Available online at http://scik.org

Engineering Mathematics Letters, 2014, 2014:2

ISSN 2049-9337

MATHEMATICAL ANALYSIS OF THE SMALL OSCILLATION OF A HEAVY HETEROGENEOUS VISCOUS LIQUID IN AN OPEN IMMOVABLE CONTAINER

H. ESSAOUINI[1,*], J. ELBAHAOUI[1], L. ELBAKKALI[1] AND P. CAPODANNO[2]

[1]University Abdelmalek Essaâdi, Faculty of Science, M2SM ER28/FS/05 ,93000, Tetuan, Morocco

[2]Université de Franche-Comté, 2B Rue des jardins, 2500 Besançon, France

Abstract : The aim of this paper is the study of the influence of the viscosity on the oscillations of a heterogeneous liquid in a container. Above all, it is proved that the presence of viscosity removes the essential spectrum which appears in the case of a heterogeneous inviscid liquid. From the equations of the system container-liquid, we deduce the variational equation of the problem, and then an operatorial equation in a suitable Hilbert space. The study of the normal oscillations is reduced to the study of an operator bundle whose kind is well known. We obtain an infinity of a periodic damped motions and, for sufficiently small viscosity, a finite number of oscillatory damped motions. The existence and uniqueness of the associated evolution problem are then proved using the weak formulation.

Key words: heterogeneous and viscous liquid; small oscillations; variational method; operator bundle; spectral and evolution problems.

2010 AMS Subject Classifications: 76D03, 76D05, 49R05, 47A75.

---

[*]Corresponding author

Received July 16, 2013

1. Introduction

Studying small oscillations of a container, partially filled by a heavy liquid is a subject of a great interest in engineering for example: construction of tanks, of trucks for the companies of transport of liquids, double-skin construction of ships, etc.

The theoretical results are very important for numerical and experimental calculations of hydroelastic properties and dynamic characteristics of such structures.

The case of an immovable container with homogeneous and viscous heavy liquid has been studied by many authors: [1], [2], [9].

The problem of a heterogeneous ideal heavy liquid was studied at first by Rayleigh by considering the density of the liquid in equilibrium under the form $\rho_0(x_3) = ke^{-\beta x_3}$ [10].

It seems that the problem has been considered next by a limited number of authors [10], [7], [3], [4].

Moreover, Capodanno and his collaborators have studied the problem when the liquid is inviscid and " almost homogeneous”, (i.e its density in the equilibrium position is a linear function of the depth, which a little from a constant) [5], [6]. This problem is more complicated because, in this case, an essential spectrum appeare, in contrast to the case of a homogeneous liquid, where the spectrum is entirely discrete.

The aim of this work is to prove that the presence of the viscosity removes the essential spectrum.

After writing the general equations of motion of the liquid, we linearize the problem assuming small displacements from an equilibrium postion. We reformulate the equations as a variational problem and finally as an operatorial problem involving non bounded linear operators on suitable function spaces.

In this way, we reduce the problem to the study of a well-known operator bundle. There are an infinity of aperiodic damped motions and, for a sufficiently small viscosity, an at most finite number of oscillatory damped motions.

Finally, using the weak formulation, we give an existence and uniqueness theorem for the solution of the evolution problem.

2. Position of the problem

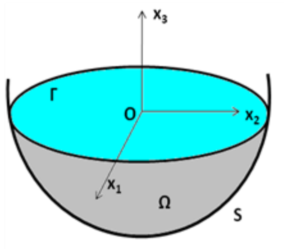

Figure 1

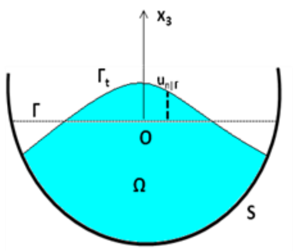

Figure 2

Let assume that a heavy heterogeneous liquid fills partially an immovable container and in the equilibrium state occupies a domain Ω that is bounded by the solid boundary S and the free surface Γ. Γ is a plane that is orthogonal to the acceleration $\vec{\mathbf{g}}$ of the gravitation field. As usual, we choose the system of coordinates $Ox_1x_2x_3$ such that $\vec{\mathbf{g}} = -g\vec{\mathbf{x}}_3$ and its center O is located on the equilibrium surface Γ (**Figure1**).

We are going to study the small oscillations of the liquid about its equilibrium position, obviously in linear theory.

As usual, we are considering that the linearized velocities and accelerations are "true" velocities and accelerations, in order to avoid writing needless formulas in the following calculations.

3. Equations of the motion

We denote with $\vec{\mathbf{u}}(x,t)$ the small displacement of a particle of the liquid which occupies the position x at the instant t from its equilibrium position. $\rho^*(x,t)$, $P^*(x,t)$ are respectively the density and pressure.

The coefficient of viscosity of the liquid at constant temperature μ^* is function of the density ρ^*, we have $\mu^* = \mu^*(\rho^*)$.

The equations of the motion of the liquid can be written in the form:

$$\begin{cases} \rho^* \ddot{u}_i = \dfrac{\partial \Sigma_{ij}\left(\dot{\vec{\mathbf{u}}}, P^*\right)}{\partial x_j} - \rho^* g\, \delta_{i3} & (i = 1,2,3) \\ \operatorname{div}\left(\dot{\vec{\mathbf{u}}}\right) = 0 & \text{in } \Omega \text{ (incompressibility)}, \end{cases} \tag{3.1}$$

where $\Sigma_{ij}\left(\dot{\vec{\mathbf{u}}}, P^*\right)$ are the components of the stress tensor and δ_{ij} indicates the Kronecker's symbol.

We add the laws between the stresses and the velocities of deformation:

$$\Sigma_{ij}\left(\dot{\vec{\mathbf{u}}}, P^*\right) = -P^*\delta_{ij} + 2\mu^*\left(\rho^*\right)\varepsilon_{ij}\left(\dot{\vec{\mathbf{u}}}\right), \tag{3.2}$$

where $\varepsilon_{ij}\left(\dot{\vec{\mathbf{u}}}\right) = \dfrac{1}{2}\left(\dfrac{\partial \dot{u}_i}{\partial x_j} + \dfrac{\partial \dot{u}_j}{\partial x_i}\right)$ are the components of the tensor of the velocities of deformation.

We write the **linearized Navier-Stokes equation of the motion of the liquid** in the following form:

$$\begin{cases} \rho^* \ddot{u}_i = -\dfrac{\partial P^*}{\partial x_j} + \dfrac{\partial}{\partial x_i}\left(2\mu^*\left(\rho^*\right)\varepsilon_{ij}\left(\dot{\vec{\mathbf{u}}}\right)\right) - \rho^* g\, \delta_{i3} & (i = 1,2,3) \\ \operatorname{div}\left(\dot{\vec{\mathbf{u}}}\right) = 0 & \text{in } \Omega \text{ (incompressibility)}, \end{cases} \tag{3.3}$$

We must add the continuity equation

$$\frac{\partial \rho^*}{\partial t} + \operatorname{div}\left(\rho^* \dot{\vec{\mathbf{u}}}\right) = 0 \qquad \text{in } \Omega \tag{3.4}$$

Since $\operatorname{div}\left(\dot{\vec{\mathbf{u}}}\right) = 0$, the equation (3.4) becomes

$$\frac{\partial \rho^*}{\partial t} + \dot{\vec{\mathbf{u}}} \cdot \overrightarrow{\mathbf{grad}}\rho^* = 0 \qquad \text{in } \Omega \tag{3.5}$$

On the other hand, in linear theory, we can integrate the equation $\operatorname{div}\left(\dot{\vec{\mathbf{u}}}\right) = 0$ from the date of the equilibrium position to the instant t , and we obtain

$$\operatorname{div}\left(\vec{\mathbf{u}}\right) = 0 \qquad \text{in } \Omega \tag{3.6}$$

- Now we consider the **boundary conditions:**

i) The no-slip condition at the rigid wall S is:

$$\dot{\vec{\mathbf{u}}}_{|S} = 0;$$

in the same way, we can replace it by

$$\vec{\mathbf{u}}_{|S} = 0; \tag{3.7}$$

In the following we denote by $\vec{\mathbf{n}}$ the unit vector of the external normal to $\partial\Omega$.

ii) In order to write the dynamic conditions, we consider the linearized equation $x_3 = \mathrm{u}_{\mathrm{n}|\Gamma}$ of the free moving surface Γ_t **(Figure2),** then we must have

$$\Sigma_{ij}\left(\dot{\vec{\mathbf{u}}}, P^*\right)\mathrm{n}_{\mathrm{t_j}} = -p_a\,\mathrm{n}_{\mathrm{t_j}} \qquad \text{on} \quad \Gamma_t \tag{3.8}$$

where p_a is the atmospheric pressure, which is assumed to be constant.

Taking into account the equation (2.3), the equation (3.8) becomes

$$\left[-P^*_{\Gamma_t}\delta_{\mathrm{ij}} + 2\mu^*\varepsilon_{\mathrm{ij}}\left(\dot{\vec{\mathbf{u}}}\right)\right]\mathrm{n}_{\mathrm{t_j}} = -p_a\,\mathrm{n}_{\mathrm{t_i}} \quad \text{on} \quad \Gamma_t \tag{3.9}$$

This equation can be be written, since the coefficients of the $\mathrm{n}_{\mathrm{t_j}}$ are small:

$$-\left(P^*_{\Gamma_t} - p_a\right)\mathrm{n}_{\mathrm{i}} + 2\mu^*\varepsilon_{\mathrm{ij}}\left(\dot{\vec{\mathbf{u}}}\right)\mathrm{n}_{\mathrm{j}} = 0 \quad \text{on} \quad \Gamma \tag{3.10}$$

On Γ, we have $\mathrm{n}_1 = \mathrm{n}_2 = 0$, $\mathrm{n}_3 = 1$, and from (3.10), we can deduce

$$\begin{cases} \varepsilon_{13}\left(\dot{\vec{\mathbf{u}}}\right) = 0, \ \varepsilon_{23}\left(\dot{\vec{\mathbf{u}}}\right) = 0 \\ -\left(P^*_{\Gamma_t} - p_a\right) + 2\mu^*\left(\rho^*\right)\varepsilon_{33}\left(\dot{\vec{\mathbf{u}}}\right) = 0 \end{cases} \qquad \text{on} \quad \Gamma \tag{3.11}$$

- We are going **to study the equilibrium of the system:**

If ρ_0 and P_{st} are the density and the pressure of the liquid in the equilibrium position, we have

$$\overrightarrow{\mathbf{grad}} P_{st} = -\rho_0 g\ \vec{\mathbf{x}}_3,$$

so that P_{st} and ρ_0 are functions of x_3, with

$$\frac{\mathrm{d}P_{st}(x_3)}{\mathrm{d}x_3} = -\rho_0\left(x_3\right)g, \tag{3.12}$$

We have

$$P_{st} = -g\int_0^{x_3} \rho_0(w)\,\mathrm{d}w + p_a$$

Setting

$$R_0(x_3) = \int_0^{x_3} \rho_0(w)\,\mathrm{d}w\ , \tag{3.13}$$

we can write

$$P_{st} = -gR_0(x_3) + p_a\,, \tag{3.14}$$

and $R_0(0) = 0\,,\quad R_0'(x_3) = -\rho_0(x_3)$

- Finally the **Navier - Stokes equations** take the form:

$$\rho^* \ddot{u}_i = \frac{\partial \sigma_{ij}}{\partial x_j} - \left(\rho^* - \rho_0(x_3)\right) g\,\delta_{i3} \quad \text{in } \Omega \tag{3.15}$$

where $p(x,t) = P^*(x,t) - P_{st}$, is the dynamic pressure, and $\sigma_{ij} = -p\delta_{ij} + 2\mu^*\varepsilon_{ij}\left(\dot{\vec{\mathbf{u}}}\right)$.

The equations (3.15), (3.5), (3.6), (3.7), (3.11) are the equations of the small motions of the system.

From these equations we want to deduce the variational equation of the problem.

4. Formal variational formulation of the problem

i) For a formal calculation, we introduce the space **of the admissible displacements**:

$$W = \left\{\vec{\mathbf{v}} \,/\, \mathrm{div}\left(\vec{\mathbf{v}}\right) = 0,\ \vec{\mathbf{v}}_{|S} = 0\right\},$$

with $\vec{\mathbf{v}}$ sufficiently smooth. This space will be précised later.

By multiplying the i-th Navier-Stokes equation (3.15) for $\overline{\mathrm{v}}_i$, by adding with respect to i, and by integrating on Ω, we obtain

$$\int_\Omega \rho^* \ddot{\vec{\mathbf{u}}} \cdot \bar{\vec{\mathbf{v}}}\, d\Omega = \int_\Omega \frac{\partial \sigma_{ij}}{\partial x_j} \cdot \overline{\mathrm{v}}_i \,\mathrm{d}\Omega - \int_\Omega \left[\rho^* - \rho_0(x_3)\right] g\, \overline{\mathrm{v}}_3 \,\mathrm{d}\Omega$$

But

$$\int_\Omega \frac{\partial \sigma_{ij}}{\partial x_j}\cdot \overline{v}_i \,d\Omega = \int_\Omega \frac{\partial}{\partial x_j}\left(\sigma_{ij}\cdot \overline{v}_i\right)d\Omega - \int_\Omega \sigma_{ij}\frac{\partial \overline{v}_i}{\partial x_j}\,d\Omega,$$

and the Green formula gives

$$\int_\Omega \frac{\partial}{\partial x_j}\left(\sigma_{ij}\cdot \overline{v}_i\right)d\Omega = \int_{S+\Gamma} \sigma_{ij}\, n_j \cdot \overline{v}_i \,d\sigma$$

From (3.11), we deduce $\sigma_{13} = \sigma_{23} = 0$ on Γ,

on the other hand, we have

$$\begin{aligned}\sigma_{33} &= -p_{|\Gamma} + 2\mu^* \varepsilon_{33|\Gamma}\\ &= -p_{|\Gamma_t} + P^*_{|\Gamma_t} - p_a\\ &= P_{st|\Gamma_t} - p_a\\ &= -gR_0\left(\mathbf{u}_{3|\Gamma}\right)\end{aligned}$$

Taking into account that $\vec{\mathbf{v}}_{|S} = 0$, we get easily

$$\int_\Omega \frac{\partial}{\partial x_j}\left(\sigma_{ij}\cdot \overline{v}_i\right)d\Omega = \int_\Gamma \sigma_{ij}\, n_j \cdot \overline{v}_i \,d\Gamma = -\int_\Gamma gR_0\left(u_{3|\Gamma}\right)\overline{v}_3\,d\Gamma$$

As σ_{ij} are symmetric,

$$\int_\Omega \sigma_{ij}\frac{\partial \overline{v}_i}{\partial x_j}\,d\Omega = \frac{1}{2}\int_\Omega \left(\sigma_{ij}\frac{\partial \overline{v}_i}{\partial x_j} + \sigma_{ji}\frac{\partial \overline{v}_j}{\partial x_i}\right)d\Omega = \int_\Omega \sigma_{ij}\varepsilon_{ij}\left(\overline{\vec{\mathbf{v}}}\right)d\Omega$$

therefore

$$\int_\Omega \sigma_{ij}\frac{\partial \overline{v}_i}{\partial x_j}\,d\Omega = \int_\Omega \left(-p\delta_{ij} + 2\mu^*\varepsilon_{ij}\left(\dot{\vec{\mathbf{u}}}\right)\right)\varepsilon_{ij}\left(\overline{\vec{\mathbf{v}}}\right)d\Omega$$

But

$$\delta_{ij}\varepsilon_{ij}\left(\vec{\mathbf{v}}\right) = \varepsilon_{ii}\left(\vec{\mathbf{v}}\right) = \operatorname{div}\left(\vec{\mathbf{v}}\right) = 0,$$

Finally, we have

$$\int_\Omega \sigma_{ij}\frac{\partial \overline{v}_i}{\partial x_j}\,d\Omega = \int_\Omega 2\mu^*\left(\rho^*\right)\varepsilon_{ij}\left(\dot{\vec{\mathbf{u}}}\right)\varepsilon_{ij}\left(\overline{\vec{\mathbf{v}}}\right)d\Omega$$

Therefore, we obtain the variational equation

$$
\begin{cases}
\int_\Omega \rho^* \ddot{\vec{\mathbf{u}}} \cdot \bar{\vec{\mathbf{v}}}\, \mathrm{d}\Omega + \int_\Omega 2\mu^*\left(\rho^*\right)\varepsilon_{\mathrm{ij}}\left(\dot{\vec{\mathbf{u}}}\right)\varepsilon_{\mathrm{ij}}\left(\bar{\vec{\mathbf{v}}}\right)\mathrm{d}\Omega + \int_\Gamma gR_0\left(\mathrm{u}_{3|\Gamma}\right)\bar{\mathrm{v}}_{3|\Gamma}\,\mathrm{d}\Gamma \\
+\int_\Omega \left[\rho^* - \rho_0\left(x_3\right)\right] g\, \bar{\mathrm{v}}_3\, \mathrm{d}\Omega = 0
\end{cases}
\tag{4.1}
$$

$\forall\ \vec{\mathbf{v}} \in W$.

ii) Reciprocally, we are proving that, from the equation (4.1), we can deduce the equations of motion and the dynamic boundary conditions of the problem.

We take $\vec{\mathbf{v}}(x)$ sufficiently smooth in Ω, such that $\vec{\mathbf{v}}_{|S} = 0$, but not verifying $\operatorname{div}\vec{\mathbf{v}} = 0$.

Then, introducing a multiplier λ_0 associated to the constraint $\operatorname{div}\vec{\mathbf{v}} = 0$, we replace the equation (4.1) by

$$
\begin{cases}
\int_\Omega \rho^* \ddot{\vec{\mathbf{u}}} \cdot \bar{\vec{\mathbf{v}}}\, \mathrm{d}\Omega + \int_\Omega 2\mu^*\left(\rho^*\right)\varepsilon_{\mathrm{ij}}\left(\dot{\vec{\mathbf{u}}}\right)\varepsilon_{\mathrm{ij}}\left(\bar{\vec{\mathbf{v}}}\right)\mathrm{d}\Omega + \int_\Gamma gR_0\left(\mathrm{u}_{3|\Gamma}\right)\bar{\mathrm{v}}_{3|\Gamma}\,\mathrm{d}\Gamma \\
+\int_\Omega \left[\rho^* - \rho_0\left(x_3\right)\right] g\, \bar{\mathrm{v}}_3\, \mathrm{d}\Omega + \int_\Omega \lambda_0 \operatorname{div}\vec{\mathbf{v}}\,\mathrm{d}\Omega = 0
\end{cases}
\tag{4.2}
$$

Setting

$$
\hat{\sigma}_{\mathrm{ij}} = -p\delta_{\mathrm{ij}} + 2\mu^*\varepsilon_{\mathrm{ij}}\left(\dot{\vec{\mathbf{u}}}\right)
$$

Taking into account that $\hat{\sigma}_{\mathrm{ij}}$ are symmetric, we have

$$
\int_\Omega \hat{\sigma}_{\mathrm{ij}} \frac{\partial \bar{\mathrm{v}}_{\mathrm{i}}}{\partial x_{\mathrm{j}}}\,\mathrm{d}\Omega = \int_\Omega \sigma_{\mathrm{ij}}\varepsilon_{\mathrm{ij}}\left(\bar{\vec{\mathbf{v}}}\right)\mathrm{d}\Omega
$$

$$
= \int_\Omega \left(\lambda_0\delta_{\mathrm{ij}} + 2\mu^*\varepsilon_{\mathrm{ij}}\left(\dot{\vec{\mathbf{u}}}\right)\right)\varepsilon_{\mathrm{ij}}\left(\bar{\vec{\mathbf{v}}}\right)\mathrm{d}\Omega
$$

and therefore

$$
\int_\Omega \hat{\sigma}_{\mathrm{ij}} \frac{\partial \bar{\mathrm{v}}_{\mathrm{i}}}{\partial x_{\mathrm{j}}}\,\mathrm{d}\Omega = \int_\Omega \lambda_0 \operatorname{div}\bar{\vec{\mathbf{v}}}\,\mathrm{d}\Omega + \int_\Omega 2\mu^*\varepsilon_{\mathrm{ij}}\left(\dot{\vec{\mathbf{u}}}\right)\varepsilon_{\mathrm{ij}}\left(\bar{\vec{\mathbf{v}}}\right)\mathrm{d}\Omega
$$

Then the equation (4.2) becomes

$$
\begin{cases}
\int_\Omega \rho^* \ddot{\vec{\mathbf{u}}} \cdot \bar{\vec{\mathbf{v}}}\, \mathrm{d}\Omega + \int_\Omega \hat{\sigma}_{\mathrm{ij}} \frac{\partial \bar{\mathrm{v}}_{\mathrm{i}}}{\partial x_{\mathrm{j}}}\,\mathrm{d}\Omega + \int_\Gamma gR_0\left(\mathrm{u}_{3|\Gamma}\right)\delta_{\mathrm{i}3}\,\bar{\mathrm{v}}_{\mathrm{i}}\,\mathrm{d}\Gamma \\
+\int_\Omega \left[\rho^* - \rho_0\left(x_3\right)\right] g\, \delta_{\mathrm{i}3}\, \bar{\mathrm{v}}_{\mathrm{i}}\, \mathrm{d}\Omega = 0
\end{cases}
\tag{4.3}
$$

On the other hand, and by applying the Green formula

$$
\int_\Omega \frac{\partial \hat{\sigma}_{\mathrm{ij}}}{\partial x_{\mathrm{j}}} \cdot \bar{\mathrm{v}}_{\mathrm{i}}\,\mathrm{d}\Omega = \int_\Omega \left[\frac{\partial}{\partial x_{\mathrm{j}}}\left(\hat{\sigma}_{\mathrm{ij}} \cdot \bar{\mathrm{v}}_{\mathrm{i}}\right) - \sigma_{\mathrm{ij}} \frac{\partial \bar{\mathrm{v}}_{\mathrm{i}}}{\partial x_{\mathrm{j}}}\right]\mathrm{d}\Omega
$$

$$= \int_{S+\Gamma} \sigma_{ij} n_j \cdot \bar{v}_i \, d\sigma - \int_{\Omega} \sigma_{ij} \frac{\partial \bar{v}_i}{\partial x_j} d\Omega$$

Taking into account that $\vec{\mathbf{v}}_{|S} = 0$, the equation (4.3) bocomes

$$\int_{\Omega} \left[\rho^* \ddot{u}_i - \frac{\partial \hat{\sigma}_{ij}}{\partial x_j} + \left[\rho^* - \rho_0 (x_3) \right] g \delta_{i3} \right] \bar{v}_i \, d\Omega + \int_{\Gamma} \left[\hat{\sigma}_{ij} n_j + g R_0 \left(u_{3|\Gamma} \right) \delta_{i3} \right] \bar{v}_{i|\Gamma} \, d\Gamma = 0 \quad (4.4)$$

We take $\vec{\mathbf{v}} \in \left[\mathcal{D}(\Omega) \right]^3$ (according to $\vec{\mathbf{v}}_{|S} = 0$), and we have

$$\int_{\Omega} \left[\rho^* \ddot{u}_i - \frac{\partial \hat{\sigma}_{ij}}{\partial x_j} + \left[\rho^* - \rho_0 (x_3) \right] g \delta_{i3} \right] \bar{v}_i \, d\Omega, \quad \forall \vec{\mathbf{v}} \in \left[\mathcal{D}(\Omega) \right]^3$$

therefore

$$\rho^* \ddot{u}_i - \frac{\partial \hat{\sigma}_{ij}}{\partial x_j} + \left[\rho^* - \rho_0 (x_3) \right] g \delta_{i3} = 0, \qquad \text{in} \quad \mathcal{D}'(\Omega); \quad (i,j=1,2,3)$$

Also, we deduce from (4.4)

$$\int_{\Gamma} \left[\hat{\sigma}_{ij} n_j + g R_0 \left(u_{3|\Gamma} \right) \delta_{i3} \right] \bar{v}_{i|\Gamma} \, d\Gamma, \text{ for every admissible } \vec{\mathbf{v}},$$

then for arbitrary $\vec{\mathbf{v}}_{|\Gamma}$.

We obtain

$$\hat{\sigma}_{ij} n_j + g R_0 \left(u_{3|\Gamma} \right) \delta_{i3} = 0 \quad \text{on} \quad \Gamma.$$

Taking into account the definition of $\hat{\sigma}_{ij}$, we have

$$\lambda_0 n_i + 2\mu^* \varepsilon_{ij} \left(\dot{\vec{\mathbf{u}}} \right) n_j + g R_0 \left(u_{3|\Gamma} \right) \delta_{i3} = 0 \quad \text{on} \quad \Gamma$$

Now for i=1, i=2, we have respectively

$$\varepsilon_{13} \left(\dot{\vec{\mathbf{u}}} \right) = 0; \ \varepsilon_{23} \left(\dot{\vec{\mathbf{u}}} \right) = 0 \qquad \text{on} \quad \Gamma$$

On the other hand, for i=3, we obtain

$$\lambda_{0|\Gamma} + 2\mu^* \varepsilon_{32} \left(\dot{\vec{\mathbf{u}}} \right) + g R_0 \left(u_{3|\Gamma} \right) = 0.$$

Setting $\lambda_{0|\Gamma} = -p$ [15], we find the equations of motion and the dynamic boundary conditions.

iii) In order to give the variational equation in the case of the small motion, we set:

$$\rho^* = \rho_0(x_3) + \tilde{\rho}(x,t) + \ldots$$

$\tilde{\rho}$ is the first order with respect to the amplitude of the oscillations and the dots indicate terms of higher order.

The continuity equation is, at the first order

$$\frac{\partial \tilde{\rho}}{\partial t} + \operatorname{div}\left(\rho_0(x_3)\dot{\vec{\mathbf{u}}}\right) = 0$$

or taking into account of $\operatorname{div}\left(\dot{\vec{\mathbf{u}}}\right) = 0$:

$$\frac{\partial \tilde{\rho}}{\partial t} + \dot{\vec{\mathbf{u}}} \cdot \overrightarrow{\mathbf{grad}}\rho_0(x_3) = 0 \quad \text{in}\,\Omega$$

Integrating from the instant of the equilibrium to the instant t, we obtain the linearized continuity equation

$$\tilde{\rho} = -\mathrm{u}_3\,\rho_0'(x_3) \tag{4.5}$$

Since $\varepsilon_{\mathrm{ij}}\left(\dot{\vec{\mathbf{u}}}\right)$ is the first order, we can replace, in linear theory, the term

$\mu^*\left(\rho^*\right)$ by $\mu^*\left(\rho_0(x_3)\right)$.

On the other hand, we have the following equations

$$\begin{aligned} R\left(\mathbf{u}_{3|\Gamma}\right) &= R(0) + \mathbf{u}_{3|\Gamma}R'(0) + \ldots = \mathbf{u}_{3|\Gamma}\rho_0(0) + \ldots; \\ \rho^* - \rho_0(x_3) &= \tilde{\rho}(x,t) + \ldots = -\mathbf{u}_3\rho_0'(x_3) + \ldots \end{aligned}$$

Since $\mathrm{u}_{3|\Gamma} = \mathrm{u}_{\mathrm{n}|\Gamma}$, we deduce the final variational equation of the problem

$$\begin{cases} \int_\Omega \rho_0(x_3)\ddot{\vec{\mathbf{u}}}\cdot\bar{\vec{\mathbf{v}}}\,\mathrm{d}\Omega + \int_\Omega 2\mu^*\left(\rho_0(x_3)\right)\varepsilon_{\mathrm{ij}}\left(\dot{\vec{\mathbf{u}}}\right)\varepsilon_{\mathrm{ij}}\left(\bar{\vec{\mathbf{v}}}\right)\mathrm{d}\Omega \\ +\rho_0(0)g\int_\Gamma \mathrm{u}_{\mathrm{n}|\Gamma}\,\bar{\mathrm{v}}_{\mathrm{n}|\Gamma}\,\mathrm{d}\Gamma + g\int_\Omega -\rho_0'(x_3)\mathrm{u}_3\,\bar{\mathrm{v}}_3\,\mathrm{d}\Omega = 0 \end{cases} \tag{4.6}$$

$\forall \vec{\mathbf{v}} \in W$.

5. Variational formulation of the problem

Theorem 5.1. *The exact variational formulation of the problem is: to find* $\vec{\mathbf{u}} \in V$ such that:

$$\left(\ddot{\vec{\mathbf{u}}},\vec{\mathbf{v}}\right)_H + \left(\dot{\vec{\mathbf{u}}},\vec{\mathbf{v}}\right)_V + \rho_0(0)g\int_\Gamma \mathrm{u}_{\mathrm{n}|\Gamma}\,\bar{\mathrm{v}}_{\mathrm{n}|\Gamma}\,\mathrm{d}\Gamma + g\int_\Omega -\rho_0'(x_3)\mathrm{u}_3\,\bar{\mathrm{v}}_3\,\mathrm{d}\Omega = 0, \quad \forall \vec{\mathbf{v}} \in V \tag{5.1}$$

Proof.

In the following, we shall assume that $\rho_0(x_3)$, $\rho_0'(x_3)$ and $\mu^*(\rho_0(x_3))$ are sufficiently smooth functions.

In this paragraph we are giving the exact variational formulation of the problem. For this reason we introduce the space (of the **admissible displacements of the liquid**):

$$V = \left\{ \vec{\mathbf{u}} \in \Xi^1(\Omega) = \left[H^1(\Omega) \right]^3 ;\ \mathrm{div}\,\vec{\mathbf{u}} = 0,\ \vec{\mathbf{u}}_{|S} = 0 \right\},$$

equipped with the scalar product

$$(\vec{\mathbf{u}}, \vec{\mathbf{v}})_V = \left(\int_\Omega 2\mu^*(\rho_0(x_3)) \varepsilon_{\mathrm{ij}}(\vec{\mathbf{u}}) \varepsilon_{\mathrm{ij}}(\bar{\vec{\mathbf{v}}}) d\Omega \right)^{1/2}$$

Its associated norm $\| \,.\, \|_V$ is equivalent to the classical norm $\| \,.\, \|_1$ of $\Xi^1(\Omega)$ by virtue of Korn inequality and under the hypothesis about $\mu^*(\rho_0(x_3))$.

We denote with H the functional completion of V for the norm associated with the scalar product

$$(\vec{\mathbf{u}}, \vec{\mathbf{v}})_H = \int_\Omega \rho_0(x_3) \vec{\mathbf{u}} \cdot \bar{\vec{\mathbf{v}}} d\Omega$$

It is easy to see that the norme $\|\vec{\mathbf{u}}\|_H$ is equivalent to the classical norm $\|\vec{\mathbf{u}}\|_{\mathbf{L}^2(\Omega)}$ of $\mathbf{L}^2(\Omega) = \left[L^2(\Omega) \right]^3$.

The embedding of V into H is, obviously, dense, continuous and it is compact (by virtue of Rellich theorem).

Then, we can deduce from (4.6) the variational equation (5.1).

6. Operatorial equation

Theorem 6.1. *The operatorial equation is:*

to find $\vec{\mathbf{U}}(\cdot) \in H$ *such that*

$$\mathrm{A}^{-1} \ddot{\vec{\mathbf{U}}} + \dot{\vec{\mathbf{U}}} + \rho_0(0) g\, \mathrm{A}^{1/2}\, \mathrm{T} \gamma_{\mathbf{n}}\, \mathrm{A}^{-1/2} \vec{\mathbf{U}} + g\, \mathrm{A}^{-1/2}\, \mathrm{K}\, \mathrm{A}^{-1/2} \vec{\mathbf{U}} = 0; \qquad (6.3)$$

where all the operators are bounded.

Proof.

In order to deduce an operatorial equation of the problem with bounded operators, we want to substitute the equation (5.1) with an equation on the space H .

- Classically [11], if A is the unbounded operator of H which is associated to the sesquilinear form $(\vec{\mathbf{u}},\vec{\mathbf{v}})_V$ and to the pair (V,H), we have

$$(\vec{\mathbf{u}},\vec{\mathbf{v}})_V=(\mathrm{A}\vec{\mathbf{u}},\vec{\mathbf{v}})_H,\qquad \forall\,\vec{\mathbf{u}}\in D(\mathrm{A})\subset V\subset H,\ \forall\,\vec{\mathbf{v}}\in V \tag{a.6}$$

- For every $X\in\tilde{L}^2(\Gamma)=\left\{w\in L^2(\Gamma),\int_\Gamma w\,\mathrm{d}\Gamma=0\right\}$, we can write

$$\left|\int_\Gamma X\ \bar{\mathrm{v}}_{\mathrm{n}|\Gamma}\,\mathrm{d}\Gamma\right|\le\|X\|_{\tilde{L}^2(\Gamma)}\left\|\mathrm{v}_{\mathrm{n}|\Gamma}\right\|_{\tilde{L}^2(\Gamma)},\quad \forall\vec{\mathbf{v}}\in V$$

By virtue of a trace theorem, we obain

$$\left|\int_\Gamma X\ \bar{\mathrm{v}}_{\mathrm{n}|\Gamma}\,\mathrm{d}\Gamma\right|\le c_1\|X\|_{\tilde{L}^2(\Gamma)}\|\vec{\mathbf{v}}\|_V,\ (c_1>0)$$

so that, by using a Riesz theorem, there exists a bounded operator T from $\tilde{L}^2(\Gamma)$ into V such that:

$$\int_\Gamma X\ \bar{\mathrm{v}}_{\mathrm{n}|\Gamma}\,\mathrm{d}\Gamma=(\mathrm{T}X,\vec{\mathbf{v}})_V,\quad \forall\vec{\mathbf{v}}\in V$$

then, we have

$$\int_\Gamma \mathrm{u}_{\mathrm{n}|\Gamma}\,\bar{\mathrm{v}}_{\mathrm{n}|\Gamma}\,\mathrm{d}\Gamma=\left(\mathrm{T}\,\mathrm{u}_{\mathrm{n}|\Gamma},\vec{\mathbf{v}}\right)_V=(\mathrm{T}\gamma_{\mathrm{n}}\vec{\mathbf{u}},\vec{\mathbf{v}})_V,\ \forall\vec{\mathbf{u}},\ \vec{\mathbf{v}}\in V \tag{b.6}$$

where γ_{n} is the restriction of the application normal trace ($\Xi^1(\Omega)\to L^2(\Gamma)$) to V.

In the same manner, we can write

$$\int_\Omega -\rho_0'(x_3)\,\mathrm{u}_3\,\bar{\mathrm{v}}_3\,\mathrm{d}\Omega=(\mathrm{K}\vec{\mathbf{u}},\vec{\mathbf{v}})_H,\qquad \forall\vec{\mathbf{u}},\ \vec{\mathbf{v}}\in H \tag{c.6}$$

where K is a non negative self-adjoint bounded operator from H into H.

Finally, from (a.6), (b.6), (c.6), we can write the variational equation of the problem:

to find $\vec{\mathbf{u}}(\cdot)\in V$

$$(\ddot{\vec{\mathbf{u}}},\vec{\mathbf{v}})_H+(\mathrm{A}\dot{\vec{\mathbf{u}}},\vec{\mathbf{v}})_H+\rho_0(0)g\,(\mathrm{A}\,\mathrm{T}\gamma_{\mathbf{n}}\vec{\mathbf{u}},\vec{\mathbf{v}})_H+g\,(\mathrm{K}\vec{\mathbf{u}},\vec{\mathbf{v}})_H=0,\qquad \forall\vec{\mathbf{v}}\in V \tag{6.1}$$

- This variational equation is equivalent to the operatorial equation

$$\ddot{\vec{\mathbf{u}}}+\mathrm{A}\left(\dot{\vec{\mathbf{u}}}+\rho_0(0)g\ \mathrm{T}\gamma_{\mathbf{n}}\vec{\mathbf{u}}\right)+g\ \mathrm{K}\vec{\mathbf{u}}=0,\ \ \vec{\mathbf{u}}\in V\ . \tag{6.2}$$

In order to eliminate the unbounded operator, setting $\mathrm{A}^{1/2}\vec{\mathbf{u}}=\vec{\mathbf{U}}\in H$ and applying the operator $\mathrm{A}^{-1/2}$, we obtain a final operatorial equation (6.3) of the problem.

Now, we are going to study the different operators which appears in the precedent operatorial equation (6.3).

i) it is well known that the operator A^{-1} is compact of H in H, self adjoint and positive definite;

ii) We consider the two operators : $\tilde{L}^2(\Gamma) \xleftarrow[\text{compact}]{\gamma_{\mathbf{n}}} V \xleftarrow[\text{bounded}]{\mathrm{A}^{-1/2}} H$;

$$H \xleftarrow[\text{bounded}]{\mathrm{A}^{1/2}} V \xleftarrow[\text{bounded}]{\mathrm{T}} \tilde{L}^2(\Gamma),$$

we have, for every $X \in \tilde{L}^2(\Gamma)$ and $\vec{\mathbf{U}} \in H$:

$$\begin{aligned}\left(\mathrm{A}^{1/2}\,\mathrm{T}X, \vec{\mathbf{U}}\right)_H &= \left(\mathrm{A}^{1/2}\,\mathrm{T}X, \mathrm{A}^{1/2}\,\vec{\mathbf{u}}\right)_H \\ &= \left(\mathrm{T}X, \vec{\mathbf{u}}\right)_V \\ &= \left(X, \mathbf{u}_{\mathbf{n}|\Gamma}\right)_{\tilde{L}^2(\Gamma)} \\ &= \left(X, \gamma_{\mathbf{n}}\vec{\mathbf{u}}\right)_{\tilde{L}^2(\Gamma)} \\ &= \left(X, \gamma_{\mathbf{n}}\,\mathrm{A}^{-1/2}\,\vec{\mathbf{U}}\right)_{\tilde{L}^2(\Gamma)}\end{aligned}$$

Then, the operators $\gamma_{\mathbf{n}}\,\mathrm{A}^{-1/2}$ and $\mathrm{A}^{1/2}\,\mathrm{T}$ are mutually adjoint, $\gamma_{\mathbf{n}}\,\mathrm{A}^{-1/2}$ is compact from H in $\tilde{L}^2(\Gamma)$. By virtue of a Schauder's theorem, $\mathrm{A}^{1/2}\,\mathrm{T}$ is compact too from $\tilde{L}^2(\Gamma)$ into H .

iii) Setting $\mathrm{B} = \mathrm{A}^{1/2}\,\mathrm{T}\gamma_{\mathbf{n}}\,\mathrm{A}^{-1/2}$, it is easy to see that B is a self-adjoint and compact operator from $\tilde{L}^2(\Gamma)$ in $\tilde{L}^2(\Gamma)$.

We have

$$\left(\mathrm{B}\vec{\mathbf{U}}, \vec{\mathbf{U}}\right)_H = \left\|\gamma_{\mathbf{n}}\,\mathrm{A}^{-1/2}\,\vec{\mathbf{U}}\right\|^2_{\tilde{L}^2(\Gamma)} \geq 0$$

From $\left(\mathrm{B}\vec{\mathbf{U}}, \vec{\mathbf{U}}\right)_H = 0$, we deduce $\gamma_{\mathbf{n}}\vec{\mathbf{u}} = \mathrm{u}_{\mathrm{n}|\mathbf{\Gamma}} = 0$, and then $\vec{\mathbf{u}}$ belongs to a space which contains

$$J_0^1(\Omega) = \left\{\vec{\mathbf{u}} \in \Xi^1(\Omega);\ \operatorname{div}\vec{\mathbf{v}} = 0,\ \vec{\mathbf{u}}_{|\partial\Omega} = 0\right\},$$

therefore, B is non negative.

iv) We have, $H \xleftarrow[\text{compact}]{\mathrm{A}^{-1/2}} H \xleftarrow[\text{bounded}]{\mathrm{K}} H \xleftarrow[\text{compact}]{\mathrm{A}^{-1/2}} H$, therefore

$\mathrm{A}^{-1/2}\,\mathrm{K}\,\mathrm{A}^{-1/2}$ is self-adjoint, non negative and compact operator from H into H .

7. Operator bundle of the problem and existence of the eigenvalues

Theorem 7.1. *The problem has a countably eigenvalues* λ *with real part positive admitting* $\lambda = 0$ *e* $\lambda = \infty$ *for points of accumulation.*

There are countably aperiodic motions arbitrary strongly damped (λ_k^+ real $\to \infty$ *) and countably aperiodic motions arbitrary weakly damped (* λ_k^- real $\to 0$ *).*

1) *If* $4\left\|\mathrm{A}^{-1}\right\|\left\|\mathrm{B}_0\right\| < 1$*, all eigenvalues are real and there are* ***no oscillatory motions*** *(it is the case in which* μ *is sufficiently great)*

2) *If* $4\left\|\mathrm{A}^{-1}\right\|\left\|\mathrm{B}_0\right\| \geq 1$*, there is at most a finite number of complex eigenvalues, in the circular ring* $\dfrac{1}{2\left\|\mathrm{A}^{-1}\right\|} \leq |\lambda| \leq 2\left\|\mathrm{B}_0\right\|$*;* ***oscillatory damped motions*** *correspond to these eigenvalues.*

Proof.

We are finding solutions of the equation (6.3) under the form:

$$\vec{\mathbf{U}}(x,t) = e^{-\lambda t}\vec{\mathbf{U}}(x), \qquad \lambda \in \mathbb{C};$$

we obtain

$$\lambda^2 \mathrm{A}^{-1}\vec{\mathbf{U}} - \lambda\vec{\mathbf{U}} + \left[\rho_0(0) g\, \mathrm{B} + g\, \mathrm{A}^{-1/2}\mathrm{K}\mathrm{A}^{-1/2}\right]\vec{\mathbf{U}} = 0 \tag{7.1}$$

From the equation $\left[\rho_0(0) g\, \mathrm{B} + g\, \mathrm{A}^{-1/2}\mathrm{K}\mathrm{A}^{-1/2}\right]\vec{\mathbf{U}} = 0$ and taking into account of the properties of the operators B and K, we deduce

$$\vec{\mathbf{u}} \in V, \quad u_{n|\Gamma} = 0, \quad u_3 = 0,$$

so that $\vec{\mathbf{u}}$ belongs to a space containing $J_0^1(\Omega)$.

Consequently $\lambda = 0$ is an eigenvalue with infinite multiplicity.

Now, we eliminate this case, and setting

$$\mathrm{B}_0 = g\left[\rho_0(0)\mathrm{B} + \mathrm{A}^{-1/2}\mathrm{K}\mathrm{A}^{-1/2}\right]$$

We obtain

$$\mathrm{L}(\lambda)\vec{\mathbf{U}} \equiv \left(\mathrm{I}_H - \lambda\mathrm{A}^{-1} - \lambda^{-1}\mathrm{B}_0\right)\vec{\mathbf{U}} = 0, \qquad \vec{\mathbf{U}} \in H \tag{7.2}$$

The equation (7.2) is of the kind considered by **Askerov, krein, Laptev** [1], i.e

$$f = \lambda \mathrm{P} f + \frac{1}{\lambda} \mathrm{Q} f \ , \qquad f \in H$$

with

$$\mathrm{P} = \mathrm{A}^{-1} \ \text{ and } \ \mathrm{Q} = \mathrm{B}_0$$

It is proved in paragraph 5 that P is self adjoint, compact, positive definite operator and Q is a self adjoint, compact and non negative operator.

We have the theorem 7.1.

8. Conclusions

i) The presence of viscosity ***removes the continuous spectrum*** which appears in the case of a ***heterogeneous inviscid*** liquid.

ii) The problem is reduced to the study of a classical Askerov, Krein, Laptev pencil. The small motions of the system depend on the viscosity coefficient. There are always damped motions, but damped oscillatory motions can appear only for weak viscosity coefficient.

9. Existence and uniqueness theorem

Theorem 9.1. *If the initial data verify:* $\vec{\mathbf{w}}^0 \in V$; $\dot{\vec{\mathbf{w}}}^0 \in H$ *, the problem:*

to find $\vec{\mathbf{w}}(t) \in V$ *such that*

$$\frac{\mathrm{d}}{\mathrm{d}t} c\left(\dot{\vec{\mathbf{w}}}, \vec{\mathbf{v}}\right) + b\left(\dot{\vec{\mathbf{w}}}, \vec{\mathbf{v}}\right) + a\left(\vec{\mathbf{w}}, \vec{\mathbf{v}}\right) = 0 \ , \qquad \forall \vec{\mathbf{v}} \in V$$

has one and only one solution such that

$$\vec{\mathbf{w}} \in \mathrm{L}^2\left(0,T ;V\right) ; \qquad \dot{\vec{\mathbf{w}}} \in \mathrm{L}^2\left(0,T ;V\right) ;$$

where T *is a positive constant.*

Proof.

Setting $\vec{\mathbf{u}} = e^t \vec{\mathbf{w}}$, the variational problem (5.1) is equivalent to:

To find $\vec{\mathbf{w}}(.)\in V$ such that

$$\begin{cases}\left(\ddot{\vec{\mathbf{w}}},\vec{\mathbf{v}}\right)_H+\left(\dot{\vec{\mathbf{w}}},\vec{\mathbf{v}}\right)_V+2\left(\dot{\vec{\mathbf{w}}},\vec{\mathbf{v}}\right)_H \\ +\left(\vec{\mathbf{w}},\vec{\mathbf{v}}\right)_V+\left(\vec{\mathbf{w}},\vec{\mathbf{v}}\right)_H+\rho_0(0)g\left(\mathrm{T}\gamma_n\vec{\mathbf{w}},\vec{\mathbf{v}}\right)_V+g\left(\mathrm{K}\vec{\mathbf{w}},\vec{\mathbf{v}}\right)_H=0\end{cases} \quad \forall\vec{\mathbf{v}}\in V \tag{9.1}$$

We define the sesquilinear forms in $V\times V$ [8,664-665]

$$c\left(\vec{\mathbf{w}},\vec{\mathbf{v}}\right)=\left(\vec{\mathbf{w}},\vec{\mathbf{v}}\right)_H,$$

$$b\left(\vec{\mathbf{w}},\vec{\mathbf{v}}\right)=\left(\vec{\mathbf{w}},\vec{\mathbf{v}}\right)_V+2\left(\vec{\mathbf{w}},\vec{\mathbf{v}}\right)_H,$$

$$a\left(\vec{\mathbf{w}},\vec{\mathbf{v}}\right)=\left(\vec{\mathbf{w}},\vec{\mathbf{v}}\right)_V+\left(\vec{\mathbf{w}},\vec{\mathbf{v}}\right)_H+\rho_0(0)g\left(\mathrm{T}\gamma_n\vec{\mathbf{w}},\vec{\mathbf{v}}\right)_V+g\left(\mathrm{K}\vec{\mathbf{w}},\vec{\mathbf{v}}\right)_H$$

The imbedding $V\subset H$ is continuous, we have:

$\exists c_0>0$; such that $\|\vec{\mathbf{v}}\|_H\le c_0\|\vec{\mathbf{v}}\|_V$. As the trace application is continuous from V into $L^2(\Gamma)$, we get

$$\left|\left(\mathrm{T}\gamma_n\vec{\mathbf{w}},\vec{\mathbf{v}}\right)_V\right|=\left|\left(\mathrm{w}_{\mathrm{n}|_\Gamma},\mathrm{v}_{\mathrm{n}|_\Gamma}\right)_{L^2(\Gamma)}\right|\le\left\|\mathrm{w}_{\mathrm{n}|_\Gamma}\right\|_{L^2(\Gamma)}\cdot\left\|\mathrm{v}_{\mathrm{n}|_\Gamma}\right\|_{L^2(\Gamma)}\le c_1\|\vec{\mathbf{w}}\|_V\cdot\|\vec{\mathbf{v}}\|_V$$

c_1 is a constant >0 ,

then the last sesquilinear forms are continuous in $V\times V$

On the other hand, we have: $\forall\left(\vec{\mathbf{w}},\vec{\mathbf{v}}\right)\in V\times V$

- $t\mapsto a\left(\vec{\mathbf{w}},\vec{\mathbf{v}}\right)$ is a $C^1([0,T])$, $a\left(\vec{\mathbf{w}},\vec{\mathbf{v}}\right)=\overline{a\left(\vec{\mathbf{v}},\vec{\mathbf{w}}\right)}$ and $a\left(\vec{\mathbf{v}},\vec{\mathbf{v}}\right)\ge\|\vec{\mathbf{v}}\|_V^2$,
- $t\mapsto b\left(\vec{\mathbf{w}},\vec{\mathbf{v}}\right)$ is a $C^1([0,T])$, $b\left(\vec{\mathbf{w}},\vec{\mathbf{v}}\right)=\overline{b\left(\vec{\mathbf{v}},\vec{\mathbf{w}}\right)}$ and $b\left(\vec{\mathbf{v}},\vec{\mathbf{v}}\right)\ge\|\vec{\mathbf{v}}\|_V^2$,
- $t\mapsto c\left(\vec{\mathbf{w}},\vec{\mathbf{v}}\right)$ is a $C^1([0,T])$, $c\left(\vec{\mathbf{w}},\vec{\mathbf{v}}\right)=\overline{c\left(\vec{\mathbf{v}},\vec{\mathbf{w}}\right)}$ and $c\left(\vec{\mathbf{v}},\vec{\mathbf{v}}\right)=\|\vec{\mathbf{v}}\|_H^2$.

From (9.1), we deduce the following problem:

to find $\vec{\mathbf{w}}(t)\in V$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}c\left(\dot{\vec{\mathbf{w}}},\vec{\mathbf{v}}\right)+b\left(\dot{\vec{\mathbf{w}}},\vec{\mathbf{v}}\right)+a\left(\vec{\mathbf{w}},\vec{\mathbf{v}}\right)=0,\quad\forall\vec{\mathbf{v}}\in V \tag{9.2}$$

Therefore, by virtue of a known theorem [8, pp 664-670], we have the proof of theorem 9.1.

REFERENCES

[1] N. G. Askerov, S. G. Krein and G. I. Laptev, A class of non selfadjoint boudary value problems, Dokl, Akad, Nauk. SSSR. 155(3) (1964), 499-502.

[2] P. Capodanno, Etude mathématique des petites oscillations de deux liquides pesants non miscibles dans un container, Mechanics Research Communications. 23(1) (1996), 75-80.

[3] P. Capodanno, Un exemple simple de problème non standard de vibration: oscillations d'un liquide hétérogène pesant dans un container, Mechanics Research Communications. 20(3) (1993), 257-262.

[4] P. Capodanno, Piccole oscillazioni piane di un liquido perfetto incompressibile pesante eterogeneo in un recipiente, Lecture in the Rome University TR- CTIT (2001), 10-16.

[5] P. Capodanno and D. Vivona, Mathematical study of the planar oscillations of a heavy almost-homogeneous liquid in a container, Proc.13th Conference on Waves And Stability In Continuous Media WASCOM'07 (Baia samuele, Sicily, Italy, 30 June – 7 July 2007) ed T. Ruggeri et all (Singapore: World Scientific) (2007), 90-95.

[6] H. Essaouini H, L. Elbakkali and P. Capodanno, Analysis of the small oscillations of a heavy almost-homogeneous liquid-gas system, Mechanics Research Communications. **37 (**2010), 257-62.

[7] S. Chandrasekhar, Hydrodynamic and Hydromagnetic stability. Clarendon Press, Oxford, 1961.

[8] R. Dautray and J. L. Lions, Analyse Mathématique et calcul numérique. Vol **8**, Masson, Paris, 1988.

[9] N.D. Kopachevskii and S.G. Krein, Operator approach to linear problems of Hydrodynamics, Vol **2**, Birkhauser, Basel, 2003.

[10] H. Lamb, Hydrodynamics. Cambridge at the University Press, Cambridge, 1932.

[11] J.L. Lions, Equations différentielles opérationnelles et problèmes aux limites. Springer, Berlin, 1961.

[12] N.N. Moiseyev and V.V. Rumyantsev, Dynamic Stability of Bodies Containing Fluid. Springer, Berlin, 1968.

[13] J. Sanchez Hubert and E. Sanchez Palencia, Vibration and coupling of continuous systems, Asymptotic methods. Springer, Berlin, 1989.

[14] R. Temam, Navier-Stokes equation. North Holland, Amsterdam, 1979.

[15] A. Sommerfeld. Mechanik der deformierbaren Medien, Akademische Verlags-gesellschaft. Leipzig, 1964.