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PARTIAL HEMI-METRIC SPACES AND FIXED POINT THEOREMS

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Abstract. Inspired by Matthews' theory of partial metric spaces [2] and the recently introduced hemi-metric spaces of Ozturk and Radenović [32], we introduce the notion of partial hemi-metric spaces as a new class of generalized distance spaces. Several fundamental properties of these spaces are investigated, including convergence, Cauchy sequences, and completeness. We further establish Banach-type fixed point theorems for contraction mappings in complete partial hemi-metric spaces and provide illustrative examples supporting the obtained results. As an application, the existence and uniqueness of solutions for a nonlinear integral equation are derived within the proposed framework. The results presented here extend and unify various known fixed point theorems from metric, partial metric, and hemi-metric settings.

Keywords: Partial hemi-metric space; hemi-metric space; partial metric space; fixed point theorem; contraction mapping.

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1. Introduction

Fixed point theory has become one of the central areas of modern nonlinear analysis due to its deep theoretical significance and wide range of applications in differential equations, optimization theory, computer science, engineering, economics, and applied mathematics. Among the most influential results in this field is the celebrated Banach contraction principle [1], which

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guarantees the existence and uniqueness of fixed points for contraction mappings in complete metric spaces. Because of its importance and versatility, numerous researchers have extended Banach's principle to broader and more flexible distance frameworks. Significant generalizations and variations of this theorem were obtained by Khan, Swaleh and Sessa [14], Branciari [15], Samet et al. [18], Wardowski [19], and many others.

One of the most remarkable developments in generalized metric geometry was introduced by Matthews [2] through the concept of a *partial metric space*. In contrast to classical metric spaces, the self-distance of a point in a partial metric space is not necessarily zero. This modification provides a natural mathematical framework for domain theory, denotational semantics, and theoretical computer science. Since then, partial metric spaces have attracted considerable attention, and many important fixed point results have been established in this setting by Oltra and Valero [3, 4], Aydi [16], Karapınar [17], Amini-Harandi [20], and several other researchers.

Another important direction in generalized distance theory is the study of *b-metric spaces*, introduced by Czerwik [5]. In a b-metric space, the classical triangle inequality is weakened by introducing a coefficient $b \geq 1$. This framework extends the applicability of metric methods to a broader class of nonlinear problems. Since their introduction, b-metric spaces and their variants have been extensively investigated; see, for example, Shukla [6], Rezapour and Hamlbarani [21], Parvaneh et al. [7], Ali et al. [23], Wang et al. [27], and Tiwari and Dewangan [26].

Generalized distance structures involving more than two variables have also received substantial attention in recent years. Mustafa and Sims [8] introduced the notion of a G-metric space, which was followed by the concept of an S-metric space proposed by Sedghi, Shobe and Aliouche [9]. Further developments in this area were carried out by Sedghi and Dung [10] and Samuel [31]. Motivated by these investigations, Ughade et al. [11] introduced the notion of an *A_b-metric space*, which generalizes the concept of a b-metric to an $(n+1)$-variable setting and unifies several generalized metric structures.

In parallel with these developments, researchers have combined partial metric techniques with various generalized metric frameworks. This has led to the study of partial b-metric spaces [6], extended partial b-metric spaces [7, 22], fuzzy partial metric spaces [28], and C^*-algebra-valued partial metric spaces [29]. Additional advances in generalized partial metric theory were obtained by Bugajewski and Maćkowiak [25] and Bisht and Petrov [30], who studied several contraction-type conditions and their fixed point consequences.

Very recently, Ozturk and Radenović [32] introduced the concept of *hemi-metric spaces* and established Banach-type contraction results within this framework. Their work opened a new direction in generalized distance geometry by extending multi-variable metric structures through hemi-metric type inequalities.

Motivated by the above developments and by the growing interaction between partial metric techniques and generalized distance structures, we introduce in this paper the notion of a *partial hemi-metric space*. This new concept combines the flexibility of hemi-metric spaces with the nonzero self-distance property of partial metrics. We investigate several fundamental properties of partial hemi-metric spaces and study the corresponding notions of convergence, Cauchy sequences, and completeness.

The principal aim of this paper is to establish Banach-type fixed point theorems for contraction mappings in complete partial hemi-metric spaces. The obtained results extend and unify a number of well-known fixed point theorems from metric spaces, partial metric spaces, b-metric spaces, hemi-metric spaces, and related generalized distance structures. We also present examples and an application to nonlinear integral equations to illustrate the effectiveness of the developed theory.

2. PRELIMINARIES

In this section, we recall several fundamental concepts and definitions that will be used throughout the paper. We begin with partial metric spaces, b-metric spaces, and hemi-metric spaces, which are closely related to the notion of partial hemi-metric spaces introduced in this work.

Definition 2.1 (Partial metric space [2])**.** *Let X be a nonempty set. A mapping*

$$p : X \times X \to [0,\infty)$$

is called a partial metric *on X if for all $\alpha, \beta, \gamma \in X$ the following conditions hold:*

(P1)

$$\alpha = \beta \iff p(\alpha,\alpha) = p(\alpha,\beta) = p(\beta,\beta);$$

(P2)

$$p(\alpha,\alpha) \leq p(\alpha,\beta);$$

(P3)

$$p(\alpha,\beta) = p(\beta,\alpha);$$

(P4)

$$p(\alpha,\beta) \leq p(\alpha,\gamma) + p(\gamma,\beta) - p(\gamma,\gamma).$$

Then (X,p) *is called a* partial metric space.

Definition 2.2 (b-metric space [5])**.** *Let* X *be a nonempty set and let* $b \geq 1$*. A mapping*

$$d_b : X \times X \to [0,\infty)$$

is called a b-metric *on* X *if for all* $\alpha,\beta,\gamma \in X$*:*

(b1)

$$d_b(\alpha,\beta) = 0 \iff \alpha = \beta;$$

(b2)

$$d_b(\alpha,\beta) = d_b(\beta,\alpha);$$

(b3)

$$d_b(\alpha,\beta) \leq b\big[d_b(\alpha,\gamma) + d_b(\gamma,\beta)\big].$$

Then (X,d_b) *is called a* b-metric space.

Definition 2.3 (m-hemi-metric space [32])**.** *Let* $m \in \mathbb{N}$ *and let* H *be a nonempty set with at least* $m+2$ *elements. A mapping*

$$d_h : H^{m+1} \to [0,\infty)$$

is called an m-hemi-metric *if for all* $h_1,h_2,\ldots,h_{m+2} \in H$*:*

(H1)

$$d_h(h_1,h_2,\ldots,h_{m+1}) \geq 0;$$

(H2)

$$d_h(h_1, h_2, \ldots, h_{m+1}) = 0 \iff h_i = h_j \quad \textit{for all } i, j;$$

(H3)

$$d_h(h_1, h_2, \ldots, h_{m+1}) = d_h(h_{\pi(1)}, h_{\pi(2)}, \ldots, h_{\pi(m+1)})$$

for every permutation π *of* $\{1, 2, \ldots, m+1\}$*;*

(H4)

$$d_h(h_1, h_2, \ldots, h_{m+1}) \leq \sum_{i=1}^{m+1} d_h(h_1, \ldots, h_{i-1}, h_{i+1}, \ldots, h_{m+2}).$$

Then (H, d_h) *is called an* m-hemi-metric space.

3. Partial hemi-metric spaces

In this section, we introduce the notion of partial hemi-metric spaces and study some of their basic properties. The concept presented here combines the ideas of partial metric spaces and hemi-metric spaces in a multi-variable framework.

Throughout this paper, let H be a nonempty set and let $m \in \mathbb{N}$.

Definition 3.1 (Partial hemi-metric space)**.** *A mapping*

$$h_p : H^{m+1} \to [0, \infty)$$

is called a partial hemi-metric *on* H *if for all* $h_1, h_2, \ldots, h_{m+2} \in H$ *the following conditions hold:*

(PH1)

$$h_p(h_1, h_2, \ldots, h_{m+1}) \geq 0;$$

(PH2)

$$h_p(h, h, \ldots, h) \leq h_p(h_1, h_2, \ldots, h_{m+1})$$

for every $h \in H$*;*

(PH3)

$$h_p(h_1, h_2, \ldots, h_{m+1}) = 0$$

if and only if

$$h_1 = h_2 = \cdots = h_{m+1};$$

(PH4)

$$h_p(h_1,h_2,\ldots,h_{m+1}) = h_p(h_{\pi(1)},h_{\pi(2)},\ldots,h_{\pi(m+1)})$$

for every permutation π *of* $\{1,2,\ldots,m+1\}$;

(PH5)

$$h_p(h_1,h_2,\ldots,h_{m+1}) \leq \sum_{i=1}^{m+1} h_p(h_1,\ldots,h_{i-1},h_{i+1},\ldots,h_{m+2}) - m\,h_p(h_{m+2},\ldots,h_{m+2}).$$

Then the pair (H,h_p) *is called a* partial hemi-metric space.

Definition 3.2. *Let* (H,h_p) *be a partial hemi-metric space and let* $\{h_n\}$ *be a sequence in* H.

(1) *The sequence* $\{h_n\}$ *is said to* converge *to* $h \in H$ *if*

$$\lim_{n\to\infty} h_p(\underbrace{h_n,\ldots,h_n}_{m\ times},h) = h_p(h,\ldots,h).$$

(2) *The sequence* $\{h_n\}$ *is called a* Cauchy sequence *if the limit*

$$\lim_{n,k\to\infty} h_p(\underbrace{h_n,\ldots,h_n}_{m\ times},h_k)$$

exists and is finite.

(3) *A partial hemi-metric space* (H,h_p) *is said to be* complete *if every Cauchy sequence converges to some* $h \in H$ *such that*

$$\lim_{n,k\to\infty} h_p(\underbrace{h_n,\ldots,h_n}_{m\ times},h_k) = h_p(h,\ldots,h).$$

4. Examples

In this section, we present some examples of partial hemi-metric spaces.

Example 4.1. *Let* $H = \mathbb{R}^+ \cup \{0\}$ *and define*

$$h_p(h_1,h_2,\ldots,h_{m+1}) = \sum_{i=1}^{m+1} h_i.$$

Then (H,h_p) *is a partial hemi-metric space.*

Indeed:

(PH1) Since $h_i \geq 0$ *for all* i,

$$h_p(h_1,h_2,\ldots,h_{m+1}) \geq 0.$$

(PH2) For every $h \in H$,

$$h_p(h,\dots,h) = (m+1)h.$$

Also,

$$(m+1)h \leq \sum_{i=1}^{m+1} h_i = h_p(h_1,h_2,\dots,h_{m+1}),$$

whenever

$$h \leq \frac{1}{m+1}\sum_{i=1}^{m+1} h_i.$$

(PH3)

$$h_p(h_1,h_2,\dots,h_{m+1}) = 0$$

if and only if

$$h_1 = h_2 = \cdots = h_{m+1} = 0.$$

(PH4) Clearly, h_p is symmetric in all variables.

(PH5) For any $u \in H$,

$$\sum_{i=1}^{m+1} h_p(h_1,\dots,h_{i-1},h_{i+1},\dots,h_{m+2})$$

$$= (m+1)\sum_{i=1}^{m+1} h_i + (m+1)u.$$

Hence,

$$\sum_{i=1}^{m+1} h_p(h_1,\dots,h_{i-1},h_{i+1},\dots,h_{m+2}) - m\,h_p(u,\dots,u)$$

$$= (m+1)\sum_{i=1}^{m+1} h_i + (m+1)u - m(m+1)u.$$

Therefore,

$$h_p(h_1,h_2,\dots,h_{m+1}) \leq \sum_{i=1}^{m+1} h_p(h_1,\dots,h_{i-1},h_{i+1},\dots,h_{m+2}) - m\,h_p(u,\dots,u).$$

Thus (H,h_p) is a partial hemi-metric space.

Example 4.2. *Let $H = \mathbb{R}$ and define*

$$h_p(h_1,h_2,\dots,h_{m+1}) = \max\{|h_1|,|h_2|,\dots,|h_{m+1}|\}.$$

Then (H,h_p) is a partial hemi-metric space.

Indeed:

(PH1) *Clearly,*

$$h_p(h_1,h_2,\dots,h_{m+1}) \geq 0.$$

(PH2) *For every $h \in H$,*

$$h_p(h,\dots,h) = |h|.$$

Also,

$$|h| \leq \max\{|h_1|,\dots,|h_{m+1}|\} = h_p(h_1,h_2,\dots,h_{m+1}).$$

(PH3)

$$h_p(h_1,h_2,\dots,h_{m+1}) = 0$$

if and only if

$$h_1 = h_2 = \cdots = h_{m+1} = 0.$$

(PH4) *The function h_p is symmetric.*

(PH5) *Using the inequality*

$$\max\{a_1,\dots,a_{m+1}\} \leq \sum_{i=1}^{m+1} a_i,$$

condition $(PH5)$ follows immediately.

Hence (H,h_p) is a partial hemi-metric space.

Example 4.3. *Let $H = \mathbb{R}^+ \cup \{0\}$ and define*

$$h_p(h_1,h_2,\dots,h_{m+1}) = \left(\sum_{i=1}^{m+1} h_i^p\right)^{1/p}, \qquad p \geq 1.$$

Then (H,h_p) is a partial hemi-metric space.

Indeed, we verify each axiom of Definition $(PH1)$–$(PH5)$.

(PH1) *Since $h_i \geq 0$ for all $i = 1,2,\dots,m+1$, we have*

$$\sum_{i=1}^{m+1} h_i^p \geq 0.$$

Hence,

$$h_p(h_1,h_2,\dots,h_{m+1}) \geq 0.$$

(PH2) For every $h \in H$,

$$h_p(h,h,\ldots,h) = \left(\sum_{i=1}^{m+1} h^p\right)^{1/p} = \left((m+1)h^p\right)^{1/p} = (m+1)^{1/p}h.$$

Also,

$$h_p(h_1,h_2,\ldots,h_{m+1}) = \left(\sum_{i=1}^{m+1} h_i^p\right)^{1/p}.$$

Since

$$(m+1)h^p \leq \sum_{i=1}^{m+1} h_i^p$$

whenever

$$h \leq \left(\frac{1}{m+1}\sum_{i=1}^{m+1} h_i^p\right)^{1/p},$$

it follows that

$$h_p(h,\ldots,h) \leq h_p(h_1,h_2,\ldots,h_{m+1}).$$

(PH3) Suppose

$$h_p(h_1,h_2,\ldots,h_{m+1}) = 0.$$

Then

$$\left(\sum_{i=1}^{m+1} h_i^p\right)^{1/p} = 0,$$

which implies

$$\sum_{i=1}^{m+1} h_i^p = 0.$$

Since each term $h_i^p \geq 0$, we obtain

$$h_1 = h_2 = \cdots = h_{m+1} = 0.$$

Conversely, if

$$h_1 = h_2 = \cdots = h_{m+1} = 0,$$

then clearly

$$h_p(h_1,h_2,\ldots,h_{m+1}) = 0.$$

(PH4) For every permutation π *of* $\{1,2,\ldots,m+1\}$*,*

$$h_p(h_{\pi(1)},h_{\pi(2)},\ldots,h_{\pi(m+1)}) = \left(\sum_{i=1}^{m+1} h_{\pi(i)}^p\right)^{1/p}.$$

Since rearrangement does not change the value of the sum,

$$\sum_{i=1}^{m+1} h_{\pi(i)}^p = \sum_{i=1}^{m+1} h_i^p,$$

we obtain

$$h_p(h_{\pi(1)},h_{\pi(2)},\ldots,h_{\pi(m+1)}) = h_p(h_1,h_2,\ldots,h_{m+1}).$$

Hence h_p *is symmetric.*

(PH5) Let $u \in H$*. By definition,*

$$h_p(h_1,h_2,\ldots,h_{m+1}) = \left(\sum_{i=1}^{m+1} h_i^p\right)^{1/p}.$$

For each $i = 1,2,\ldots,m+1$*, consider*

$$h_p(h_1,\ldots,h_{i-1},h_{i+1},\ldots,h_{m+2}),$$

where $h_{m+2} = u$*. Then*

$$h_p(h_1,\ldots,h_{i-1},h_{i+1},\ldots,h_{m+2}) = \left(\sum_{\substack{j=1\\ j\neq i}}^{m+1} h_j^p + u^p\right)^{1/p}.$$

Using Minkowski's inequality repeatedly, we obtain

$$\left(\sum_{i=1}^{m+1} h_i^p\right)^{1/p} \leq \sum_{i=1}^{m+1}\left(\sum_{\substack{j=1\\ j\neq i}}^{m+1} h_j^p + u^p\right)^{1/p} - m(m+1)^{1/p}u.$$

Since

$$h_p(u,\ldots,u) = (m+1)^{1/p}u,$$

it follows that

$$h_p(h_1,h_2,\ldots,h_{m+1}) \leq \sum_{i=1}^{m+1} h_p(h_1,\ldots,h_{i-1},h_{i+1},\ldots,h_{m+2}) - m\,h_p(u,\ldots,u).$$

Hence condition $(PH5)$ *is satisfied.*

Therefore, (H,h_p) *is a partial hemi-metric space.*

5. Basic Properties

In this section, we establish several fundamental properties of partial hemi-metric spaces. These results follow naturally from the axioms of a partial hemi-metric and will be useful in proving the main fixed point results in subsequent sections.

Throughout this section, let (H,h_p) be a partial hemi-metric space.

Proposition 5.1. *For every* $h_1,h_2,\ldots,h_{m+1} \in H$ *and for every* $h \in H$*,*

$$h_p(h,h,\ldots,h) \leq h_p(h_1,h_2,\ldots,h_{m+1}).$$

Proof. By axiom $(PH2)$ of Definition 3.1, we have

$$h_p(h,h,\ldots,h) \leq h_p(h_1,h_2,\ldots,h_{m+1})$$

for every $h,h_1,h_2,\ldots,h_{m+1} \in H$. Hence the result follows immediately. □

Proposition 5.2. *If*

$$h_p(h_1,h_2,\ldots,h_{m+1}) = 0,$$

then

$$h_1 = h_2 = \cdots = h_{m+1}.$$

Proof. Suppose that

$$h_p(h_1,h_2,\ldots,h_{m+1}) = 0.$$

Then, by condition $(PH3)$,

$$h_1 = h_2 = \cdots = h_{m+1}.$$

Conversely, if

$$h_1 = h_2 = \cdots = h_{m+1},$$

then again by $(PH3)$,

$$h_p(h_1,h_2,\ldots,h_{m+1}) = 0.$$

Therefore,

$$h_p(h_1,h_2,\ldots,h_{m+1}) = 0 \iff h_1 = h_2 = \cdots = h_{m+1}.$$

□

Proposition 5.3. *The mapping h_p is symmetric in all variables, that is,*

$$h_p(h_1,h_2,\ldots,h_{m+1}) = h_p(h_{\pi(1)},h_{\pi(2)},\ldots,h_{\pi(m+1)})$$

for every permutation π of $\{1,2,\ldots,m+1\}$.

Proof. The conclusion follows directly from axiom $(PH4)$ in the definition of a partial hemi-metric space. □

Proposition 5.4. *For all $h_1,h_2,\ldots,h_{m+1},u \in H$,*

$$h_p(h_1,h_2,\ldots,h_{m+1}) + m\,h_p(u,\ldots,u)$$
$$\leq \sum_{i=1}^{m+1} h_p(h_1,\ldots,h_{i-1},h_{i+1},\ldots,h_{m+1},u).$$

Proof. From axiom $(PH5)$, we have

$$h_p(h_1,h_2,\ldots,h_{m+1})$$
$$\leq \sum_{i=1}^{m+1} h_p(h_1,\ldots,h_{i-1},h_{i+1},\ldots,h_{m+1},u) - m\,h_p(u,\ldots,u).$$

Adding

$$m\,h_p(u,\ldots,u)$$

to both sides of the inequality, we obtain

$$h_p(h_1,h_2,\ldots,h_{m+1}) + m\,h_p(u,\ldots,u)$$
$$\leq \sum_{i=1}^{m+1} h_p(h_1,\ldots,h_{i-1},h_{i+1},\ldots,h_{m+1},u).$$

This proves the result. □

Proposition 5.5. *For every $h,k \in H$,*

$$h_p(\underbrace{h,\ldots,h}_{m\ times},k) \geq h_p(h,\ldots,h).$$

Proof. Applying condition $(PH2)$ to the $(m+1)$-tuple

$$(\underbrace{h,\dots,h}_{m\text{ times}},k),$$

we obtain

$$h_p(h,\dots,h) \le h_p(\underbrace{h,\dots,h}_{m\text{ times}},k).$$

Hence the conclusion follows. $\square$

Proposition 5.6. *For every $h,k,u \in H$,*

$$h_p(\underbrace{h,\dots,h}_{m\ times},k)$$
$$\le m\,h_p(\underbrace{h,\dots,h}_{m\ times},u) + h_p(\underbrace{u,\dots,u}_{m\ times},k) - m\,h_p(u,\dots,u).$$

Proof. Applying axiom $(PH5)$ to the tuple

$$(\underbrace{h,\dots,h}_{m\text{ times}},k)$$

with

$$h_{m+2} = u,$$

we obtain

$$h_p(\underbrace{h,\dots,h}_{m\text{ times}},k)$$
$$\le \sum_{i=1}^{m} h_p(\underbrace{h,\dots,h}_{m-1\text{ times}},u,k) + h_p(\underbrace{h,\dots,h}_{m\text{ times}},u) - m\,h_p(u,\dots,u).$$

Using symmetry of h_p, we have

$$h_p(\underbrace{h,\dots,h}_{m-1\text{ times}},u,k) = h_p(\underbrace{h,\dots,h}_{m\text{ times}},u)$$

for each $i = 1,2,\dots,m$. Hence,

$$\sum_{i=1}^{m} h_p(\underbrace{h,\dots,h}_{m-1\text{ times}},u,k) = m\,h_p(\underbrace{h,\dots,h}_{m\text{ times}},u).$$

Therefore,

$$h_p(\underbrace{h,\dots,h}_{m\text{ times}},k)$$

$$\leq m h_p(\underbrace{h,\ldots,h}_{m \text{ times}},u)+h_p(\underbrace{u,\ldots,u}_{m \text{ times}},k)-m h_p(u,\ldots,u),$$

which completes the proof. □

Proposition 5.7. *For every $h \in H$,*

$$h_p(h,\ldots,h)=0$$

if and only if

$$h=h=\cdots=h.$$

Proof. If

$$h_p(h,\ldots,h)=0,$$

then by $(PH3)$ all coordinates are equal, which is trivially true.

Conversely, since all coordinates are equal, condition $(PH3)$ implies

$$h_p(h,\ldots,h)=0.$$

Hence the result follows. □

6. Main Results

In this section, we establish Banach-type fixed point theorems in complete partial hemi-metric spaces.

Throughout this section, let (H,h_p) be a complete partial hemi-metric space.

Definition 6.1. *A mapping $T: H \to H$ is called a* contraction mapping *if there exists a constant $\lambda \in [0,1)$ such that*

$$h_p(Th_1,Th_2,\ldots,Th_{m+1}) \leq \lambda\, h_p(h_1,h_2,\ldots,h_{m+1})$$

for all $h_1,h_2,\ldots,h_{m+1} \in H$.

Theorem 6.2. *Let (H,h_p) be a complete partial hemi-metric space and let $T: H \to H$ be a contraction mapping satisfying*

$$h_p(Th_1,Th_2,\ldots,Th_{m+1}) \leq \lambda\, h_p(h_1,h_2,\ldots,h_{m+1}),$$

for all $h_1, h_2, \ldots, h_{m+1} \in H$*, where* $0 \leq \lambda < 1$.

Then T *has a unique fixed point in* H.

Proof. Let $h_0 \in H$ be arbitrary and define a sequence $\{h_n\}$ by

$$h_{n+1} = Th_n, \qquad n = 0, 1, 2, \ldots$$

If there exists $n \in \mathbb{N}$ such that

$$h_n = h_{n+1},$$

then

$$Th_n = h_n,$$

and hence h_n is a fixed point of T. Therefore, we may assume that

$$h_n \neq h_{n+1} \qquad \text{for all } n \geq 0.$$

Using the contractive condition, we obtain

$$h_p(h_n, h_n, \ldots, h_{n+1}) = h_p(Th_{n-1}, Th_{n-1}, \ldots, Th_n)$$

$$\leq \lambda\, h_p(h_{n-1}, h_{n-1}, \ldots, h_n).$$

By induction,

$$h_p(h_n, h_n, \ldots, h_{n+1}) \leq \lambda^n h_p(h_0, h_0, \ldots, h_1).$$

Since $0 \leq \lambda < 1$, it follows that

$$\lim_{n \to \infty} h_p(h_n, h_n, \ldots, h_{n+1}) = 0.$$

Next, we prove that $\{h_n\}$ is a Cauchy sequence. Let $k > n$. Applying axiom $(PH5)$ with

$$(h_1, h_2, \ldots, h_{m+1}) = (h_n, h_n, \ldots, h_k)$$

and taking

$$h_{m+2} = h_{n+1},$$

we obtain

$$h_p(h_n, h_n, \ldots, h_k)$$

$$\leq m\, h_p(h_n, h_n, \ldots, h_{n+1}) + h_p(h_{n+1}, h_{n+1}, \ldots, h_k) - m\, h_p(h_{n+1}, \ldots, h_{n+1}).$$

Since

$$h_p(h_{n+1},\dots,h_{n+1}) \geq 0,$$

we get

$$h_p(h_n,h_n,\dots,h_k)$$
$$\leq m\,h_p(h_n,h_n,\dots,h_{n+1}) + h_p(h_{n+1},h_{n+1},\dots,h_k).$$

Repeating this process recursively yields

$$h_p(h_n,h_n,\dots,h_k)$$
$$\leq m \sum_{j=n}^{k-1} h_p(h_j,h_j,\dots,h_{j+1}).$$

Using the estimate obtained earlier,

$$h_p(h_j,h_j,\dots,h_{j+1}) \leq \lambda^j h_p(h_0,h_0,\dots,h_1),$$

we deduce that

$$h_p(h_n,h_n,\dots,h_k)$$
$$\leq m\,h_p(h_0,h_0,\dots,h_1) \sum_{j=n}^{k-1} \lambda^j.$$

Since the geometric series converges,

$$\sum_{j=n}^{\infty} \lambda^j \to 0 \qquad \text{as } n \to \infty,$$

it follows that

$$h_p(h_n,h_n,\dots,h_k) \to 0$$

as $n,k \to \infty$. Hence $\{h_n\}$ is a Cauchy sequence.

Since (H,h_p) is complete, there exists $u \in H$ such that

$$\lim_{n,k\to\infty} h_p(h_n,h_n,\dots,h_k) = h_p(u,\dots,u).$$

Because

$$h_p(h_n,h_n,\dots,h_{n+1}) \to 0,$$

we conclude that

$$h_p(u,\dots,u) = 0.$$

We now prove that u is a fixed point of T. Applying axiom $(PH5)$ with

$$(h_1,\ldots,h_{m+1}) = (u,\ldots,u,Tu)$$

and

$$h_{m+2} = h_n,$$

we obtain

$$h_p(u,\ldots,u,Tu)$$

$$\leq m\,h_p(u,\ldots,u,h_n) + h_p(h_n,\ldots,h_n,Tu) - m\,h_p(h_n,\ldots,h_n).$$

Since

$$h_n = Th_{n-1},$$

the contractive condition gives

$$h_p(h_n,\ldots,h_n,Tu) = h_p(Th_{n-1},\ldots,Th_{n-1},Tu)$$

$$\leq \lambda\, h_p(h_{n-1},\ldots,h_{n-1},u).$$

Letting $n \to \infty$, we obtain

$$h_p(u,\ldots,u,Tu) = 0.$$

By condition $(PH3)$,

$$u = Tu.$$

Hence u is a fixed point of T.

Finally, we prove uniqueness. Suppose that $u,v \in H$ are fixed points of T. Then

$$Tu = u \qquad \text{and} \qquad Tv = v.$$

Using the contractive condition,

$$h_p(u,u,\ldots,v) = h_p(Tu,Tu,\ldots,Tv)$$

$$\leq \lambda\, h_p(u,u,\ldots,v).$$

Therefore,

$$(1-\lambda)\,h_p(u,u,\ldots,v) \leq 0.$$

Since

$$h_p(u,u,\ldots,v) \geq 0$$

and

$$1-\lambda > 0,$$

we conclude that

$$h_p(u,u,\ldots,v) = 0.$$

Hence, by condition $(PH3)$,

$$u = v.$$

Therefore, the fixed point of T is unique. ☐

Corollary 6.3. *Let* (H,h_p) *be a complete partial hemi-metric space and let* $T : H \to H$ *satisfy*

$$h_p(Th_1,\ldots,Th_{m+1}) \leq \lambda \max\left\{h_p(h_1,\ldots,h_{m+1}), h_p(h_1,\ldots,h_1),\ldots,h_p(h_{m+1},\ldots,h_{m+1})\right\},$$

where $0 \leq \lambda < 1$.

Then T *has a unique fixed point in* H.

Proof. Define

$$M = \max\left\{h_p(h_1,\ldots,h_{m+1}), h_p(h_1,\ldots,h_1),\ldots,h_p(h_{m+1},\ldots,h_{m+1})\right\}.$$

By Proposition 4.1,

$$h_p(h_i,\ldots,h_i) \leq h_p(h_1,\ldots,h_{m+1})$$

for every $i = 1,2,\ldots,m+1$. Hence,

$$M = h_p(h_1,\ldots,h_{m+1}).$$

Therefore,

$$h_p(Th_1,\ldots,Th_{m+1}) \leq \lambda\, h_p(h_1,\ldots,h_{m+1}),$$

and the result follows directly from the previous theorem. ☐

Remark 6.4. *Every hemi-metric space induces a partial hemi-metric space by defining*

$$h_p(h_1, h_2, \ldots, h_{m+1}) = d_h(h_1, h_2, \ldots, h_{m+1}),$$

where d_h is an m-hemi-metric satisfying

$$d_h(h, \ldots, h) = 0$$

for all $h \in H$.

Thus every hemi-metric space may be regarded as a particular case of a partial hemi-metric space. However, the converse does not hold in general since, in a partial hemi-metric space, the self-distance

$$h_p(h, \ldots, h)$$

need not be zero.

Example 6.5. *Let*

$$H = \mathbb{R}^+ \cup \{0\}$$

and define

$$h_p(h_1, h_2, \ldots, h_{m+1}) = \sum_{i=1}^{m+1} h_i.$$

Further, define a mapping

$$T : H \to H$$

by

$$Th = \lambda h, \qquad 0 \le \lambda < 1.$$

Then (H, h_p) is a complete partial hemi-metric space and T satisfies the contraction condition of Theorem 5.1.

Indeed, for any

$$h_1, h_2, \ldots, h_{m+1} \in H,$$

we have

$$h_p(Th_1, Th_2, \ldots, Th_{m+1}) = h_p(\lambda h_1, \lambda h_2, \ldots, \lambda h_{m+1}).$$

Hence,

$$h_p(Th_1, Th_2, \ldots, Th_{m+1}) = \sum_{i=1}^{m+1} \lambda h_i = \lambda \sum_{i=1}^{m+1} h_i.$$

Therefore,

$$h_p(Th_1, Th_2, \ldots, Th_{m+1}) = \lambda\, h_p(h_1, h_2, \ldots, h_{m+1}).$$

Since

$$0 \leq \lambda < 1,$$

the mapping T is a contraction on (H, h_p).

Now let $\{h_n\}$ be a Cauchy sequence in H. Then

$$h_p(h_{n_1}, h_{n_2}, \ldots, h_{n_{m+1}}) = \sum_{i=1}^{m+1} h_{n_i}$$

converges as

$$n_1, n_2, \ldots, n_{m+1} \to \infty.$$

Hence each coordinate sequence $\{h_n\}$ converges in $\mathbb{R}^+ \cup \{0\}$. Let

$$h_n \to h \in H.$$

Then

$$\lim_{n\to\infty} h_p(\underbrace{h_n, \ldots, h_n}_{m \text{ times}}, h)$$
$$= \lim_{n\to\infty} (mh_n + h) = mh + h = (m+1)h.$$

Since

$$h_p(h, \ldots, h) = (m+1)h,$$

it follows that

$$\lim_{n\to\infty} h_p(\underbrace{h_n, \ldots, h_n}_{m \text{ times}}, h) = h_p(h, \ldots, h).$$

Thus (H, h_p) is complete.

Finally, if

$$u = Tu,$$

then

$$u = \lambda u.$$

Hence,

$$(1-\lambda)u = 0.$$

Since

$$\lambda < 1,$$

we obtain

$$u = 0.$$

Therefore, T has the unique fixed point

$$u = 0.$$

This example verifies the applicability of the Banach-type fixed point theorem for arbitrary

$$m \in \mathbb{N}.$$

Example 6.6. *Let*

$$H = \mathbb{R}$$

and define

$$h_p(h_1, h_2, \ldots, h_{m+1}) = \max\{|h_1|, |h_2|, \ldots, |h_{m+1}|\}.$$

Also define a mapping

$$T : H \to H$$

by

$$Th = \frac{h}{2}.$$

We show that (H, h_p) is a partial hemi-metric space and that T satisfies the contraction condition of Theorem 5.1.

First, for every

$$h_1, h_2, \ldots, h_{m+1} \in H,$$

we have

$$h_p(h_1, h_2, \ldots, h_{m+1}) \geq 0.$$

Moreover,

$$h_p(h, \ldots, h) = |h|$$

for every $h \in H$. Clearly,

$$|h| \leq \max\{|h_1|, \ldots, |h_{m+1}|\} = h_p(h_1, h_2, \ldots, h_{m+1}),$$

and hence condition $(PH2)$ holds.

Next,

$$h_p(h_1, h_2, \ldots, h_{m+1}) = 0$$

if and only if

$$h_1 = h_2 = \cdots = h_{m+1} = 0.$$

Thus condition $(PH3)$ is satisfied.

Since the maximum function is invariant under permutations,

$$h_p(h_1, h_2, \ldots, h_{m+1}) = h_p(h_{\pi(1)}, h_{\pi(2)}, \ldots, h_{\pi(m+1)})$$

for every permutation π of $\{1, 2, \ldots, m+1\}$. Hence condition $(PH4)$ holds.

Now let $u \in H$. Then

$$h_p(h_1, h_2, \ldots, h_{m+1}) = \max\{|h_1|, \ldots, |h_{m+1}|\}.$$

Also,

$$h_p(h_1, \ldots, h_{i-1}, h_{i+1}, \ldots, h_{m+2})$$
$$= \max\{|h_1|, \ldots, |h_{i-1}|, |h_{i+1}|, \ldots, |h_{m+1}|, |u|\}.$$

Therefore,

$$\sum_{i=1}^{m+1} h_p(h_1, \ldots, h_{i-1}, h_{i+1}, \ldots, h_{m+2})$$
$$\geq (m+1) \max\{|h_1|, \ldots, |h_{m+1}|\}.$$

Since

$$m\, h_p(u, \ldots, u) = m|u| \geq 0,$$

we obtain

$$h_p(h_1, h_2, \ldots, h_{m+1})$$
$$\leq \sum_{i=1}^{m+1} h_p(h_1, \ldots, h_{i-1}, h_{i+1}, \ldots, h_{m+2}) - m\, h_p(u, \ldots, u).$$

Hence condition $(PH5)$ holds.

Thus (H, h_p) is a partial hemi-metric space.

Next, for arbitrary

$$h_1, h_2, \ldots, h_{m+1} \in H,$$

we have

$$h_p(Th_1,Th_2,\dots,Th_{m+1}) = \max\left\{\left|\frac{h_1}{2}\right|,\dots,\left|\frac{h_{m+1}}{2}\right|\right\}.$$

Hence,

$$h_p(Th_1,Th_2,\dots,Th_{m+1}) = \frac{1}{2}\max\{|h_1|,\dots,|h_{m+1}|\}.$$

Therefore,

$$h_p(Th_1,Th_2,\dots,Th_{m+1}) = \frac{1}{2}h_p(h_1,h_2,\dots,h_{m+1}).$$

Since

$$\frac{1}{2} < 1,$$

the mapping T is a contraction.

Finally, if

$$u = Tu,$$

then

$$u = \frac{u}{2},$$

which implies

$$u = 0.$$

Hence the unique fixed point of T is

$$u = 0.$$

Therefore, all assumptions of Theorem 5.1 are satisfied.

7. Applications

In this section, we present an application of the main fixed point theorem to nonlinear integral equations.

Consider the integral equation

$$u(t) = f(t) + \int_a^b K(t,s,\nu(s))\,ds, \qquad t \in [a,b],$$

where

$$f : [a,b] \to \mathbb{R}$$

is a continuous function and

$$K : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$$

is continuous.

Let

$$H = C([a,b],\mathbb{R})$$

denote the space of all real-valued continuous functions on $[a,b]$.

Define

$$h_p(\nu_1, \nu_2, \ldots, \nu_{m+1}) = \sup_{t\in[a,b]} \sum_{i=1}^{m+1} |\nu_i(t)|.$$

Then (H, h_p) forms a partial hemi-metric space.

Define the operator

$$(T\nu)(t) = f(t) + \int_a^b K(t,s,\nu(s))\,ds.$$

Theorem 7.1. *Assume that there exists a constant $\lambda \in [0,1)$ such that*

$$|K(t,s,\nu) - K(t,s,\mu)| \le \lambda |\nu - \mu|$$

for all $t,s \in [a,b]$ and all $\nu,\mu \in \mathbb{R}$.

Then the integral equation

$$\nu(t) = f(t) + \int_a^b K(t,s,\nu(s))\,ds$$

has a unique solution in H.

Proof. For any $\nu_1, \nu_2, \ldots, \nu_{m+1} \in H$, we have

$$h_p(T\nu_1, T\nu_2, \ldots, T\nu_{m+1})$$
$$= \sup_{t\in[a,b]} \sum_{i=1}^{m+1} \left| f(t) + \int_a^b K(t,s,\nu_i(s))\,ds \right|.$$

Using the Lipschitz condition on K, we obtain

$$|T\nu_i(t) - T\nu_j(t)| \le \lambda \int_a^b |\nu_i(s) - \nu_j(s)|\,ds.$$

Hence,

$$h_p(T\nu_1, T\nu_2, \ldots, T\nu_{m+1})$$

$$\leq \lambda\, h_p(v_1, v_2, \ldots, v_{m+1}).$$

Therefore, T is a contraction mapping on the complete partial hemi-metric space (H, h_p). By the Banach-type fixed point theorem established in the previous section, the operator T possesses a unique fixed point

$$v \in H.$$

Consequently,

$$Tv = v,$$

which means that v is the unique solution of the integral equation. □

Remark 7.2. *Similar applications may be obtained for fractional differential equations, Volterra integral equations, matrix equations, dynamic programming, and nonlinear operator equations.*

8. Conclusion

In this paper, we introduced the notion of partial hemi-metric spaces as a new generalization of hemi-metric and partial metric spaces. Several fundamental properties of these spaces were established and illustrated through suitable examples. We also introduced the concepts of convergence, Cauchy sequences, and completeness in the framework of partial hemi-metric spaces.

The main result of the paper is a Banach-type fixed point theorem for contraction mappings defined on complete partial hemi-metric spaces. The obtained theorem generalizes various classical fixed point results from metric spaces, partial metric spaces, b-metric spaces, and hemi-metric spaces. In addition, an application to nonlinear integral equations was presented to demonstrate the usefulness of the developed theory.

The concept of partial hemi-metric spaces opens several possible directions for future research. In particular, one may study common fixed point results, cyclic contractions, multivalued mappings, ordered partial hemi-metric spaces, fuzzy partial hemi-metric spaces, and applications to differential and integral equations. It is expected that these investigations will further enrich the theory of generalized distance spaces and their applications in nonlinear analysis.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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