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A HOLLING TYPE COMMENSAL SYMBIOSIS MODEL INVOLVING ALLEE EFFECT

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Abstract. A two species commensal symbiosis model with Holling type functional response and Allee effect on the second species takes the form

$$\begin{aligned}\frac{dx}{dt} &= x\Big(a_1 - b_1 x + \frac{c_1 y^p}{1+y^p}\Big),\\ \frac{dy}{dt} &= y(a_2 - b_2 y)\frac{y}{u+y}\end{aligned}$$

is investigated, where $a_i, b_i, i = 1,2$ p, u and c_1 are all positive constants, $p \geq 1$. Local and global stability property of the equilibria is investigated. Our study indicates that the unique positive equilibrium is globally stable and the system always permanent, consequently, Allee effect has no influence on the final density of the species. However, numeric simulations show that the stronger the Allee effect, the longer the for the system to reach its stable steady-state solution.

Keywords: commensal symbiosis model; Holling type functional response; Allee effect; stability.

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1. Introduction

The aim of this paper is to investigate the dynamic behaviors of the following two species commensal symbiosis model with Holling type functional response and Allee effect on the second species:

$$\begin{aligned} \frac{dx}{dt} &= x\Big(a_1 - b_1 x + \frac{c_1 y^p}{1+y^p}\Big), \\ \frac{dy}{dt} &= y(a_2 - b_2 y)\frac{y}{u+y}, \end{aligned} \tag{1.1}$$

where $a_i, b_i, i = 1,2$ p, u and c_1 are all positive constants, $p \geq 1$.

During the lase decades, many scholars investigated the dynamic behaviors of the mutualism model or commensalism model ([1]-[28]). Such topic as the stability of the positive equilibrium, the persistent of the system, the existence of the positive periodic solution etc are extensively investigated.

Sun and Wei[21] first time proposed a intraspecific commensal model:

$$\begin{aligned} \frac{dx}{dt} &= r_1 x\Big(\frac{k_1 - x + ay}{k_1}\Big), \\ \frac{dy}{dt} &= r_2 y\Big(\frac{k_2 - y}{k_2}\Big). \end{aligned} \tag{1.2}$$

They investigated the local stability of all equilibrium points. Han and Chen[22] incorporating the feedback control variables to the above system, and they showed that system admits a unique globally stable positive equilibrium, which means that feedback control variables has no influence on the stability property of the system (1.2). Corresponding to system (1.2), Xie et al. [24] proposed a discrete commensal symbiosis model, they investigated the positive ω-periodic solution of the system. Xue et al[25] further proposed a discrete commensalism model with the delay, they investigated the almost periodic solution of the system. Miao et al[26] studied the persistent property of the periodic Lotka-Volterra commensal symbiosis model with impulsive, their results indicates that impulsive is one of the important reasons that can change the long time behaviors of species. Recently, we [16] argued that it may be more suitable to assume the relationship between two species is nonlinear type instead of linear, and we established the following two species commensal symbiosis model

$$\begin{aligned}\frac{dx}{dt} &= x\Big(a_1 - b_1 x + \frac{c_1 y^p}{1+y^p}\Big),\\ \frac{dy}{dt} &= y(a_2 - b_2 y),\end{aligned} \tag{1.3}$$

where $a_i, b_i, i = 1,2$ p and c_1 are all positive constants, $p \geq 1$.

Allee effect, which describe the fact that the reduce per-capita population growth rate at low densities, such phenomenon can be caused by difficulties in finding a mate or predator avoid danger or defense. During the last decades, many scholars studied the dynamic behaviors of the the predator-prey system and competition system with Allee effect, see [28]-[34] and the references cited therein. H*ü*seyin Merdan [34] investigated the influence of the Allee effect on the Lotka-Volterra type predator-prey system. To do so, the author proposed the following predator-prey system with Allee effect for prey species

$$\frac{dx}{dt} = rx(1-x)\frac{x}{\beta + x} - axy, \ \ \frac{dy}{dt} = ay(x-y), \tag{1.4}$$

where $\dfrac{x}{\beta+x}$ represents the Allee effect, β is positive constant. He showed that: (1) The system subject to an Allee effect takes a longer time to reach its steady-state solution; (2) The Allee effect reduces the population densities of both predator and prey at the steady-state.

It bring to our attention that, to this day, still no scholars study the influence of Allee effect to the commensalism model. Stimulated by the works of [16, 34], we propose the system (1.1).

The aim of this paper is to investigate the local and global stability property of the possible equilibria of system (1.1). We arrange the paper as follows: In the next section, we will investigate the existence and local stability property of the equilibria of system (1.1), we also discuss the persistent property of the system. In Section 3, by constructing some suitable Dulac function, we will investigate the global stability property of the positive equilibrium of the system; In Section 4, an example together with its numeric simulations is presented to show the feasibility of our main results. We end this paper by a briefly discussion.

2. The existence and local stability of the equilibria

The equilibria of system (1.1) is determined by the system

$$
\begin{aligned}
&x\Big(a_1-b_1x+\frac{c_1y^p}{1+y^p}\Big)=0,\\
&y(a_2-b_2y)\frac{y}{u+y}=0.
\end{aligned}
\tag{2.1}
$$

Hence, system (1.1) admits four possible equilibria, $A_0(0,0), A_1\big(\frac{a_1}{b_1},0\big), A_2\big(0,\frac{a_2}{b_2}\big)$ and $A_3\big(x^*,y^*\big)$, where

$$
x^*=\frac{a_1(\frac{a_2}{b_2})^p+c_1(\frac{a_2}{b_2})^p+a_1}{b_1\big(1+(\frac{a_2}{b_2})^p\big)},\ y^*=\frac{a_2}{b_2}. \tag{2.2}
$$

Concerned with the local stability property of the above four equilibria, we have

Theorem 2.1. *$A_0(0,0)$ and $A_1(\frac{a_1}{b_1},0)$ are non-hyperbolic, and $A_2\big(0,\frac{a_2}{b_2}\big)$ is unstable; $A_3\big(x^*,y^*\big)$ is locally stable.*

Proof. The Jacobian matrix of the system (1.1) is calculated as

$$
J(x,y)=\begin{pmatrix} a_1-2b_1x+\dfrac{c_1y^p}{1+y^p} & \dfrac{c_1pxy^{p-1}}{(1+y^p)^2}\\ 0 & \Gamma \end{pmatrix}. \tag{2.3}
$$

where

$$
\Gamma=\frac{y\Big(-3b_2uy-2b_2y^2+2a_2u+a_2y\Big)}{(u+y)^2}.
$$

Then the Jacobian matrix of the system (1.1) about the equilibria $A_0(0,0), A_1(\frac{a_1}{b_1},0)$ and $A_2(0,\frac{a_2}{b_2})$ are given by

$$
\begin{pmatrix} a_1 & 0\\ 0 & 0\end{pmatrix}. \tag{2.4}
$$

$$
\begin{pmatrix} -a_1 & 0\\ 0 & 0\end{pmatrix}. \tag{2.5}
$$

and

$$
\begin{pmatrix} a_1+\dfrac{c_1(\frac{a_2}{b_2})^p}{1+(\frac{a_2}{b_2})^p} & 0\\ 0 & -\dfrac{a_2^2}{b_2(u+\frac{a_2}{b_2})}\end{pmatrix} \tag{2.6}
$$

respectively. (2.4)-(2.6) shows that A_0, A_1 are non-hyperbolic, and $A_2(0, \frac{a_2}{b_2})$ is unstable.

The Jacobian matrix about the equilibrium A_3 is given by

$$\begin{pmatrix} -\dfrac{a_1(\frac{a_2}{b_2})^p + c_1(\frac{a_2}{b_2})^p + a_1}{1+(\frac{a_2}{b_2})^p} & F_{12} \\ 0 & -\dfrac{a_2^2}{b_2 u + a_2} \end{pmatrix}, \tag{2.7}$$

where

$$F_{12} = \frac{c_1 p b_2 \Big(a_1(\frac{a_2}{b_2})^{2p} + c_1(\frac{a_2}{b_2})^{2p} + a_1(\frac{a_2}{b_2})^p\Big)}{b_1 a_2 \Big(1+(\frac{a_2}{b_2})^p\Big)^3}. \tag{2.8}$$

The eigenvalues of the above matrix are

$$\lambda_1 = -\frac{a_1(\frac{a_2}{b_2})^p + c_1(\frac{a_2}{b_2})^p + a_1}{1+(\frac{a_2}{b_2})^p} < 0, \lambda_2 = -\frac{a_2^2}{b_2 u + a_2} < 0.$$

Hence, $A_3(x^*, y^*)$ is locally stable.

This ends the proof of Theorem 2.1.

Compared with the equilibria of system (1.1) and (1.3), we found the system admits the same equilibria. However, for system (1.3), the three boundary equilibria are all unstable, while for system (1.1), A_0 and A_1 are non-hyperbolic, and we could not obtain the stability information above these two equilibrium by investigating the eigenvalues of the Jacobian matrix. Following we will try to solve this problem by investing the persistent property of the system.

Lemma 2.1 *Consider the following equation*

$$\frac{dy}{dt} = y(a_2 - b_2 y)\frac{y}{u+y}, \tag{2.9}$$

the unique positive equilibrium $y^* = \dfrac{a_2}{b_2}$ *is global stability.*

Proof. Set

$$F(y) = (a_2 - b_2 y)\frac{y}{u+y}.$$

Then

(1)There is a $y^* = \dfrac{a_2}{b_2}$, such that $F(y^*) = 0$;

(2)For all $y^* > y > 0, F(y) > 0$;

(3)For all $y > y^* > 0, F(y) < 0$.

Now let's consider the Lyapunov function

$$V = y - y^* - y^* \ln \frac{y}{y^*}.$$

Direct calculation shows that

$$\frac{dV}{dt} = (y - y^*)(F(y) - F(y^*)) < 0.$$

Thus, y^* is global stability. This ends the proof of Lemma 2.1.

Theorem 2.2. *System (1.1) is permanent.*

Proof. It follows from the second equation of system (1.1) and Lemma 2.1 that

$$\lim_{t\to+\infty} y(t) = \frac{a_2}{b_2}. \tag{2.10}$$

For $\varepsilon > 0$ enough small, there exists $T > 0$, when $t > T$, we have

$$\frac{a_2}{b_2} - \varepsilon < y(t) < \frac{a_2}{b_2} + \varepsilon. \tag{2.11}$$

From the first equation of system (1.1) and the right part of the inequalities (2.11), when $t \geq T$, we obtain

$$\frac{dx}{dt} \leq x\left(a_1 - b_1 x + c_1 \frac{(\frac{a_2}{b_2}+\varepsilon)^p}{1+(\frac{a_2}{b_2}+\varepsilon)^p}\right). \tag{2.12}$$

Therefore,

$$\limsup_{t\to+\infty} x(t) \leq \frac{a_1 + c_1 \frac{(\frac{a_2}{b_2}+\varepsilon)^p}{1+(\frac{a_2}{b_2}+\varepsilon)^p}}{b_1}.$$

Setting $\varepsilon \to 0$ in above inequality leads to

$$\limsup_{t\to+\infty} x(t) \leq \frac{a_1 + c_1 \frac{(\frac{a_2}{b_2})^p}{1+(\frac{a_2}{b_2})^p}}{b_1}. \tag{2.13}$$

When $t \geq 0$, from the first equation of system (1.1), we also have

$$\frac{dx}{dt} \geq x(a_1 - b_1 x), \tag{2.14}$$

and so

$$\liminf_{t\to+\infty} x(t) \geq \frac{a_1}{b_1}. \tag{2.15}$$

(2.13) and (2.15) show that the first species of system (1.1) is permanent.

This ends the proof of Theorem 2.2.

Since the permanence of the system implies that all the solutions will be bounded above and below by positive constants, thus, it's impossible for the solutions to approach to $A_0(0,0)$ and $A_1(\frac{a_1}{b_1},0)$, which means that $A_0(0,0)$ and $A_1(\frac{a_1}{b_1},0)$ are locally unstable.

Corollary 2.1. *The equilibria A_0 and A_1 of system (1.1) are all unstable.*

3. Global stability of the positive equilibrium

Theorem 2.1 shows that the system always admits a positive equilibrium, and this equilibrium is locally stable. Theorem 2.2 shows that all other three boundary equilibria are unstable. One interesting thing is whether the system (1.1) could have limit cycle or not, which means that the two species could be coexistent in a periodic oscillation form. The aim of this section is to show that such phenomenon could not be happened.

Theorem 3.1. *$A_3\big(x^*,y^*\big)$ is globally stable.*

Proof. Theorem 2.2 shows that every solution of system (1.1) starts in R^2_+ is uniformly bounded on

$$D=\left\{(x,y)|x<\frac{a_1+c_1\dfrac{(\frac{a_2}{b_2})^p}{1+(\frac{a_2}{b_2})^p}}{b_1}+\varepsilon,\ y<\frac{a_2}{b_2}+\varepsilon\right\}.$$

Also, from Theorem 2.1 and 2.2 there is a unique local stable positive equilibrium $A_3(x^*,y^*)$. To show that $A_3(x^*,y^*)$ is globally stable, it's enough to show that the system admits no limit cycle in the area D, Let's consider the Dulac function $u(x,y)=x^{-1}y^{-2}$, then

$$\begin{aligned}
&\frac{\partial(uP)}{\partial x}+\frac{\partial(uQ)}{\partial y}\\
=&\ \frac{1}{xy^2}\left(a_1-2b_1x+\frac{c_1y^p}{1+y^p}\right)-\frac{1}{x^2y^2}\left(a_1x-b_1x^2+\frac{c_1xy^p}{1+y^p}\right)\\
&+\frac{-2b_2y+a_2}{y(u+y)x}-\frac{-b_2y^2+a_2y}{y^2(u+y)x}-\frac{-b_2y^2+a_2y}{y(u+y)^2x}\\
=&\ -\frac{b_1u^2x+2b_1uxy+b_1xy^2+b_2uy^2+a_2y^2}{x(u+y)^2y^2}<0,
\end{aligned}$$

where

$$\begin{aligned}
P(x,y) &= x\Big(a_1-b_1x+\frac{c_1y^p}{1+y^p}\Big),\\
Q(x,y) &= y(a_2-b_2y)\frac{y}{u+y}.
\end{aligned}$$

By Dulac Theorem[28], there is no closed orbit in area D. Consequently, $A_3(x^*,y^*)$ is globally asymptotically stable. This completes the proof of Theorem 3.1.

4. Numeric simulations

Now let us consider the following example.

Example 4.1. Consider the following system

$$\begin{aligned}
\frac{dx}{dt} &= x\Big(1-2x+\frac{y}{1+y}\Big),\\
\frac{dy}{dt} &= y(1-2y)\frac{y}{u+y}.
\end{aligned}\tag{4.1}$$

In this system, corresponding to system (1.1), we take $a_1=a_2=c_1=1, b_1=b_2=2$. From Theorem 3.1, the unique positive equilibrium $\left(\frac{2}{3},\frac{1}{2}\right)$ is globally stable. Numeric simulation (Fig.1) also support this assertion. Now let's take $u=1,5$ and 10, respectively, Fig. 2 and 3 show that with the increasing of the u (i. e., the increasing of the Allee effect), the solution takes much time to reach its steady state.

5. Conclusion

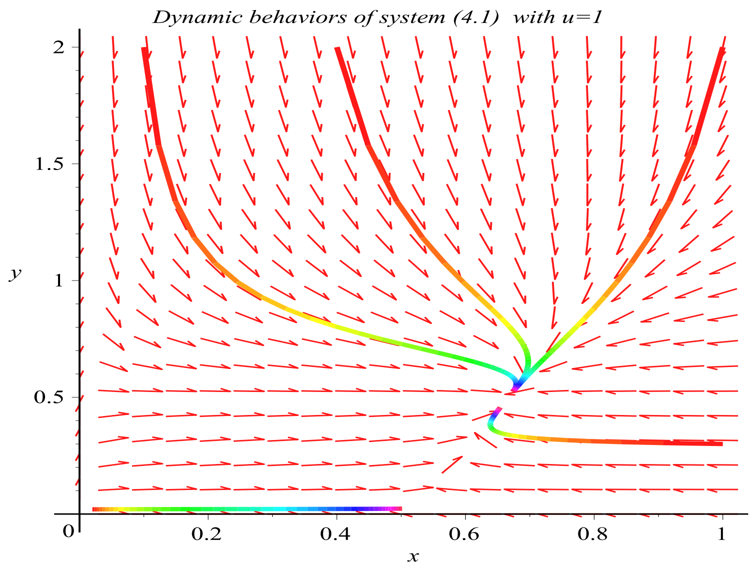

FIGURE 1. Numeric simulations of system (4.1) with $u = 1$, the initial conditions $(x(0), y(0)) = (0.4, 2), (1, 0.3), (0.02, 0.02), (1, 2)$ and $(0.1, 2)$, respectively.

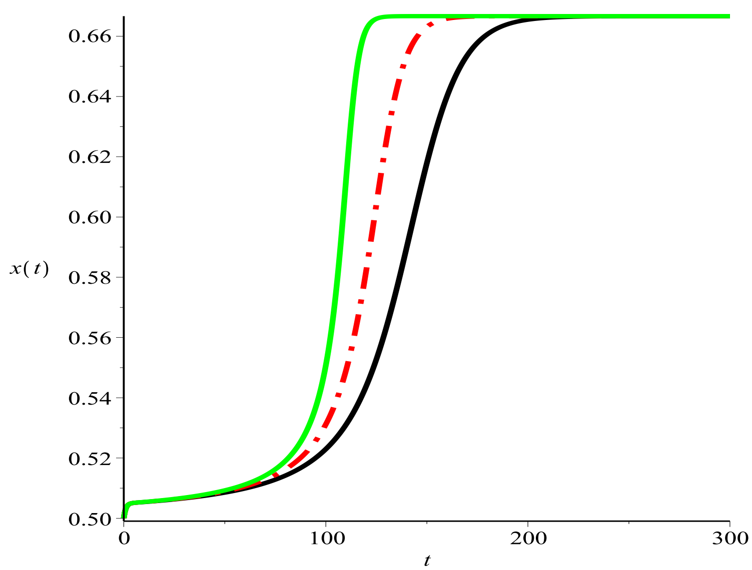

FIGURE 2. Numeric simulations of $x(t)$, with $u = 1, 5, 10$ and $(x(0), y(0)) = (0.5, 0.01)$, respectively, where black curve is the solution of $u = 10$, green curve is the solution of $u = 1$, and red curve is the solution of $u = 5$.

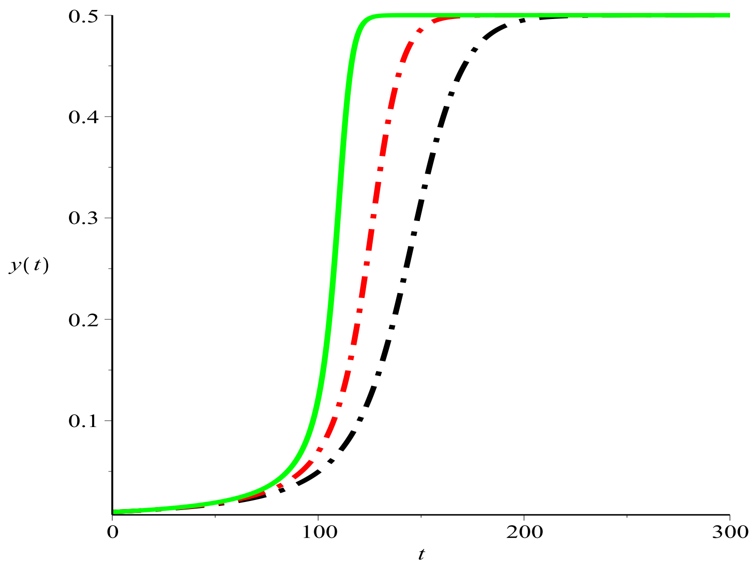

FIGURE 3. Numeric simulations of $y(t)$, with $u = 1, 5, 10$ and $(x(0), y(0)) = (0.5, 0.01)$, respectively, where black curve is the solution of $u = 10$, green curve is the solution of $u = 1$, and red curve is the solution of $u = 5$.

We propose a two species commensal symbiosis model with Holling type functional response and Allee effect to the second species, our study shows that the dynamic behaviors of the system is similar to the system without Allee effect, i.e., the dynamic behaviors of the system (1.1) is similar to that of the dynamic behaviors of the system (1.3). The system always admits a unique globally stable positive equilibrium. However, by introducing the Allee effect, the stability property of the boundary equilibria become complicated, since $A_0(0,0)$ and $A_1(\frac{a_1}{b_1}, 0)$ are non-hyperbolic type equilibria. We solve this problem by discuss the persistent property of the system. We showed that the system is always permanent, and consequently, $A_0(0,0)$ and $A_1(\frac{a_1}{b_1}, 0)$ are unstable.

Already, H*ü*seyin Merdan [34] had showed that the Allee effect reduces the population densities of both predator and prey at the steady-state, while in system (1.1), this property does not hold. Allee effect has no influence on the final density of the both species. On the other hand, Numeric simulations (Fig. 2, Fig. 3) also show that the stronger the Allee effect (u become large), the system takes a longer time to reach its steady-state solution. Such a property is coincidence with that of the findings of H*ü*seyin Merdan [34].

At the end of the paper we would like to mention that whether the system has similar dynamics for $0 < p < 1$ is still unknown, we leave this for future discussion.

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Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] K. Yang, Z. S. Miao, F. D. Chen, X. D. Xie, Influence of single feedback control variable on an autonomous Holling-II type cooperative system, J. Math. Anal. Appl. 435(1)(2016), 874-888.

[2] F. D. Chen, X. D. Xie, X. F. Chen, Dynamic behaviors of a stage-structured cooperation model, Commun. Math. Biol. Neurosci. 2015 (2015), Article ID 4.

[3] K. Yang, X. D. Xie, F. D. Chen, Global stability of a discrete mutualism model, Abstr. Appl. Anal. 2014 (2014), Article ID 709124, 7 pages.

[4] F. D. Chen, M. S. You, Permanence for an integrodifferential model of mutualism, Appl. Math. Comput. 186(1)(2007)30-34.

[5] L. J. Chen, X. D. Xie, Feedback control variables have no influence on the permanence of a discrete N-species cooperation system, Discrete Dyn. Nat. Soc. 2009(2009), Article ID 306425, 10 pages.

[6] F. D. Chen, Permanence for the discrete mutualism model with time delays, Math. Comput. Modelling 47(2008), 431-435.

[7] F. D. Chen, J. H. Yang, L. J. Chen, X. D. Xie, On a mutualism model with feedback controls, Appl. Math. Comput. 214(2009), 581-587.

[8] L. J. Chen, L. J. Chen, Z. Li, Permanence of a delayed discrete mutualism model with feedback controls, Math. Comput. Modelling 50(2009), 1083-1089.

[9] Y. K. Li, T. Zhang, Permanence of a discrete N-species cooperation system with time-varying delays and feedback controls, Math. Comput. Modelling 53(2011), 1320-1330.

[10] X. D. Xie, F. D. Chen, Y. L. Xue, Note on the stability property of a cooperative system incorporating harvesting, Discrete Dyn. Nat. Soc. 2014 (2014), Article ID 327823, 5 pages.

[11] X. D. Xie, F. D. Chen, K. Yang and Y. L. Xue, Global attractivity of an integrodifferential model of mutualism, Abstr. Appl. Anal. 2014(2014), Article ID 928726, 6 pages.

[12] X. P. Li, W. S. Yang, Permanence of a discrete model of mutualism with infinite deviating arguments, Discrete Dyn. Nat. Soc. 2010(2010), Article ID 931798, 7 pages.

[13] F. D. Chen, X. D. Xie, Study on the dynamic behaviors of cooperation population modeling. Beijing: Science Press, 2014.

[14] Z. J. Liu, J. H. Wu, R. H. Tan, et al., Modeling and analysis of a periodic delays two-species model of facultative mutualism, Appl. Math. Comput., 217(2010), 893-903.

[15] W. S. Yang, X. P. Li, Permanence of a discrete nonlinear N-species cooperation system with time delays and feedback controls, Appl. Math. Comput., 218(7)(2011), 3581-3586.

[16] R. X. Wu, L. Li, X. Y. Zhou, A commensal symbiosis model with Holling type functional response, J. Math. Comput. Sci. 16(2016), 364-371.

[17] K. Yang, Z. S. Miao, F. D. Chen, X. D. Xie, Influence of single feedback control variable on an autonomous Holling-II type cooperative system, J. Math. Anal. Appl. 435(1)(2016), 874-888.

[18] F. D. Chen, X. D. Xie, Study on the dynamic behaviors of cooperation population modeling. Beijing: Science Press, 2014.

[19] G. C. Sun, W. L. Wei, The qualitative analysis of commensal symbiosis model of two populations, Math. Theory Appl. 23(3)(2003), 64-68.

[20] R. Y. Han, F. D. Chen, Global stability of a commensal symbiosis model with feedback controls, Commun. Math. Biol. Neurosci. 2015 (2015), Article ID 15.

[21] X. D. Xie, Z. S. Miao, Y. L. Xue, Positive periodic solution of a discrete Lotka-Volterra commensal symbiosis model, Commun. Math. Biol. Neurosci. 2015 (2015), Article ID 2.

[22] Y. L. Xue, X. D. Xie, F. D. Chen, R. Y. Han, Almost periodic solution of a discrete commensalism system, Discrete Dyn. Nat. Soc. 2015 (2015), Article ID 295483, 11 pages.

[23] Z. S. Miao, X. D. Xie, L. Q. Pu, Dynamic behaviors of a periodic Lotka-Volterra commensal symbiosis model with impulsive, Commun. Math. Biol. Neurosci. 2015 (2015), Article ID 3.

[24] J. H. Chen, R. X. Wu, A commensal symbiosis model with non-monotonic functional response, Commun. Math. Biol. Neurosci. 2017 (2017), Article ID 5.

[25] F. D. Chen, L. Q. Pu, L. Y. Yang, Positive periodic solution of a discrete obligate Lotka-Volterra model Commun. Math. Biol. Neurosci. 2015 (2015), Article ID 14.

[26] Z. F. Zhu, F. Xu, Y. A. Li, Mathematical analysis on commensalism Lotka-Volterra model of populations, J. Chongqing Inst. Tech. Nat. Sci. Ed., 21(10)(2007), 59-62.

[27] Y. L. Xue, F. D. Chen, X. D. Xie, et al. Dynamic behaviors of a discrete commensalism system, Ann. Appl. Math. 31(4)(2015), 452-461.

[28] Y. C. Zhou, Z. Jin, J. L. Qin, Ordinary differential equaiton and its application, Science Press, 2003.

[29] M. A. McCarthy, ,The Allee effect, fingding mates and theoretical models, Ecol. Modelling, 103 (1997), 99-102.

[30] C Çelik, O. Duman, Allee effect in a discrete-time predator-prey system, Chaos Solitons Fractals, 90 (2009), 1952-1956.

[31] G. Eduardo, R. P. Alejandro, G. Y. Betsabe, Multiple limit cycles in a Leslie-Gower-type predator-prey model considering weak Allee effect on prey, Nonlinear Anal., Model. Control, 22(2017), 347-365.

[32] K. K. Gurubilli, P. N. D. Srinivasu, M. Banerjee, Global dynamics of a prey-predator model with Allee effect and additional food for the predators, Int. J. Dyn. Control, 5 (13) (2016), 1-14.

[33] Y. Kang, A. A. Yakubu, Weak Allee effects and species coexistence, Nonlinear Anal., Real World Appl. 12 (2011), 3329-3345.

[34] H. Merdan, O. Duman,On the stability analysis of a general discrete-time population model predation and Allee effects, Chaos Solitons Fractals, 40 (2009), 1169-1175.