analysis of 100 readings on the breaking stress of carbon fibers (in GPa) data set for various distribution

**Table 5** offers the descriptive statistics for data set I. The positive skewness (3.1824 ) indicates a slight rightward asymmetry, and the kurtosis (17.4240) suggests a distribution with lighter tails than a normal distribution.

TABLE 5. Descriptive Statistics of breaking stress of carbon fibers (in GPa)

| Min | Max | Mean | Median | Mode | Variance | Skewness | Kurtosis |
|---|---|---|---|---|---|---|---|
| 0.9200 | 5.3060 | 1.6578 | 1.5440 | 1.4871 | 0.3592 | 3.1824 | 17.4240 |

**Table 6 and 7** display the AIC, BIC, HQIC, and CAIC values used to evaluate and compare different models. The Marshall-Olkin Exponentiated Weibull distribution consistently has the lowest values in these criteria, indicating the best fit to the data []. It also excels over the other models with the lowest K-S, $W^*$ and $A^*$ values, the highest log-likelihood, and the highest P-value for the K-S statistic [].

TABLE 6. Values of Information Criteria for various distributions for data set I

| Distributions | AIC | BIC | CAIC | HQIC |
|---|---|---|---|---|
| MOEW | 111.6690 | 122.0897 | 112.0901 | 115.8865 |
| MOW | 118.7789 | 129.1995 | 119.1999 | 122.9963 |
| EW | 436.9058 | 444.7213 | 437.1558 | 440.0689 |
| Weibull | 184.2978 | 189.5081 | 184.4215 | 186.4065 |

TABLE 7. The Goodness-of-fit statistics, log-likelihood, and maximum likelihood estimates (MLEs) for data set I

| Distributions | Estimates(SEs) | $\ell$ | $W^*$ | $A^*$ | K-S (p-value) |
|---|---|---|---|---|---|
| **MOEW** | $\alpha = 42.15775(1.96552)$<br>$\beta = 0.67428(0.74065)$<br>$\lambda = 2.30782(2.06734)$<br>$\theta = 0.0145(2.17308)$ | $-51.68115$ | 0.06325 | 0.44388 | 0.06174 (0.84038) |
| **MOW** | $\alpha = 0.00001(36.06059)$<br>$\beta = 6.83201(0.08395)$<br>$\lambda = 0.00000(36.06976)$<br>$\theta = 1.00000(11869.32536)$ | $-55.38943$ | 33.33234 | 1061.36337 | 0.99999 ($2.78 \times 10^{-87}$) |
| **EW** | $\alpha = 222051.62313(4.04290)$<br>$\beta = 0.35289(0.31550)$<br>$\lambda = 871.01347(3.16156)$ | $-53.37855$ | 0.08020 | 0.63473 | 0.07857 (0.56756) |
| **Weibull** | $\alpha = 2.63199(0.16339)$<br>$\beta = 1.85099(0.07491)$ | $-90.14889$ | 1.19090 | 7.23756 | 0.19495 (0.00100) |

### 6.2. Data set II: The relief times (in minutes) of 20 patients who received an analgesic.

The second real-life data was originally reported by as reported by Gross and Clark. The following are the observations of this dataset: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6,2.0. This data set is also transformed by dividing each observation by 3, to get data values between 0 and 1.

**Figures 12-15** depict the TTT, histogram, violin, and box plots for the relief times (in minutes) of 20 patients who received an analgesic dataset. The TTT plot indicates an increasing hazard rate, while the histogram reveals a right-skewed distribution of precipitation data. The box plot suggests the presence of outliers, and the violin plot illustrates a concentration of values around the median.

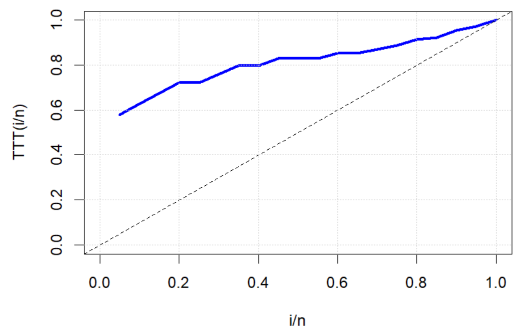


FIGURE 12. TTT Plot for the relief times (in minutes) of 20 patients who received an analgesic

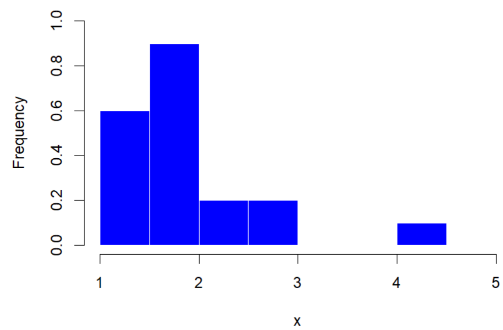


FIGURE 13. Histogram for the relief times (in minutes) of 20 patients who received an analgesic

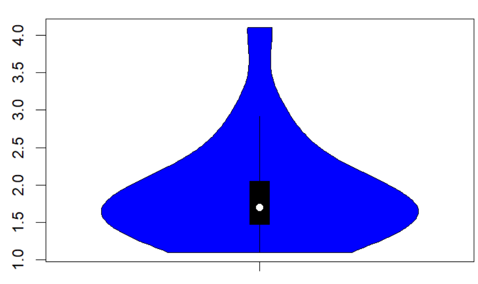


FIGURE 14. Violin Plot of the relief times (in minutes) of 20 patients who received an analgesic

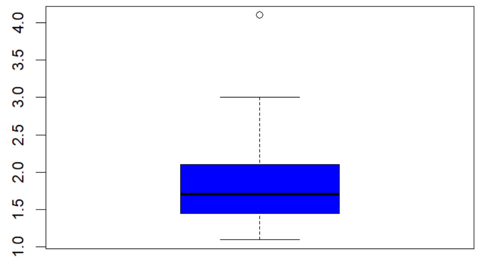


FIGURE 15. Box plot the relief times (in minutes) of 20 patients who received an analgesic

**Figures 16 and 17** present the estimated probability density function (PDF) and cumulative distribution function (CDF) of the MOEW distribution for elief times (in minutes) of 20 patients who received an analgesic dataset. The PDF plot reveals a right-skewed shape suggests that the MOEW distribution effectively captures the asymmetry in the data. The strong alignment between the estimated PDF and CDF with the empirical data confirms that the MOEW distribution effectively models the precipitation pattern in this dataset.

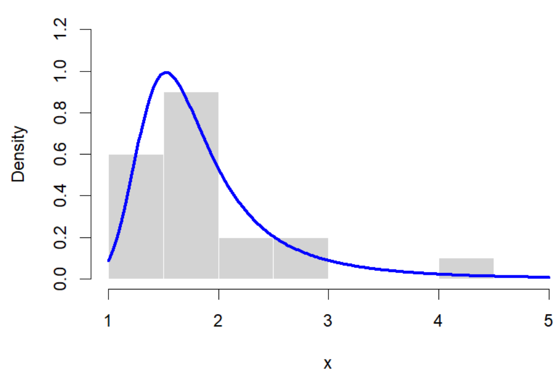


FIGURE 16. Plot of estimated PDF of the relief times (in minutes) of 20 patients who received an analgesic

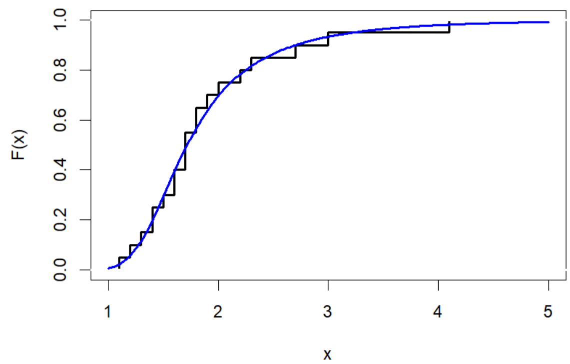


FIGURE 17. Plot of estimated CDF of the relief times (in minutes) of 20 patients who received an analgesic

**Figures 18 and 19** present the Kaplan-Meier survival curve and the probability-probability (PP) plot, respectively. The Kaplan-Meier curve closely follows the model's estimated survival function, indicating a strong fit to the data. Meanwhile, the PP plot assesses the goodness-of-fit by comparing the empirical and theoretical distributions. The close alignment of points along the diagonal in the PP plot suggests that the model effectively represents the data.

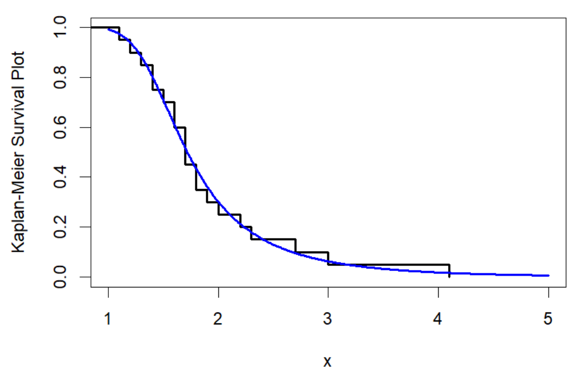


FIGURE 18. Plot of Kaplan-Meier of the relief times (in minutes) of 20 patients who received an analgesic

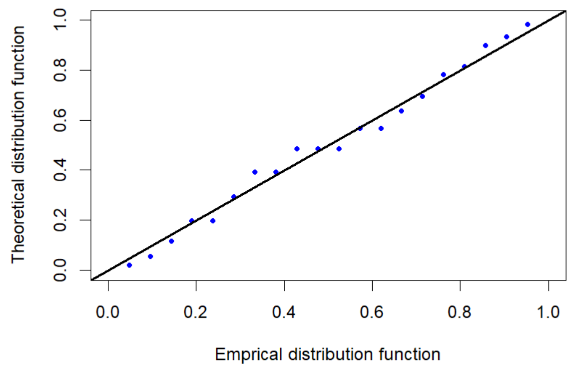


FIGURE 19. PP Plot of the relief times (in minutes) of 20 patients who received an analgesic

**Figure 20** compares various probability distributions (MOEW,MOW,EW,and Weibull) with the empirical data by overlaying their fitted density curves on a histogram. The MOEW distribution aligns most closely with the histogram, indicating that it provides the best fit.

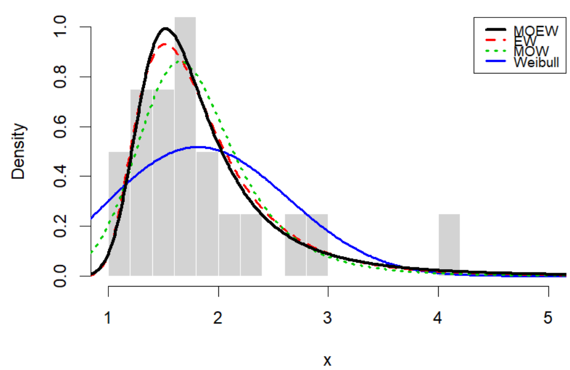


FIGURE 20. Estimated fitted densities ofthe relief times (in minutes) of 20 patients who received an analgesic data set for various distribution

**Table 8** offers the descriptive statistics for data set II. A skewness value is 1.7197 which indicates that the distribution is positively skewed and the kurtosis of 5.9241 suggests that the distribution is platykurtic.

TABLE 8. Descriptive Statistics patients who received an analgesic

| Min | Max | Mean | Median | Mode | Variance | Skewness | Kurtosis |
|---|---|---|---|---|---|---|---|
| 1.1 | 4.1 | 1.9 | 1.7 | 1.651 | 0.49582 | 1.7197 | 5.9241 |

**Tables 9 and 10** present the AIC, BIC, HQIC, and CAIC values used to evaluate and compare different models. The Marshall-Olkin Exponentiated Weibull distribution consistently demonstrates the lowest values in these criteria, suggesting the best fit to the data []. Additionally, it outperforms the other models with the lowest K-S, $W^*$, and $A^*$ values, the highest log-likelihood, and the highest P-value for the K-S statistic.

TABLE 9. Values of Information Criteria for various distributions for data set I

| **Distributions** | **AIC** | **BIC** | **CAIC** | **HQIC** |
|---|---|---|---|---|
| MOEW | 39.0731 | 43.0560 | 41.7398 | 39.8506 |
| MOW | 40.9270 | 44.9099 | 43.5936 | 41.7045 |
| EW | 98.1910 | 101.1782 | 99.6910 | 98.7742 |
| Weibull | 45.1728 | 47.1643 | 45.8787 | 45.5616 |

TABLE 10. The Goodness-of-fit statistics, log-likelihood, and maximum likelihood estimates (MLEs) for data set I

| **Distributions** | **Estimates(SEs)** | $\ell$ | $W^*$ | $A^*$ | **K-S (p-value)** |
|---|---|---|---|---|---|
| **MOEW** | $\alpha = 114.63209(11.4144)$<br>$\beta = 0.64031(2.41058)$<br>$\lambda = 5.22146(9.96127)$<br>$\theta = 0.16432(3.80335)$ | $-15.39305$ | 0.02233512 | 0.1388038 | 0.08814292 (0.9977342) |
| **MOW** | $\alpha = 0.00272(6.3286)$<br>$\beta = 5.86089(0.19415)$<br>$\lambda = 0.0001(6.58045)$<br>$\theta = 0.99932(NaN)$ | $-16.46348$ | 0.03487309 | 0.2836677 | 0.11137380 (0.9651607) |
| **EW** | $\alpha = 328088.60113(NaN)$<br>$\beta = 0.3104(NaN)$<br>$\lambda = 2280.46975(NaN)$ | $-15.43798$ | 0.02624560 | 0.1620998 | 0.09528317 (0.9934111) |
| **Weibull** | $\alpha = 2.78683(0.42728)$<br>$\beta = 2.12991(0.18203)$ | $-20.58640$ | 0.18339114 | 1.0834354 | 0.18492758 (0.5008691) |

## 7. Conclusion

This paper applies a new four-parameter model known as the Marshall-Olkin Exponentiated Weibull distribution(MOEW) to two real datasets. We explore the mathematical and statistical properties of this distribution, deriving expressions for its cumulative distribution function,probability density function, survival function, hazard rate function, Cumulative hazard function, reversed hazard rate function, Odds function, quantile function, moment and order statistics. To estimate the parameters of the MOEW distribution, we utilize maximum likelihood estimation (MLE). Monte Carlo simulations are employed to evaluate the performance of the MLEs. Our study finds that the MLEs are both accurate and reliable in estimating model parameters. As the sample size increases, the MLEs converge to the true parameter values, as evidenced by the reduction in average biases (ABs). Additionally, root mean square errors (RMSEs) decrease with larger sample sizes. The analysis reveals that the MOEW distribution surpasses existing distributions in modeling the datasets used in this study. Moreover,various plots show that the MOEW distribution can adapt to different shapes, highlighting its flexibility in fitting diverse data sets.


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## Conflict of Interests

The authors declare that there is no conflict of interests.

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