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CONVERGENCE RESULTS FOR ZAMFIRESCU OPERATORS AND (α,β) NONEXPANSIVE TYPE 1 MAPPING WITH APPLICATIONS

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Abstract. In this paper, we introduce a new iterative scheme for approximating fixed points of Zamfirescu operators and generalized (α,β)-nonexpansive type-1 mappings in uniformly convex Banach spaces. We establish strong and weak convergence results under suitable conditions. Furthermore, we prove stability and compare the rate of convergence with several existing iterative methods, showing that the proposed scheme converges faster. Numerical examples and an application to a nonlinear integral equation arising in infectious disease modeling are provided to support the theoretical results.

Keywords: Banach space; Zamfirescu operator; iterative scheme; $(\alpha,\beta)-$nonexpansive type 1 mapping; nonlinear integral equation.

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1. Introduction

Fixed point theory is essential in numerous analytical and computational methods, particularly in fields such as optimization, game theory, economics, and differential equations [6]. A fixed point may not always exist. Even when it exists, it can be difficult to compute. Approximating fixed points depends on three essential factors: iterative schemes, the underlying space,

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and the mappings involved. Iterative schemes play an important role in fixed-point theory, both for proving existence and for approximation. One of the earliest methods for approximating the fixed point of contraction mappings in a complete metric space is the Picard iterative process. It was shown by Krasnosel'skii [13] that the Picard iterative process fails to converge to the fixed point of a nonexpansive mapping, even when the fixed point exists. Consequently, researchers have developed various generalized mappings and iterative techniques to overcome these limitations and to obtain better convergence behavior. In recent years, several iterative schemes have been developed to approximate fixed points of nonlinear operators in various spaces ([18], [19], [20], [21]).

However, there are several iteration processes in which studies have been done with different types of mapping and space. In which researchers compare their iteration process with all other iteration processes, establishing fast convergence, finding some strong and weak convergence theorems, and fixed points ([3], [12], [14], [22], [23], [28], [34], [36], [37]).

On the other hand, Zamfirescu operators, introduced as a generalization of contraction mappings, unify several contractive conditions into a single framework. These operators satisfy at least one of a set of contractive-type inequalities. The class of Zamfirescu operator Φ is one of the most studied class of quasi-contractive type operators. Zamfirescu showed in [41] that an operator satisfying Z has a unique fixed point that can be approximated using Picard iteration. Later Rhoades [26], [27] proved that the Mann [16] and Ishikawa [10] iterations can also be used to approximate fixed points of Zamfirescu operators.

A mapping $\Phi : \mathscr{M} \to \mathscr{M}$ is called an $\kappa-$contractive if

(z$_1$) $\|\Phi\varkappa - \Phi\nu\| \leq \kappa\|\varkappa - \nu\|$ for all $\varkappa, \nu \in \mathscr{M}$, where $\kappa \in [0,1)$.

The mapping $\Phi : \mathscr{M} \to \mathscr{M}$ is called a Kannan mapping [11] if there exists $\vartheta \in [0, \frac{1}{2})$ such that

(z$_2$) $\|\Phi\varkappa - \Phi\nu\| \leq \vartheta[\|\varkappa - \Phi\varkappa\| + \|\nu - \Phi\nu\|]$ for all $\varkappa, \nu \in \mathscr{M}$.

The mapping $\Phi : \mathscr{M} \to \mathscr{M}$ is called a Chatterjea mapping [7] if there exists $\mu \in [0, \frac{1}{2})$ such that

(z$_3$) $\|\Phi\varkappa - \Phi\nu\| \leq \mu[\|\varkappa - \Phi\nu\| + \|\nu - \Phi\varkappa\|]$ for all $\varkappa, \nu \in \mathscr{M}$.

An operator Φ satisfying (z_1), (z_2) and (z_3) are independent contractive conditions is called a Zamfirescu operator [41].

We present some of the iterative schemes here.

In 2000, Noor [17] defined the following iterative scheme (1) read as follows:

$$\begin{cases} w_1 \in \mathscr{M}, \\ w_{n+1} = (1-\vartheta_n)\Phi w_n + \vartheta_n \Phi t_n, \\ t_n = (1-\rho_n) w_n + \rho_n \Phi u_n, \\ u_n = (1-\sigma_n) w_n + \sigma_n \Phi w_n, \quad n \in \mathbb{I}^+, \end{cases} \tag{1}$$

where $\{\vartheta_n\}$, $\{\rho_n\}$ and $\{\sigma_n\}$ are sequences in (0,1).

In 2007, Agarwal et al. [4] introduced a generalized form of the Ishikawa iterative scheme (3) read as follows:

$$\begin{cases} w_1 \in \mathscr{M}, \\ t_n = (1-\rho_n) w_n + \rho_n \Phi w_n, \\ w_{n+1} = (1-\vartheta_n)\Phi w_n + \vartheta_n \Phi t_n, \quad n \in \mathbb{I}^+, \end{cases} \tag{2}$$

where $\{\vartheta_n\}$, and $\{\rho_n\}$ are sequences in (0,1).

In 2014, Gursoy and Karakay [9] introduced a new iterative scheme called the Picard-S iterative scheme (3) read as follows:

$$\begin{cases} w_1 \in \mathscr{M}, \\ w_{n+1} = \Phi t_n, \\ t_n = (1-\sigma_n)\Phi w_n + \sigma_n \Phi u_n, \\ u_n = (1-\vartheta_n) w_n + \vartheta_n \Phi w_n, \quad n \in \mathbb{I}^+, \end{cases} \tag{3}$$

where $\{\vartheta_n\}$, and $\{\rho_n\}$ are sequences in (0,1).

In 2016, Thakur et al. [35] proposed a new iterative scheme (4) read as follows:

$$\begin{cases} w_1 \in \mathscr{M}, \\ w_{n+1} = \Phi t_n, \\ t_n = \Phi((1-\sigma_n)w_n + \sigma_n u_n), \\ u_n = (1-\vartheta_n)w_n + \vartheta_n \Phi w_n, \quad n \in \mathbb{I}^+, \end{cases} \tag{4}$$

where $\{\vartheta_n\}$, and $\{\rho_n\}$ are sequences in (0,1).

In 2018, Ullah and Arshad [38] introduced M-iterative scheme (5) read as follows:

$$\begin{cases} w_1 \in \mathscr{M}, \\ u_n = (1-\vartheta_n)w_n + \vartheta_n \Phi w_n, \\ t_n = \Phi u_n, \\ w_{n+1} = \Phi t_n, \quad n \in \mathbb{I}^+, \end{cases} \tag{5}$$

where $\{\vartheta_n\}$ is a sequence in (0,1).

Motivated by the preceding literature, we introduce a new iterative scheme and establish its weak and strong convergence for approximating fixed points of generalized (α,β)-nonexpansive type-1 mappings in a uniformly convex Banach space. Furthermore, we extend the applicability of the proposed scheme by studying its convergence for Zamfirescu operators as well as generalized (α,β)-nonexpansive type-1 mappings.

Although numerous iterative methods have been developed in the literature, many of them exhibit slow convergence or lack stability when applied to nonlinear integral equations. To address these limitations, we propose a new iterative scheme with improved convergence characteristics. The efficiency of the method is validated through theoretical analysis and numerical experiments.

The main objective of this work is to establish convergence results within a uniformly convex Banach space framework and to demonstrate the effectiveness of the proposed method through suitable numerical examples. The proposed iteration is formulated as follows:

$$\begin{cases} w_1 \in \mathscr{M}, \\ w_{n+1} = \Phi((1-\vartheta_n)v_n + \vartheta_n \Phi v_n), \\ v_n = \Phi((1-\sigma_n)u_n + \sigma_n \Phi u_n), \\ u_n = \Phi((1-\rho_n)w_n + \rho_n \Phi w_n), \quad n \in \mathbb{I}^+, \end{cases} \tag{6}$$

In addition, we present several fixed-point results for generalized (α,β)-nonexpansive type-1 mappings, which are based on the work of F. Akutsah and O. K. Narain [2]. Zamfirescu operators constitute an important class of nonlinear mappings that generalize several well-known contractions; the development of efficient iterative methods for approximating their fixed points remains an active area of research. Motivated by this, we introduce a new iterative scheme and analyze its convergence behavior.

2. PRELIMINARIES

In this section, we review some definitions and lemmas that will be useful in the sequel.

In 1967, Ostrowski [24] coined the concept of stability for fixed point iterative schemes and proved that Picard-iterative scheme is stable with respect to contraction. The following definition is due to Ostrowski.

Definition 2.1. *Let $\{w_n\}$ be an approximate sequence of a theoretical sequence $\{v_n\}$ in a convex subset $\mathscr{M}$ of a Banach space $\mathscr{B}$. Then an iterative scheme $w_{n+1} = f(\Phi, w_n)$ for some function f, converging to a fixed point q, is said to be $\Phi-$ stable or stable with respect to Φ, if for $\varepsilon_n = \|s_{n+1} - f(\Phi, s_n)\|$, $n \in \mathbb{I}^+$, we have*

$$\lim_{n\to\infty} \varepsilon_n = 0 \iff \lim_{n\to\infty} s_n = q.$$

Lemma 2.2. [39] *Let $\{a_n\}$ and $\{b_n\}$ be sequences in $\mathbb{R}_+$ satisfying the following inequality*

$$a_{n+1} \le (1-\gamma_n)a_n + b_n,$$

where $\gamma_n \in (0,1)$, for all $n \in \mathbb{I}^+$ with $\sum_{n=0}^{\infty} \gamma_n$. If $\lim_{n\to\infty} \frac{b_n}{\gamma_n} = 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3. [32] *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying the following inequality*

$$s_{n+1} \leq (1-\lambda_n)s_n.$$

If $\lambda_n \in (0,1)$ and $\sum_{n=0}^{\infty} \lambda_n = \infty$, then $\lim_{n\to\infty} s_n = 0$.

Definition 2.4. [40] *A Banach Space $\mathscr{B}$ is said to satisfy the Opial's property if, for each sequence $\{\varkappa_n\} \subset \mathscr{B}$, converges weakly to $\varkappa \in \mathscr{B}$, we have*

$$\limsup_{n\to\infty} \|\varkappa_n - \varkappa\| < \limsup_{n\to\infty} \|\varkappa_n - \nu\|,$$

for all $\nu \in \mathscr{B}$ with $\nu \neq \varkappa$.

Definition 2.5. [8] *Let $\mathscr{B}$ be a UCBS. For every ε in the interval $(0,2]$, there exists a $\delta > 0$ such that for any vectors $\varkappa$ and ν in $\mathscr{B}$, if $\|\varkappa\| \leq 1$, $\|\nu\| \leq 1$, and $\|\varkappa - \nu\| > \varepsilon$, then it follows that $\left\|\frac{\varkappa+\nu}{2}\right\| < (1-\delta)$.*

Definition 2.6. [5, 35] *Let $\mathscr{M}$ be a nonempty closed convex subset of a Banach space $\mathscr{B}$ and $\{\varkappa_n\}$ a bounded sequence in $\mathscr{B}$. Then*

(1) *An asymptotic radius of $\{\varkappa_n\}$ at $\varkappa$ in $\mathscr{M}$ is defined as*

$$r(\varkappa, \{\varkappa_n\}) = \limsup_{n\to\infty} \|\varkappa_n - \varkappa\|,$$

(2) *An asymptotic radius of $\{\varkappa_n\}$ relative to $\mathscr{M}$ is defined as*

$$r(\mathscr{M}, \{\varkappa_n\}) = \inf\{r(\varkappa, \{\varkappa_n\}) : \varkappa \in \mathscr{M}\},$$

(3) *An asymptotic center of $\{\varkappa_n\}$ with respect to $\mathscr{M}$ is defined as*

$$A(\mathscr{M}, \{\varkappa_n\}) = \{\varkappa \in \mathscr{M} : r(\varkappa, \{\varkappa_n\}) = r(\mathscr{M}, \{\varkappa_n\})\}.$$

The set $A(\mathscr{M}, \{\varkappa_n\})$ is singleton.

Definition 2.7. [2] *Let Φ be a self generalized (α, β)-nonexpansive type 1 mapping defined on a nonempty closed convex subset $\mathscr{M}$ of a UCBS $\mathscr{B}$. if there exist $\alpha, \beta, \lambda \in [0,1)$, with $\alpha \leq \beta$ and $\alpha + \beta < 1$ such that for all $\varkappa, \nu \in \mathscr{M}$, $\lambda\|\Phi\varkappa - \varkappa\| \leq \|\varkappa - \nu\|$, then*

$$\|\Phi\varkappa - \Phi\nu\| \leq \alpha\|\nu - \Phi\varkappa\| + \beta\|\varkappa - \Phi\nu\| + (1-(\alpha+\beta))\|\varkappa - \nu\|.$$

Definition 2.8. [33] *A mapping $\Phi : \mathscr{M} \to \mathscr{M}$ is considered to satisfy condition (I) if a continuous nondecreasing function $\zeta : [0,\infty) \to [0,\infty)$ such that $\zeta(0) = 0$ and $\zeta(t) > 0$ for all $t \in (0,\infty)$ such that $\|\varkappa - \Phi\varkappa\| \geq \zeta(d(\varkappa, F(\Phi)))$. for all $\varkappa \in \mathscr{M}$, where $d(\varkappa, F(\Phi))$ represents the distance between $\varkappa$ and $F(\Phi)$.*

Proposition 2.9. [2] *Let $\mathscr{M}$ be a nonempty subset of a Banach space $\mathscr{B}$ and $\Phi : \mathscr{M} \to \mathscr{M}$ be a generalized $(\alpha,\beta)-$ nonexpansive type 1 mapping with $F(\Phi) \neq \emptyset$. Then Φ is quasi-nonexpansive.*

Lemma 2.10. [30] *Suppose $\mathscr{B}$ is a UCBS and $0 < a \leq \alpha_n \leq b < 1$ for all $n \in \mathbb{N}$. $\{u_n\}$ and $\{v_n\}$ be two sequences of $\mathscr{B}$ such that $\limsup_{n\to\infty} \|u_n\| \leq r$, $\limsup_{n\to\infty} \|v_n\| \leq r$ and*

$$\limsup_{n\to\infty} \|\alpha_n u_n + (1-\alpha_n)v_n\| = r$$

hold some $r \geq 0$. Then $\lim_{n\to\infty} \|u_n - v_n\| = 0$.

Lemma 2.11. [2] *Let $\emptyset \neq \mathscr{M} \subset \mathscr{B}$ and $\Phi : \mathscr{M} \to \mathscr{M}$ be a generalized $(\alpha,\beta)-$ nonexpansive type 1 mapping. Then for all $\varkappa, \nu \in \mathscr{M}$,*

$$\|\varkappa - \Phi\nu\| \leq \left(\frac{2+\alpha+\beta}{1-\beta}\right)\|\varkappa - \Phi\varkappa\| + \|\varkappa - \nu\|.$$

Lemma 2.12. [2]*Let Φ be a self generalized (α,β)-nonexpansive type 1 mapping defined on a nonempty closed convex subset $\mathscr{M}$ of a UCBS $\mathscr{B}$ with the Opial property, and $\lambda = \frac{\gamma}{2}$, $\gamma \in [0,1)$. If $\{\varkappa_n\}$ converges weakly to $\varkappa$ and $\lim_{n\to\infty} \|\Phi\varkappa_n - \varkappa_n\| = 0$, then it follows that $\Phi\varkappa = \varkappa$. This implies that the operator $I - \Phi$ is demiclosed at zero, where I denotes the identity on the Banach space $\mathscr{B}$.*

Lemma 2.13. [2] *Let $\emptyset \neq \mathscr{M} \subset \mathscr{B}$ and $\Phi : \mathscr{M} \to \mathscr{M}$ be a generalized (α,β)-nonexpansive type 1 mapping. It follows that $F(\Phi)$ is closed. In addition, if $\mathscr{B}$ is strictly convex and $\mathscr{M}$ is convex, then $F(\Phi)$ is also convex.*

3. Rate of convergence

In this section, we show both analytically and numerically that the proposed iterative scheme (6) converges faster for a Zamfirescu operators.

Theorem 3.1. *Let $\mathscr{M}$ be a nonempty closed and convex subset of a Banach space $\mathscr{B}$ and $\Phi : \mathscr{M} \to \mathscr{M}$, a Zamfirescu operator. Then the sequence $\{w_n\}$ defined by iterative scheme (6) converges strongly to a fixed point of Φ.*

Proof. Since Φ is a Zamfirescu operator, it follows that,

$$\|\Phi w - \Phi v\| \leq \delta\|w - v\| + 2\delta\|\Phi w - w\|,$$

for all $w, v \in \mathscr{M}$, where $0 < \delta < 1$. Thus for all $q \in F(\Phi)$ and $w \in \mathscr{M}$, we have

$$\|\Phi w - q\| \leq \delta\|w - q\|. \tag{7}$$

Now, by iterative scheme (6) and equation (7), we get

$$\begin{aligned} \|u_n - q\| &= \|\Phi((1-\rho_n)w_n + \rho_n\Phi w_n) - q\| \\ &\leq \delta\left(\|(1-\rho_n)w_n + \rho_n\Phi w_n) - q\|\right) \\ &\leq \delta\left((1-\rho_n)\|w_n - q\| + \rho_n\|\Phi w_n - q\|\right) \\ &\leq \delta\left((1-\rho_n)\|w_n - q\| + \delta\rho_n\|w_n - q\|\right) \\ &\leq \delta(1-(1-\delta)\rho_n)\|w_n - q\|. \end{aligned} \tag{8}$$

Using equation (8), we obtain

$$\begin{aligned} \|v_n - q\| &= \Phi\|((1-\sigma_n)u_n + \sigma_n\Phi u_n) - q\| \\ &\leq \delta\left(\|((1-\sigma_n)u_n + \sigma_n\Phi u_n) - q\|\right) \\ &\leq \delta\left((1-\sigma_n)\|u_n - q\| + \sigma_n\|\Phi u_n - q\|\right) \\ &\leq \delta\left((1-\sigma_n)\|u_n - q\| + \delta\sigma_n\|u_n - q\|\right) \\ &\leq \delta^2(1-(1-\delta)\sigma_n)(1-(1-\delta)\rho_n)\|w_n - q\|. \end{aligned} \tag{9}$$

And using equation (9), we obtain

$$\begin{aligned} \|w_{n+1} - q\| &= \|\Phi((1-\vartheta_n)v_n + \vartheta_n\Phi v_n) - q\| \\ &\leq \delta\left((1-\vartheta_n)\|v_n - q\| + \vartheta_n\|\Phi v_n - q\|\right) \\ &\leq \delta(1-(1-\delta)\vartheta_n)\|v_n - q\| \end{aligned} \tag{10}$$

$$\leq \delta^3(1-(1-\delta)\vartheta_n)(1-(1-\delta)\sigma_n)(1-(1-\delta)\rho_n)\|w_n-q\|.$$

Since $0<\delta<1$ and $\{\vartheta_n\},\{\sigma_n\},\{\rho_n\}\in(0,1)$, therefore using the fact that $(1-(1-\delta)\vartheta_n)\leq 1$, $(1-(1-\delta)\sigma_n)\leq 1$ and $(1-(1-\delta)\rho_n)\leq 1$, we have

$$\|w_{n+1}-q\|\leq \delta^3\|w_n-q\|.$$

Inductively, we get

$$\|w_{n+1}-q\|\leq \delta^{3n}\|w_0-q\|. \tag{11}$$

It concludes that $\{w_n\}$ converges strongly to q. This completes the proof. □

Remark 3.2. *Theorem 3.1 guarantees the existence and convergence of the iterative sequence to the unique fixed point of the Zamfirescu operator. This result forms the theoretical foundation for the stability analysis and error estimates established in the subsequent results.*

Theorem 3.3. *Let $\mathscr{M}$ be a nonempty closed convex subset of a Banach space $\mathscr{B}$ and let $\Phi:\mathscr{M}\to\mathscr{M}$ be a Zamfirescu operator having a unique fixed point $q\in F(\Phi)$. Suppose that the sequence $\{w_n\}$ is generated by the iterative scheme (6). Then the iterative scheme (6) is Φ-stable. That is, for any sequence $\{s_n\}\subset\mathscr{M}$ satisfying $\varepsilon_n=\|s_{n+1}-f(\Phi,s_n)\|$, where $f(\Phi,s_n)$ denotes the iteration operator associated with (6), the following statements are equivalent:*

$$\lim_{n\to\infty}\varepsilon_n=0$$

and $\lim_{n\to\infty}s_n=q$. Consequently, the fixed point q is stable with respect to the proposed iterative scheme.

Proof. Let $\{s_n\}$ be an approximate sequence of $\{w_n\}$ in $\mathscr{M}$, the sequence defined by iterative scheme (6) is $w_{n+1}=f(\Phi,w_n)$ and $\varepsilon_n=\|s_{n+1}-f(\Phi,s_n)\|,\quad n\in\mathbb{I}^+$.

Now, we show that $\lim_{n\to\infty}\varepsilon_n=0\Longleftrightarrow\lim_{n\to\infty}s_n=q$.

Let $\lim_{n\to\infty}\varepsilon_n=0$, then by the iterative scheme (6), we have

$$\begin{aligned}\|s_{n+1}-q\| &\leq \|s_{n+1}-f(\Phi,s_n)\|+\|f(\Phi,s_n)-q\| \\ &= \varepsilon_n+\delta^3(1-(1-\delta)\vartheta_n)(1-(1-\delta)\sigma_n)(1-(1-\delta)\rho_n)\|s_n-q\|.\end{aligned} \tag{12}$$

Define $l_n = \|s_n - q\|$, $r_n = (1-\delta)\vartheta_n \in (0,1)$, $m_n = (1-\delta)\sigma_n \in (0,1)$, and $p_n = (1-\delta)\rho_n \in (0,1)$, then

$$w_{n+1} \leq \delta^3(1-r_n)(1-m_n)(1-p_n)w_n + \varepsilon_n.$$

Define $c_n = 1-\delta^3(1-r_n)(1-m_n)(1-p_n)$, so that

$$w_{n+1} \leq (1-c_n)w_n + \varepsilon_n.$$

Since $0 < \delta < 1$, we have

$$c_n = 1-\delta^3(1-r_n)(1-m_n)(1-p_n) \geq 1-\delta^3 > 0.$$

Hence $\{c_n\}$ is bounded away from zero. Therefore,

$$\frac{\varepsilon_n}{c_n} \to 0,$$

whenever $\varepsilon_n \to 0$. Thus by Lemma 2.11, $\lim_{n\to\infty} w_n = 0$ and $\lim_{n\to\infty} s_n = q$. Conversely, let $\lim_{n\to\infty} s_n = q$, we have

$$\begin{aligned} \varepsilon_n &= \|s_{n+1} - f(\Phi, s_n)\| \\ &\leq \|s_{n+1} - q\| + \|f(\Phi, s_n) - q\| \qquad (13) \\ &\leq \|s_{n+1} - q\| + \delta^3(1-(1-\delta)\vartheta_n)(1-(1-\delta)\sigma_n)(1-(1-\delta)\rho_n)\|s_n - q\|. \end{aligned}$$

This implies that $\lim_{n\to\infty} \varepsilon_n = 0$. Hence iterative scheme (6) is Φ-stable. □

Remark 3.4. *Theorem 3.2 guarantees that small computational or approximation errors do not affect the convergence of the proposed iterative scheme. Therefore, the method is numerically robust and reliable for practical implementations.*

Theorem 3.5. *Let $\Phi : \mathscr{M} \to \mathscr{M}$ be a Zamfirescu operator with fixed point q. Assume that $\{w_n\}$, $\{w_{1,n}\}$, $\{w_{2,n}\}$, $\{w_{3,n}\}$, $\{w_{4,n}\}$ and $\{w_{5,n}\}$ are generated by the proposed, Noor, S, Picard-S, Thakur and M iterative schemes, respectively. Then all the considered iterative schemes converge strongly to the fixed point q.*

Proof. In view of equation (11) in Theorem 3.1, we have

$$\|w_{n+1}-q\| \leq \delta^{3n}\|w_0-q\| = a_n, \quad n\in\mathbb{I}^+.$$

Now by iterative scheme (1) and equation (7), we have

$$\begin{aligned}\|u_n-q\| &= \|(1-\sigma_n)w_n+\sigma_n\Phi w_n-q\|\\ &\leq (1-\sigma_n)\|w_n-q\|+\sigma_n\|\Phi w_n-q\|\\ &\leq (1-\sigma_n)\|w_n-q\|+\sigma_n\delta\|w_n-q\|\\ &\leq (1-(1-\delta)\sigma_n)\|w_n-q\|.\end{aligned}$$

Since $\delta\in(0,1)$ and $\sigma_n\in(0,1)$, so by using the fact that $0<1-(1-\delta)\sigma_n\leq 1$, we have

$$\|u_n-q\| \leq \|w_n-q\|. \tag{14}$$

Thus using the equation (14), we obtain that

$$\begin{aligned}\|t_n-q\| &= \|(1-\rho_n)w_n+\rho_n\Phi u_n-q\|\\ &\leq (1-\rho_n)\|w_n-q\|+\rho_n\|\Phi u_n-q\|\\ &\leq (1-\rho_n)\|w_n-q\|+\rho_n\delta\|u_n-q\|\\ &\leq (1-\rho_n)\|w_n-q\|+\rho_n\delta\|w_n-q\|\\ &\leq (1-(1-\delta)\rho_n)\|w_n-q\|.\end{aligned} \tag{15}$$

Thus using the equation (4), (15) and (7), we obtain that

$$\begin{aligned}\|w_{n+1}-q\| &= (1-\vartheta_n)\Phi w_n+\vartheta_n\Phi t_n-q\|\\ &\leq (1-\vartheta_n)\|\Phi w_n-q\|+\vartheta_n\|\Phi t_n-q\|\\ &\leq \delta((1-\vartheta_n)\|w_n-q\|+\vartheta_n\|t_n-q\|)\\ &\leq \delta(1-(1-\delta)\rho_n\vartheta_n)\|w_n-q\|.\end{aligned} \tag{16}$$

Inductively, we get

$$\|w_{n+1}-q\| \leq \delta^n(1-(1-\delta)\rho_n\vartheta_n)^n\|w_0-q\|.$$

Let

$$\|w_{1,n}-q\| \leq \delta^n\|w_{1,0}-q\| = a_{1,n}, \quad n\in\mathbb{I}^+.$$

As proved by Sahu ([29], Theorem 3.6)

$$\|w_{2,n}-q\| \leq \delta^n(1-(1-\delta)\sigma_n\rho_n)^n\|w_{2,0}-q\| = a_{2,n}, \quad n\in\mathbb{I}^+.$$

The following inequality is in ([9], inequality (2.24) of Theorem 3)

$$\|w_{3,n}-q\| \leq \delta^{2n}(1-(1-\delta)\sigma_n\rho_n)^n\|w_{3,0}-q\| = a_{3,n}, \quad n\in\mathbb{I}^+.$$

Now by iterative scheme (4) and equation (7), we have

$$\begin{aligned}\|u_n-q\| &= \|(1-\vartheta_n)w_n+\vartheta_n\Phi w_n-q\|\\ &\leq (1-\vartheta_n)\|w_n-q\|+\vartheta_n\|\Phi w_n-q\|\\ &\leq (1-\vartheta_n)\|w_n-q\|+\delta\vartheta_n\|w_n-q\|\\ &\leq (1-(1-\delta)\vartheta_n)\|w_n-q\|.\end{aligned}$$

Since $\delta\in(0,1)$ and $\vartheta_n\in(0,1)$, so by using the fact that $0<1-(1-\delta)\vartheta_n\leq 1$, we have

$$\|u_n-q\| \leq \|w_n-q\|. \tag{17}$$

Using the equation (17), we get

$$\begin{aligned}\|t_n-q\| &= \|(\Phi((1-\sigma_n)w_n+\sigma_n u_n))-q\|\\ &\leq \delta\left(\|((1-\sigma_n)w_n+\sigma_n u_n))-q\|\right)\\ &\leq \delta\left((1-\sigma_n)\|w_n-q\|+\sigma_n\|u_n-q\|\right)\\ &\leq \delta\left((1-\sigma_n)\|w_n-q\|+\sigma_n\|w_n-q\|\right)\\ &\leq \delta\|w_n-q\|.\end{aligned} \tag{18}$$

Thus using the equation (18), we obtain that

$$\begin{aligned}\|w_{n+1}-q\| &= \|\Phi t_n-q\|\\ &\leq \delta\|t_n-q\| \leq \delta^2\|w_n-q\|.\end{aligned}$$

Inductively, we get

$$\|w_{4,n+1}-q\| \leq \delta^{2n}\|w_0-q\|.$$

Let

$$\|w_{4,n}-q\| \leq \delta^{2n}\|w_{4,0}-q\| = a_{4,n} \quad n\in\mathbb{I}^+.$$

Again using iterative scheme (5) and equation (7), we obtain

$$\begin{aligned}
\|u_n - q\| &= \|(1-\rho_n)w_n + \rho_n \Phi w_n - q\| \\
&\leq (1-\rho_n)\|w_n - q\| + \rho_n \|\Phi w_n - q\| \\
&\leq (1-\rho_n)\|w_n - q\| + \delta\rho_n \|w_n - q\| \\
&\leq (1-(1-\delta)\rho_n)\|w_n - q\|.
\end{aligned}$$

Since $\delta \in (0,1)$ and $\rho_n \in (0,1)$, so by using the fact that $0 < 1-(1-\delta)\rho_n \leq 1$, we have

$$\|u_n - q\| \leq \|w_n - q\|. \tag{19}$$

Using the equation (19), we get

$$\begin{aligned}
\|t_n - q\| &= \|\Phi u_n - q\| \\
&\leq \delta\|u_n - q\| \\
&\leq \delta\|w_n - q\|.
\end{aligned} \tag{20}$$

Thus using the equation (20), we obtain

$$\begin{aligned}
\|w_{n+1} - q\| &= \|\Phi t_n - q\| \\
&\leq \delta\|t_n - q\| \\
&\leq \delta^2\|w_n - q\|.
\end{aligned}$$

Inductively, we get

$$\|w_{5,n+1} - q\| \leq \delta^{2n}\|w_0 - q\|.$$

Let

$$\|w_{5,n} - q\| \leq \delta^{2n}\|w_{5,0} - q\| = a_{5,n}, \quad n \in \mathbb{I}^+.$$

Finally, by Theorem 3.1,

$$\|w_n - q\| \leq (\delta^3)^n\|w_0 - q\|.$$

Since $0 < \delta < 1$, we have

$$\lim_{n\to\infty}(\delta^3)^n = 0.$$

Therefore,

$$0 \leq \lim_{n\to\infty} \|w_n - q\| \leq \lim_{n\to\infty} (\delta^3)^n \|w_0 - q\| = 0.$$

Hence,

$$\lim_{n\to\infty} \|w_n - q\| = 0,$$

which implies that

$$\lim_{n\to\infty} w_n = q.$$

Similarly,

$$\lim_{n\to\infty} \|w_{i,n} - q\| = 0, \qquad i = 1,2,3,4,5.$$

Therefore,

$$\lim_{n\to\infty} w_{i,n} = q, \qquad i = 1,2,3,4,5.$$

Consequently,

$$\lim_{n\to\infty} w_n = \lim_{n\to\infty} w_{i,n} = q, \qquad i = 1,2,3,4,5.$$

Hence all the considered iterative schemes converge strongly to the same fixed point q. $\square$

Remark 3.6. *The above theorem provides a geometric error estimate for the proposed iterative scheme. Numerical comparisons with Noor, S, Picard-S, Thakur and M iterations are presented in Section 5. The results indicate that the proposed method attains the prescribed tolerance using fewer iterations in the considered examples.*

Example 3.7. *Let $\mathscr{B} = \mathbb{R}$ be a Banach space with usual norm and $\mathscr{M} = [0,1] \subset \mathbb{R}$. Let $\Phi : \mathscr{M} \to \mathscr{M}$ be a self map defined by*

$$\Phi w = \sqrt{\frac{w+1}{2}}, \quad \forall w \in \mathscr{M}.$$

It can be verified that Φ is not a contraction while it satisfies the Zamfirescu condition and has a fixed point $w = 1 \in \mathscr{M}$. Now, we choose control sequences

$$\rho_n = \frac{2n}{7n+9}, \quad \vartheta_n = \frac{1}{(n+2)^{1/3}}, \quad \sigma_n = \frac{1}{(n+3)^{1/3}}$$

with the initial guess $w_1 = 0.5$.

TABLE 1. Comparing the convergence behaviour of different iterative schemes

Steps	Noor	S	Picard-S	Thakur et al.	M	Proposed
w_1	0.50000000	0.50000000	0.50000000	0.50000000	0.50000000	0.50000000
w_2	0.59604989	0.88111128	0.97737622	0.97169410	0.97169410	0.99822739
w_3	0.70850232	0.97433841	0.99897352	0.99864480	0.99864480	0.99999374
w_4	0.80378361	0.99462326	0.99995132	0.99993911	0.99993911	1.00000000
w_5	0.87396895	0.99888468	0.99999761	0.99999738	0.99999738	1.00000000
w_6	0.92171649	0.99976937	0.99999988	0.99999989	0.99999989	1.00000000
w_7	0.95257292	0.99995232	0.99999999	1.00000000	1.00000000	1.00000000
w_8	0.97181253	0.99999013	1.00000000	1.00000000	1.00000000	1.00000000
w_9	0.98349835	0.99999795	1.00000000	1.00000000	1.00000000	1.00000000
w_{10}	0.99045610	0.99999957	1.00000000	1.00000000	1.00000000	1.00000000
w_{11}	0.99453474	0.99999991	1.00000000	1.00000000	1.00000000	1.00000000
w_{12}	0.99689605	0.99999998	1.00000000	1.00000000	1.00000000	1.00000000
w_{13}	0.99824932	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{14}	0.99901839	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{15}	0.99945239	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{16}	0.99969584	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{17}	0.99983171	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{18}	0.99990720	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{19}	0.99994897	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{20}	0.99997202	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{21}	0.99998469	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{22}	0.99999164	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{23}	0.99999545	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{24}	0.99999752	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{25}	0.99999865	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{26}	0.99999927	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{27}	0.99999960	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{28}	0.99999979	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{29}	0.99999988	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{30}	0.99999994	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{31}	0.99999997	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{32}	0.99999998	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{33}	0.99999999	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{34}	0.99999999	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{35}	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000

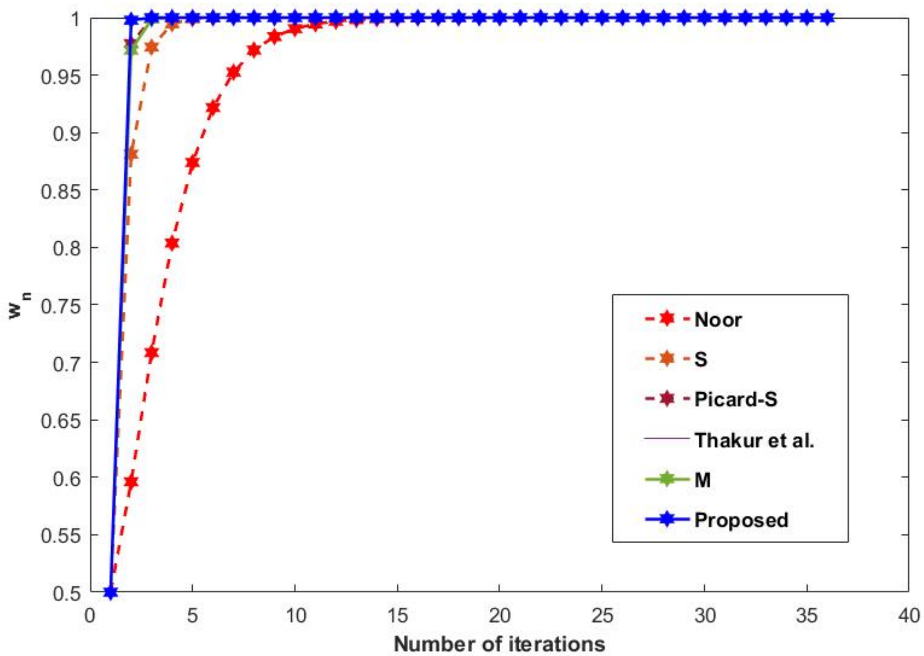

FIGURE 1. Convergence behaviour of Noor, S, Picard-S, Thakur et al., M and proposed iterative schemes towards the fixed point of the mapping Φ.

4. CONVERGENCE RESULTS

In this section, we prove the weak and strong convergence results of the proposed iterative algorithm (6) for the approximation of the fixed points of the class of generalized $(\alpha,\beta)-$nonexpansive type-1 mapping.

Lemma 4.1. *Let Φ be a self generalized $(\alpha,\beta)-$ nonexpansive type 1 mapping defined on a nonempty closed convex subset $\mathscr{M}$ of a UCBS with $F(\Phi)\neq\emptyset$. Suppose that $\{w_n\}$ is defined by (6). Then, for each $w^*\in F(\Phi)$, the limit $\lim_{n\to\infty}\|w_n-w^*\|$ exists.*

Proof. Let $w^*\in F(\Phi)$, using (6) and Proposition 2.9, we obtain

$$\begin{aligned}
\|u_n-w^*\| &= \|\Phi((1-\rho_n)w_n+\rho_n\Phi w_n)-w^*\| \\
&\leq \|(1-\rho_n)w_n+\rho_n\Phi w_n-w^*\| \qquad (21)\\
&\leq (1-\rho_n)\|w_n-w^*\|+\rho_n\|\Phi w_n-w^*\|
\end{aligned}$$

$$\le (1-\rho_n)\|w_n-w^*\|+\rho_n\|w_n-w^*\|$$

$$\le \|w_n-w^*\|.$$

Also, using (6), (21) and Proposition 2.9, we obtain

$$\begin{aligned}\|v_n-w^*\| &= \|\Phi((1-\sigma_n)u_n+\sigma_n\Phi u_n)-w^*\|\\ &\le \|((1-\sigma_n)u_n+\sigma_n\Phi u_n)-w^*\|\\ &\le (1-\sigma_n)\|u_n-w^*\|+\sigma_n\|\Phi u_n-w^*\|\\ &\le (1-\sigma_n)\|u_n-w^*\|+\sigma_n\|u_n-w^*\|\\ &\le \|u_n-w^*\|\\ &\le \|w_n-w^*\|.\end{aligned} \tag{22}$$

Lastly, using (6), (21), (22) and Proposition 2.9, we obtain

$$\begin{aligned}\|w_{n+1}-w^*\| &= \|\Phi((1-\vartheta_n)v_n+\vartheta_n\Phi v_n)-w^*\|\\ &\le \|((1-\vartheta_n)v_n+\vartheta_n\Phi v_n)-w^*\|\\ &\le (1-\vartheta_n)\|v_n-w^*\|+\vartheta_n\|\Phi v_n-w^*\|\\ &\le (1-\vartheta_n)\|v_n-w^*\|+\vartheta_n\|v_n-w^*\|\\ &\le \|v_n-w^*\|\\ &\le \|w_n-w^*\|.\end{aligned} \tag{23}$$

This shows that the sequence $\{\|w_n-w^*\|\}$ is bounded and non-increasing for all $w^*\in F(\Phi)$. Thus, $\{w_n\}$ is bounded and $\lim_{n\to\infty}\|w_n-w^*\|$ exists. □

Theorem 4.2. *Let Φ be a self generalized $(\alpha,\beta)-$ nonexpansive type 1 mapping defined on a nonempty closed convex subset $\mathscr{M}$ of a UCBS. Suppose that $\{w_n\}$ is defined by (6). Then, $F(\Phi)\neq\emptyset$. iff $\{w_n\}$ is bounded and* $\lim_{n\to\infty}\|\Phi w_n-w_n\|=0$.

Proof. Assume that $w^*\in F(\Phi)$. It is derived from Lemma 4.1 that $\{w_n\}$ is bounded and $\lim_{n\to\infty}\|w_n-w^*\|$ exists for all $w^*\in F(\Phi)$.

Let ℓ be a real value,

$$\lim_{n\to\infty}\|u_n-w^*\|\le \ell. \tag{24}$$

From Proposition 2.9, we obtain that $\|\Phi w_n - w^*\| \leq \|w_n - w^*\|$. We have

$$\limsup_{n\to\infty} \|\Phi w_n - w^*\| \leq \ell. \tag{25}$$

From (21) and (23), we have

$$\begin{aligned} \|w_{n+1} - w^*\| &\leq (1-\vartheta_n)\|v_n - w^*\| + \vartheta_n \|v_n - w^*\| \\ &\leq (1-\vartheta_n)\|u_n - w^*\| + \vartheta_n \|w_n - w^*\|. \end{aligned} \tag{26}$$

After rearranging the inequalities and takin the lim inf of both sides, we obtain

$$\ell \leq (1-\vartheta_n) \liminf_{n\to\infty} \|u_n - w^*\| + \vartheta_n \ell,$$

i.e.,

$$\ell \leq \liminf_{n\to\infty} \|u_n - w^*\|. \tag{27}$$

From (6), (24) and (27), we obtain that $\|u_n - w^*\| = \ell$.

$$\begin{aligned} \ell &= \lim_{n\to\infty} \|u_n - w^*\| \\ &= \lim_{n\to\infty} \|\Phi((1-\rho_n)w_n + \rho_n \Phi w_n) - w^*\| \\ &= \lim_{n\to\infty} \|(1-\rho_n)w_n + \rho_n \Phi w_n - w^*\| \\ &= \lim_{n\to\infty} \|(1-\rho_n)(w_n - w^*) + \rho_n(\Phi w_n - w^*)\| \\ &\leq \ell. \end{aligned} \tag{28}$$

Thus,

$$= \lim_{n\to\infty} \|(1-\rho_n)(w_n - w^*) + \rho_n(\Phi w_n - w^*)\| = \ell. \tag{29}$$

Thus, by Lemma 2.10, we have $\lim_{n\to\infty} \|w_n - \Phi w_n\| = 0$.

Conversely, suppose that $\{w_n\}$ is bounded and $\lim_{n\to\infty} \|\Phi w_n - w_n\| = 0$. Let $\varkappa \in A(\mathscr{M}, \{w_n\})$. According to Lemma 2.11, we have

$$
\begin{aligned}
r(\Phi w^*,\{w_n\}) &= \limsup_{n\to\infty}\|w_n-\Phi w^*\| \\
&\leq \left(\frac{2+\alpha+\beta}{1-\beta}\right)\limsup_{n\to\infty}\|\Phi w_n-w_n\|+\limsup_{n\to\infty}\|w_n-w^*\| \\
&= \limsup_{n\to\infty}\|w_n-w^*\| \\
&= r(w^*,\{w_n\}).
\end{aligned}
$$

This implies that $\Phi w^*\in A(\mathscr{M},\{w_n\})$. Given that $\mathscr{B}$ is uniformly convex, then $A(\mathscr{M},\{w_n\})$ has only one element, therefore we obtain $\Phi w^*=w^*$. □

Theorem 4.3. *Let $\Phi,\mathscr{M}$ and $\mathscr{B}$ be as stated in Lemma 4.1. Let $\{w_n\}$ be a sequence defined by (6) and $F(\Phi)\neq\emptyset$. Assume that $\mathscr{B}$ satisfies Opial condition and $I-\Phi$ is demiclosed with respect to zero. Then, $\{w_n\}$ converges weakly to a fixed point of Φ.*

Proof. According to Lemma 4.1, $\lim_{n\to\infty}\|w_n-w^*\|$ exists and $\{w_n\}$ is bounded. As a uniformly convex Banach space, $\mathscr{M}$ can be reflexive. We can find a subsequence, say w_{n_i}, of $\{w_n\}$ that converges weakly in $\mathscr{M}$.

Now, we prove that the sequence $\{w_n\}$ has a unique weak subsequential limit in $F(\Phi)$. Let w^* and ν denote the weak limits of the subsequences $\{w_{n_i}\}$ and $\{w_{n_j}\}$ of $\{w_n\}$, respectively. According to Theorem 4.2, we have $\|w_n-\Phi w_n\|=0$. Therefore it follows by applying Theorem 3.15. [2], we can conclude that $I-\Phi$ is demiclosed with respect to zero; thus, we conclude that $\Phi w^*=w^*$. Following a similar approach, we can demonstrate that $\nu=\Phi\nu$. From Lemma 4.1, we conclude that $\lim_{n\to\infty}\|w_n-\nu\|$ exists. Now, assume that $w^*\neq\nu$; then, according to the Opial condition [40].

$$
\begin{aligned}
\lim_{n\to\infty}\|w_n-w^*\| &= \lim_{i\to\infty}\|w_{n_i}-w^*\| \\
&< \lim_{i\to\infty}\|w_{n_i}-\nu| \\
&= \lim_{n\to\infty}\|w_n-\nu| \\
&= \lim_{j\to\infty}\|w_{n_j}-\nu\| \\
&< \lim_{i\to\infty}\|w_{n_i}-\nu\| \\
&= \lim_{n\to\infty}\|w_n-w^*\|.
\end{aligned}
$$

The statement presents a contradiction, leading to the conclusion that $w^* = \nu$. Consequently, the sequence w_n converges weakly to a fixed point of $F(\Phi)$. This concludes the proof. □

Theorem 4.4. *Let $\Phi, \mathscr{M}$ and $\mathscr{B}$ be as stated in Lemma 4.1. Let $\{w_n\}$ be defined by (6) and $F(\Phi) \neq \emptyset$. Then, $\{w_n\}$ converges strongly to a point of $F(\Phi)$ iff $\lim_{n\to\infty} d(w_n, F(\Phi)) = 0$, where $d(w, F(\Phi)) = \inf\{\|w - w^*\| : w^* \in F(\Phi)\}$.*

Proof. Let w^* be a fixed point of Φ at which the sequence $\{w_n\}$ converges. Thus, we have $\lim_{n\to\infty} d(w_n, w^*) = 0$. Since it holds that $0 \leq d(w_n, F(\Phi)) \leq d(w_n, w^*)$, we can conclude that $\lim_{n\to\infty} d(w_n, F(\Phi)) = 0$. Consequently, it follows that $\liminf_{n\to\infty} d(w_n, F(\Phi)) = 0$.

Conversely, suppose that $\liminf_{n\to\infty} d(w_n, F(\Phi)) = 0$. It follows Lemma 4.1 that $\lim_{n\to\infty} \|w_n - w^*\|$ exists and that $\lim_{n\to\infty} d(w_n, F(\Phi))$ exists for all $w^* \in F(\Phi)$. We suppose that $\liminf_{n\to\infty} d(w_n, F(\Phi)) = 0$. Let $\{w_{n_m}\}$ be any arbitrary subsequence of $\{w_n\}$ and $\{w^*_m\}$ be a sequence in $F(\Phi)$ such that for each $n \in \mathbb{N}$,

$$\|w_{n_m} - w^*_m\| < \frac{1}{2^m},$$

it follows from (23) that $\|w_{n+1} - w^*_m\| \leq \|w_n - w^*_m\| < \frac{1}{2^m}$, hence

$$\begin{aligned}
\|w^*_{m+1} - w^*_m\| &\leq \|w^*_{m+1} - w_{n+1}\| + \|w_{n+1} - w^*_m\| \\
&< \frac{1}{2^{m+1}} + \frac{1}{2^m} \\
&< \frac{1}{2^{m-1}}.
\end{aligned}$$

Therefore, $\{w^*_m\}$ is a cauchy sequence in $F(\Phi)$. In addition, by applying Theorem 3.7. [2], we can conclude that $F(\Phi)$ is closed. Thus, $\{w^*_m\}$ is convergent sequence in $F(\Phi)$. Consider that $\{w^*_m\}$ converges to $w^* \in F(\Phi)$. Since $\|w_{n_m} - w^*\| \leq \|w_{n_m} - w^*_m\| + \|w^*_m - w^*\| \to 0$ as $m \to \infty$, we find that $\lim_{m\to\infty} \|w_{n_m} - w^*\| = 0$ and thus $\{w_{n_m}\}$ converges strongly to $w^* \in F(\Phi)$. Since $\lim_{n\to\infty} \|w_n - w^*\|$ exists, $\{w_n\}$ strongly converges to w^*. □

Theorem 4.5. *Let $\Phi, \mathscr{M}$, and $\mathscr{B}$ be as stated in Lemma 4.1. Let $F(\Phi) \neq \emptyset$, and let $\{w_n\}$ be a sequence defined by the iterative algorithm (6). Then, if the mapping meets condition (I), $\{w_n\}$ converges strongly to a fixed point $w^* \in F(\Phi)$.*

Proof. It is shown in Theorem 4.2 that

$$\lim_{n\to\infty} \|w_n - \Phi w_n\| = 0. \tag{30}$$

By (30) and Definition 2.8, we have

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} \zeta(d(w_n, F(\Phi))) \\ &\leq \lim_{n\to\infty} \|w_n - \Phi w_n\| \\ &= 0. \end{aligned}$$

As a result, we have $\zeta(d(w_n, F(\Phi))) = 0$. Given that the function $\zeta : [0,\infty) \to [0,\infty)$ is nondecreasing, with the properties that $\zeta(0) = 0$ and $\zeta(t) > 0$ for all $t > 0$, we can conclude that

$$\lim_{n\to\infty} d(w_n, F(\Phi)) = 0.$$

Thus, by applying Theorem 4.4, we conclude that the sequence $\{w_n\}$ converges strongly to $w^* \in F(\Phi)$. $\square$

Example 4.6. *Let $\mathscr{B} = \mathbb{R}$ with the usual norm and $\mathscr{M} = [0,1]$ and the mapping $\Phi : \mathscr{M} \to \mathscr{M}$ be defined by*

$$\Phi(w) \begin{cases} \frac{1-w}{2}, & if\ 0 \leq w < \frac{1}{5} \\ \frac{w+7}{8}, & if\ \frac{1}{5} \leq w \leq 1, \end{cases} \tag{31}$$

For any $w \in \mathscr{M}$ take $\alpha_n = 0.3$, $\beta_n = 0.4$ and $\vartheta_n = 0.5$, $\rho_n = 0.4$, $\sigma_n = 0.3$. The fixed point is 1, and initial point $w_1 = 0.5$. Therefore Φ is a generalized $(\alpha,\beta)-$ nonexpansive type 1 mapping.

Next, we will present a numerical comparison of our proposed iterative scheme with several well- known iterative schemes.

TABLE 2. Comparing the convergence behaviour of different iterative schemes

Steps	Noor	S	Picard-S	Thakur et al.	M	Proposed
w_1	0.50000000	0.50000000	0.50000000	0.50000000	0.50000000	0.50000000
w_2	0.73009766	0.94843750	0.99300781	0.99300781	0.99560547	0.99973667
w_3	0.85430545	0.99468262	0.99990222	0.99990222	0.99996138	0.99999986
w_4	0.92135340	0.99945164	0.99999563	0.99999563	0.99999966	1.00000000
w_5	0.95754620	0.99994345	0.99999998	0.99999998	1.00000000	1.00000000
w_6	0.97708324	0.99999417	1.00000000	1.00000000	1.00000000	1.00000000
w_7	0.98762942	0.99999940	1.00000000	1.00000000	1.00000000	1.00000000
w_8	0.99332231	0.99999994	1.00000000	1.00000000	1.00000000	1.00000000
w_9	0.99639535	0.99999999	1.00000000	1.00000000	1.00000000	1.00000000
w_{10}	0.99805419	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{11}	0.99894964	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{12}	0.99943301	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{13}	0.99969394	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{14}	0.99983479	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{15}	0.99991082	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{16}	0.99995186	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{17}	0.99997401	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{18}	0.99998597	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{19}	0.99999243	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{20}	0.99999591	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{21}	0.99999779	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{22}	0.99999881	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{23}	0.99999936	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{24}	0.99999965	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{25}	0.99999981	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{26}	0.99999990	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{27}	0.99999995	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{28}	0.99999997	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{29}	0.99999998	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{30}	0.99999999	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
w_{31}	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000

TABLE 3. The number of iterations needed to get a fixed point

$\vartheta_n = \frac{n}{2n+3}$, $\rho_n = 1 - \frac{1}{\sqrt{n^3+3}}$, $\sigma_n = 1 - \frac{n}{\sqrt{n^3+3}}$

Initial value	Noor	S	Picard-S	Thakur et al.	M	Proposed
0.2	36	10	6	6	6	4
0.5	35	10	6	6	6	4
0.7	34	9	5	5	6	4
0.9	32	9	5	5	5	4

TABLE 4. The number of iterations needed to get a fixed point

$\vartheta_n = \frac{3n}{5n+1}$, $\rho_n = \frac{1}{\sqrt{n^5+30}}$, $\sigma_n = \frac{n}{\sqrt{n^2+100}}$

Initial value	Noor	S	Picard-S	Thakur et al.	M	Proposed
0.2	28	10	6	6	5	4
0.5	27	10	6	6	5	4
0.7	27	10	6	6	5	4
0.9	25	9	6	6	5	4

TABLE 5. The number of iterations needed to get a fixed point

$\vartheta_n = \frac{3n}{4n+5}$, $\rho_n = \frac{1}{\sqrt{n^3+700}}$, $\sigma_n = \frac{n}{\sqrt{n^3+700}}$

Initial value	Noor	S	Picard-S	Thakur et al.	M	Proposed
0.2	23	10	6	6	6	4
0.5	23	10	6	6	5	4
0.7	22	10	6	6	5	4
0.9	21	9	6	6	5	4

5. APPLICATION TO INFECTIOUS DISEASES MODEL

Mathematical models of infectious disease transmission are used to model the spread of disease, and effort to control it, in a population. The models may vary in complexity due to the nature of the disease being modeled, and the nature of the population being affected. Such basic models include the SIR model, SEIR models, SIS model, stochastic model, and the network models in ([1], [15], [31]). Inspired by these models we consider the following nonlinear

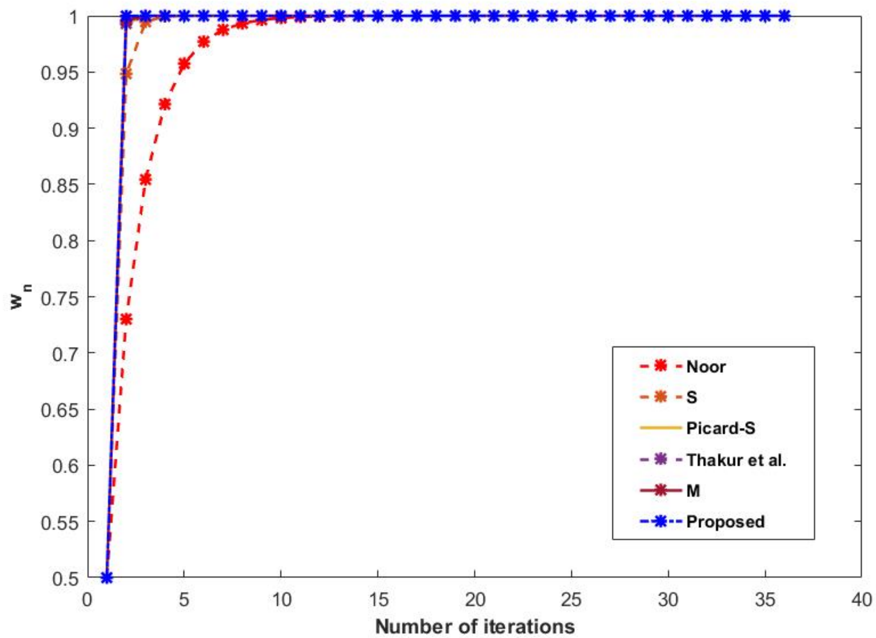

FIGURE 2. Convergence behaviour of Noor, S, Picard-S, Thakur et al., M and proposed iterative schemes towards the fixed point of the mapping Φ.

integral equation:

$$w(t) = \int_{t-\mu}^{t} \beta\left(1 + a\sin p\right) w(p)\left(1 - w(p)\right) dp, \quad t \in [0, \Phi], \tag{32}$$

where $\beta > 0$, $a \in [0,1]$, and $\mu > 0$ represents the infectious period.

Theorem 5.1. *The nonlinear integral equation defined above has a unique solution in $C([0,\Phi])$, provided that*

$$\beta(1+a)\mu < 1.$$

Proof. Let $\mathscr{M} = C([0,\Phi])$ be the Banach space endowed with the supremum norm

$$\|w\| = \sup_{t\in[0,\Phi]} |w(t)|.$$

Define the operator $\Phi : \mathscr{M} \to \mathscr{M}$ by

$$\Phi w(t) = \int_{t-\mu}^{t} \beta\left(1 + a\sin p\right) w(p)\left(1 - w(p)\right) dp. \tag{33}$$

First, we show that Φ maps a bounded subset of $\mathscr{M}$ into itself. Suppose that $w \in \Phi$ satisfies $0 \leq w(t) \leq 1$ for all $t \in [0,\Phi]$. Then

$$|w(p)(1-w(p))| \leq 1,$$

and hence

$$|\Phi w(t)| \leq \int_{t-\mu}^{t} \beta(1+a)\,dp = \beta(1+a)\mu.$$

Thus, Φ maps bounded functions into bounded functions.

Next, we establish the continuity of Φ. Since the function

$$\zeta(p,w) = \beta(1+a\sin p)w(1-w)$$

is continuous in both variables, it follows that the operator Φ is continuous on $\mathscr{M}$. Now, for any $w,v \in \mathscr{M}$, we estimate

$$|(\Phi w)(t)-(\Phi v)(t)| \leq \int_{t-\mu}^{t} \beta(1+a)\,|w(p)-v(p)|\,dp.$$

Taking the supremum over $t \in [0,\Phi]$, we obtain

$$\|\Phi w - \Phi v\| \leq \beta(1+a)\mu\|w-v\|.$$

It follows from the contraction condition $\beta(1+a)\mu < 1$ that the operator Φ admits a unique fixed point. Hence, the nonlinear integral equation has a unique solution. □

Remark 5.2. *The above model incorporates seasonal variation through the term $(1+a\sin p)$ and accounts for the infectious period via the delay parameter μ. The proposed iterative schemes can be employed to approximate the unique solution numerically.*

5.1. Example.

To illustrate the applicability of the theoretical results, we consider the nonlinear integral equation

$$w(t) = \int_{t-\mu}^{t} \beta\big(1+a\sin p\big)\,w(p)\big(1-w(p)\big)\,dp, \tag{34}$$

where $\beta = 1$, $a = 0.5$, and $\mu = 0.5$. We observe that

$$\beta(1+a)\mu = 1\times(1+0.5)\times 0.5 = 0.75 < 1.$$

Thus, the contraction condition of Theorem 5.1 is satisfied.

The parameters are chosen as $\beta = 1, a = 0.5, \mu = 0.5$. It is easy to verify that $\beta(1+a)\mu = 0.75 < 1$, which ensures that the operator is a contraction and guarantees the existence and uniqueness of the solution. For the iterative schemes, we select $\vartheta_n = 0.6, \sigma_n = 0.7, \rho_n = 0.8$. The computations are performed on the interval $[0, 10]$ using a uniform discretization with $N = 300$ grid points. The initial function is taken as $w_0(t) = 0.5$ for all $t \in [0, 10]$.

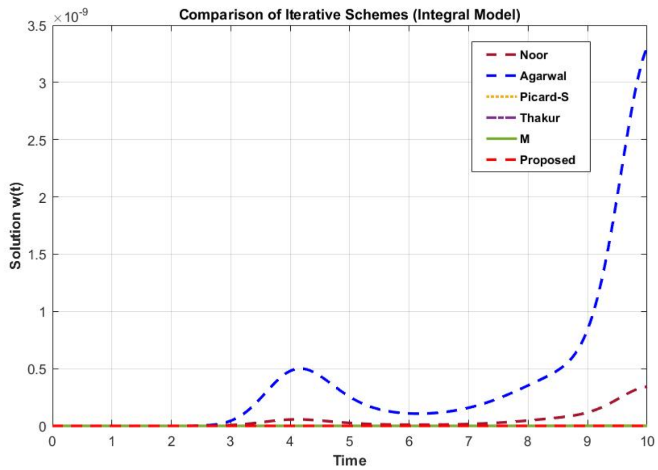

FIGURE 3. Comparison of the proposed iterative scheme with Noor, Agarwal et al., Picard-S, Thakur et al., M iterations for the nonlinear integral equation.

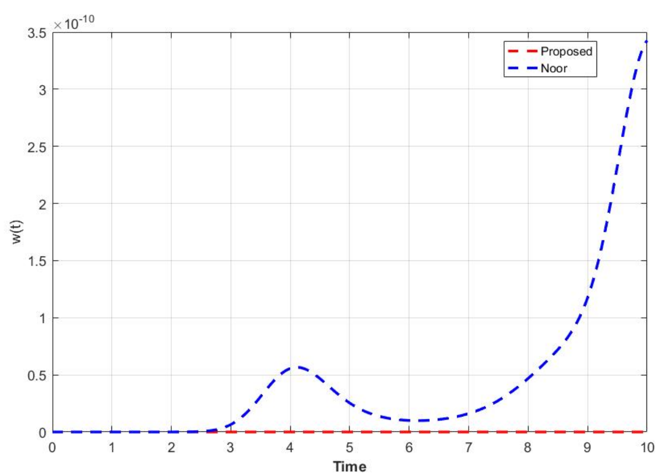

FIGURE 4. Comparison of the proposed iterative scheme with Noor iteration for the nonlinear integral equation.

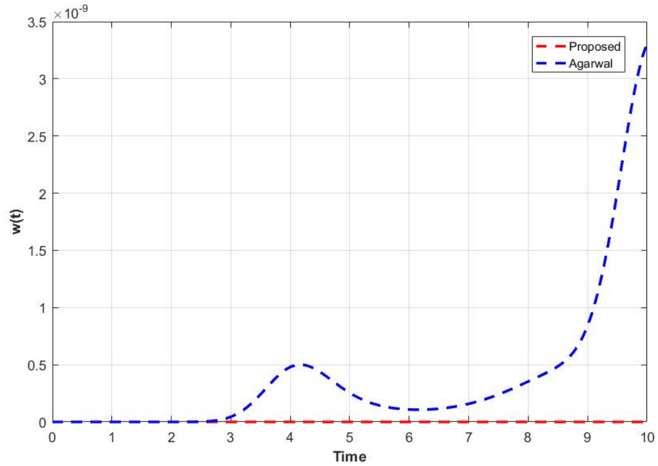

FIGURE 5. Comparison of the proposed iterative scheme with Agarwal et al. iteration for the nonlinear integral equation.

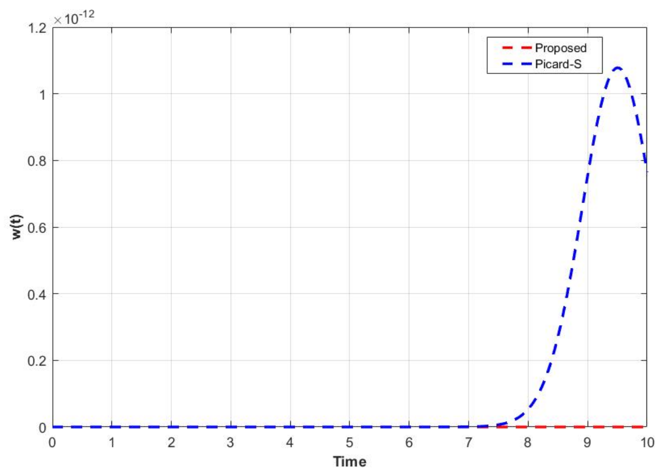

FIGURE 6. Comparison of the proposed iterative scheme with Picard-S iteration for the nonlinear integral equation.

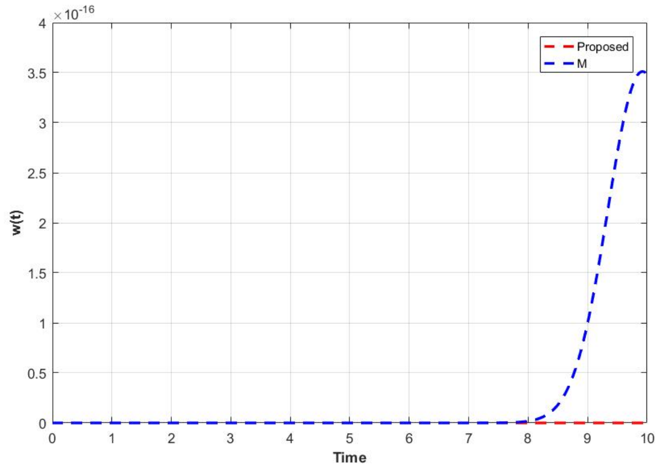

FIGURE 8. Comparison of the proposed iterative scheme with M- iteration for the nonlinear integral equation. The proposed method shows faster and smoother convergence.

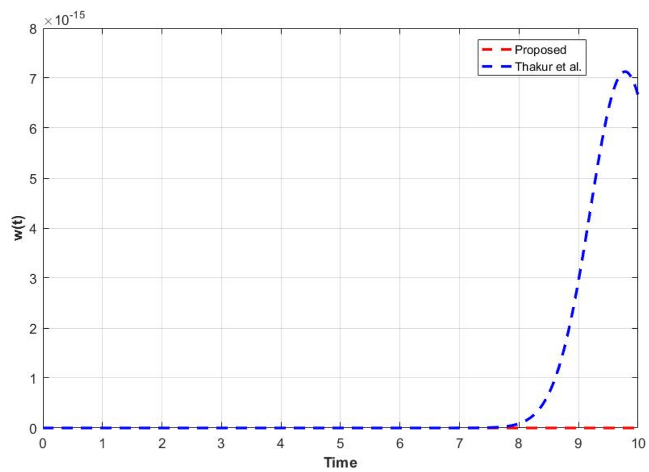

FIGURE 7. Comparison of the proposed iterative scheme with Thakur et al. iteration for the nonlinear integral equation .

5.2. Graphical Interpretation. Figures 3–8 present the graphical comparison between the proposed iterative scheme and the existing methods, namely Noor, Agarwal, Picard-S, Thakur, and M iterations, for solving the considered nonlinear integral equation.

From the figures, it is evident that all iterative schemes attempt to approximate the same solution; however, their convergence behaviors differ significantly. Thus, in every case we considered, we see that the proposed iteration follows a smooth and convergent path towards the solution, in contrast to the Agarwal iteration whose corresponding curve is shown to move unstably with the selected values of the parameters, showing an increase without stabilization, whereas it appears that the Noor method, converges slower than the scheme proposed here. The Picard-S, Thakur, and M algorithms also converge slowly since their curves are relatively flat or the curves only gradually approach the solution. In contrast, the proposed method offers more consistent performance than the other methods and has a faster convergence rate. As these graphics show, the iterative scheme proposed is more efficient and strong than the previously known numerical methods for solving nonlinear integral equations.

6. Conclusion

In this paper, we introduced a new iterative scheme for approximating fixed points of generalized (α,β)-nonexpansive type-1 mappings and Zamfirescu operators in Banach spaces. Strong convergence results were established under appropriate assumptions, and several convergence properties of the proposed method were analyzed theoretically. The efficiency of the proposed iterative scheme was further demonstrated through numerical comparisons with Noor, Agarwal, Picard-S, Thakur, and M iterations. The obtained results indicate that the proposed method exhibits faster and more stable convergence behavior compared to the existing iterative methods. As an application, the developed approach was applied to a nonlinear integral equation arising in infectious disease modeling. The graphical and numerical results confirmed the effectiveness and robustness of the proposed scheme for solving nonlinear problems of practical importance.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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