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THE DUAL INTEGRAL EQUATIONS METHOD INVOLVING HEAT EQUATION WITH MIXED BOUNDARY CONDITIONS

NASER A. HOSHAN

Department of Mathematics, TafilaTechnical University, P.O. Box 179, Tafila-jordan

Abstract. Paper is devoted to determine a solution of a steady state heat conduction equation in axial symmetric cylindrical coordinates subject to a mixed discontinuous boundary conditions of the third kind acted along a surfaces of a solid unbounded plate. The solution of the considered problem is obtained with the use of a Hankel integral transform, dual integral equations method and introduced to a Fredholm integral equation of the second kind.

Key Words: Dual integral equations, mixed boundary conditions.

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1.Introduction

Mixed boundary value problems of mathematical physics equations as well known introduced to discuss dual series or dual integral equation DIE [1-5,9]. In this paper we present an exact solution of two-dimensional heat equation in axially symmetrical cylindrical coordinates with discontinuous mixed boundary conditions of the third kind on the level surface of a semi-infinite solid cylindrical coordinates plate. The solution of the given mixed boundary value problem, is obtained with the use of classical methods (separation of variables, Hankel integral transform) and based on the application of DIE method with Bessel unction as a kernel. The solution of the DIE is introduced to some type of a Fredholm integral equation of the second kind with kernel and free term given in form of improper integral, such types of integral equation can be solved numerically.

Notice that there are several methods for solving DIE, such as substitution method or method of discontinuous integrals, regularization method, infinite series method and other methods[8,9], however, integral transform method is an effect useful tool when a mixed

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boundary conditions of the third kind are given with different types of coordinates systems and different areas of physical and technical of applications.

2. Mathematical formulation of the problem

Suppose that it is required to determine a harmonic function $\theta = \theta(r,z)$, satisfies two dimensional steady state heat conduction equation with no heat generation in axially symmetrical coordinates for a plate of height h, under the following boundary conditions

$$\theta\big|_{r=0} = \theta\big|_{r\to\infty} = \theta\big|_{z\to\infty} = 0 \tag{2.1}$$

Along the boundary the surface $z=0,\ 0<r<R$ of unbounded plate, a linear combination non homogeneous discontinuous mixed boundary condition of the third kind are given

$$\theta_z - H_1\theta = -f_1(r), \qquad 0<r<R, \quad z=0 \tag{2.2}$$

$$\theta_z - H_2\theta = -f_2(r), \qquad R<r<\infty, \quad z=0,\ \theta_z = \partial\theta/\partial z. \tag{2.3}$$

Where $f_1(r), f_2(r)$ given functions. On a surface $z=h, r\ge 0$, a homogeneous unmixed boundary conditions is given

$$\theta_z - H_3\theta = 0, r\ge 0, z=h \quad H_i = \alpha_i/\lambda_i \qquad 0<H_i<\infty, \tag{2.4}$$

where $H_i, i=1,2,3$ is the heat exchange coefficients. The physical significance of (2.2) and (2.3) is that the boundary surface under consideration dissipates heat by convection according to Newton low of cooling , the heat transfer is proportional to temperature difference to surrounding temperature that varies with position, $f_1(r), f_2(r)$ whereas, the physical situation on $z=h, r\ge 0$ is that of a heat dissipation by convection from boundary surface into surrounding at zero temperature. The constants α_i is the heat exchange coefficients , λ_i is the heat conductivity coefficients [10] and R is the radius of a disk(line of discontinuity). On the level surface $z=0, 0<r<R$, inside the disk, a heat exchange function $f_1(r)$ is given different from the heat exchange function $f_2(r)$ which s acted outside the disk $z=0, R<r<\infty$. R is the line of discontinuity. The functions $f_i(r), i=1,2$ have limited variation with any interval , furthermore

$$\int_0^\infty |f_i(r)|\sqrt{r}\,dr < \infty \text{ and } f_i(r) = \frac{1}{2}\{f_i(r+0) + f_i(r-0)\},$$

these restrictions permit to apply Hankel transform with respect to $r, r\in(0,\infty)$ [4]. Separating variables in the heat conduction problem, then using (2.1),the general of the solution of the boundary-value problem is

$$\theta(r,z)=\int_0^\infty A(p)\ J_0(pr)\frac{(p+H_3)\exp(-(2h-z))+(p-H_3)\exp(-pz)}{p-H_3}dp \qquad (2.5)$$

where $J_0(pr)$ is the Bessel function of the first kind of order zero [5] , p is the separation of variables parameter. Applying a mixed boundary condition (2.2) and (2.3) to (2.5) we get the following DIE to determine the unknown function $A(p)$

$$\int_0^\infty A(p)\,J_0(pr)\,G_1(p)\,dp = f_1(r), 0<r<R, \qquad (2.6)$$

$$\int_0^\infty A(p)\,J_0(pr)\,G_1(p)\,dp = f_2(r), \quad R<r<\infty \qquad (2.7)$$

where

$$G_1(p)=\frac{p(p+H_3)\exp(-2hp)-p(p-H_3)-H_1(p+H_3)\exp(-2hp)-H_1(p-H_3)}{p-H_3},$$

$$G_2(p)=\frac{p(p+H_3)\exp(-2hp)-p(p-H_3)-H_2(p+H_3)\exp(-2hp)-H_2(p-H_3)}{p-H_3}.$$

To solve the DIE (2.6) and (2.7), let us to write these equation in standard form

$$\int_0^\infty C(p)\,J_0(pr)\,L(p)\,dp=-f_1(r), \quad 0<r<R, \qquad (2.8)$$

$$\int_0^\infty C(p)\,J_0(pr)\,dp=-f_2(r), \qquad R<r<\infty, \qquad (2.9)$$

where $L(p)=G_1(p)/G_2(p),\ C(p)=A(p)G_2(p),\ \lim_{p\to\infty} L(p)=1,\ M(p)=L(p)-1$.

$$M(p)=\frac{(H_2-H_1)((p-H_3)+\exp(-2hp)(p+H_3))}{p(p+H_3)\exp(-2hp)-p(p-H_3)-H_2(p+H_3)\exp(-2hp)-H_2(p-H_3)},$$

and $\lim_{p\to\infty} M(p)=0$. Equation (2.7) can be written in the following form

$$\int_0^\infty C(p)\,J_0(pr)\,dp=\begin{cases}\phi(r), & 0<r<R,\\ -f_2(r), & R<r<\infty.\end{cases} \qquad (2.10)$$

Where $\phi(r)$ is unknown function defined over $(0,R)$.Applying a Hankel inverse transform to (2.10), to determine the function $C(p)$, we have

$$C(p)=\int_0^R ypJ_0(py)\phi(y)\ dy-\int_R^\infty ypJ_0(py)\ f_2(y)\ dy. \qquad (2.11)$$

Substituting (2.11) into (2.8), and interchanging the order of integration, a second kind Fredholm integral equation is obtained to determine the unknown function $\phi(r)$

$$\phi(r)+\int_0^R \phi(y)\ K(r,y)\ dy=F(r) \qquad ,0<r<R \tag{2.12}$$

with the kernel

$$K(r,y)=\int_0^\infty y\,p\ J_0(py)\ J_0(pr)\ M(p)\ dp \tag{2.13}$$

and the free term

$$F(r)=-\ f_1(r)+\int_R^\infty\int_0^\infty\ y\,p\ f_2(y)\ J_0(pr)J_0(py)\ M(p)\ dp\,dy \tag{2.14}$$

Expressions (2.13) and (2.14) must be satisfied the property [7]

$$\int_0^R |F(r)|\ dr<\infty \quad , \int_0^R\int_0^R K^2(r,y)\,dr\,dy<\infty \tag{2.15}$$

Finally the same procedure should be used for nonsymmetrical cylindrical coordinates involving Bessel function of the first kind $J_\nu(x)$ where $\nu>-1/2$, the DIE (2.8) and (2.9) become

$$\int_0^\infty C(p)J_\nu(pr)L(p)\,dp=-f_1(r), \qquad 0<r<R \tag{2.16}$$

$$\int_0^\infty C(p)J_\nu(pr)\,dp=-f_2(r), \qquad R<r<\infty \tag{2.17}$$

By using the inverse Hankel transform to (2.16), we have

$$C(p)=\int_0^R\ ypJ_\nu(py)\phi(y)\ dy-\int_R^\infty ypJ_\nu(py)\ f_2(y)\ dy \quad . \tag{2.18}$$

Substitution (2.18) into (2.16) ,then interchanging the order of obtaining expression a second kind integral equation is obtained to evaluate $\phi(r)$

$$\phi(r)+\int_0^R \phi(y)\ K(r,y)\ dy=F(r) \qquad ,0<r<R \tag{2.19}$$

where $$K(r,y)=\int_0^\infty y\,p\ J_\nu(py)\ J_\nu(pr)\ M(p)\ dp\,,$$

$$F(r)=-\ f_1(r)+\int_R^\infty\int_0^\infty\ y\,p\ f_2(y)\ J_\nu(pr)J_\nu(py)\ M(p)\ dp\,dy\,.$$

In an integral equation (2.19), the kernel and free term satisfy (2.15).

As $v = \pm 1/2$ the DIE (2.16) and (2.17) contain trigonometric functions instead of Bessel function and the Hankle integral transform replaced by sine or cosine Fourier transforms.

There is an extensive literature on the numerical solution of integral equations (2.12), (2.219), see for example monographs [6,7] .One method on fair generality is to replace the single integral equation by a set of simultaneous algebraic equations; again matrix techniques are invoked. For the non-homogeneous Fredholm integral equation (2.12) and (2.19) this method works well with the help of mathematical software packages such *Mathematica or matlab.* The integral equations (2.12) and (2.19) can be evaluated by quadrature techniques since the kernel $K(r,t)$ is continuous and has continuous derivatives with respect to r, t . The numerical quadrature replaces the integral by a summation

$$\phi(r_i) + \sum_{j=0}^{n} A_j K(r_i, y_j)\phi(r_j) = F(r_j) + R_i \qquad (2.20)$$

with A_i the quadrature coefficients, $R_i = R[K(r_i, y)\psi(y)]$ is the error resulted by replacing integral by series. In expression (2.20) neglecting the error R_i ,in effect we are changing from function description to a vector-matrix description with the n components of the vector $F(r_i)$ define as the value of the function at the nodes $r_i, i = 0,1,....,n$ $, K(r_i, y_j)$ are the values of $K(r, y)$ at $r_i, y_j, i, j = 0,1,....,n$ .The quadrature –matrix technique is applied to the symmetric kernel $K(r, y)$ and the free term $F(r)$ by replacing the integral by a set of simultaneous algebraic equations, we have

$$\phi_i + \sum_{j=0}^{n} A_j K_{ij} \phi_j = F_i \ , i = 0,...,n \qquad (2.21)$$

where $\phi_i = \phi(r_i)$,the values of $F_i = F(r_i)$, $K_{i,j} = K(r_i, y_j)$ for specific values of $H_i, i = 1,2,3,$ R, h and $f_i(r), i = 1,2$ .The numerical values of the heat exchange $H_i, i = 1,2,3$ have physical meaning discussed in details in [10] .Final approximation $\tilde{\phi}(r)$ of the unknown function $\phi(r)$ by using (2.20) and (2.21) is

$$\tilde{\phi}(r) = F(r) - \sum_{j=0}^{n} A_j K(r, r_j)\phi_j$$

where $\phi_j, i = 1,....,n$ the unknowns determined from linear algebraic equations.

Thus, integral transform method can be used to investigate several mixed boundary value problems of mathematical physics equations by using dual integral equation with different areas of applications (heat theory, elasticity theory, electromagnetic and others fields).

Finally the above technique of the boundary value problem and dual integral equations (2.6), (2.7),(2.16),(2.17) can be used

(i) For solving dual integral equations involving sine and cosine Fourier integral transform with the use some known relations between Bessel functions and trigonometric functions [8,9].

(ii) For solving Helmholtz equation in cylindrical coordinates under mixed boundary conditions of the third kind by replacing p by $\sqrt{p^2+k^2}$, k constant , in a function $L(p)$ in (2.8),(2.9).

(iii) For solving non homogeneous Laplace equation and Helmholtz equation with mixed boundary condition first, second, third kind with the use of Green's function.

(iv) For solving triple integral equations with Bessel function as a kernel, by applying the integral transform then converting a given triple equations to a system of dual equations.

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