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ON A GENERAL SYSTEM OF VARIATIONAL INEQUALITIES

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Abstract. In this paper, we investigate the problem of finding common elements of the set of solutions of a general system of variational inequalities for relaxed cocoercive mappings and of the set of fixed points of a strict pseudo-contraction based on iterative methods. Strong convergence theorems are established.

Keywords: inverse-strongly monotone mapping; nonexpansive mapping; fixed point; variational inequality.

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1. Introduction-Preliminaries

Variational inequalities, which include many important problems in nonlinear analysis and optimization such as the Nash equilibrium problem, complementarity problems, vector optimization problems, fixed point problems, saddle point problems and game theory, recently have been studied as an effective and powerful tool for studying many real world problems which arise in economics, finance, image reconstruction, ecology, transportation, and network; see [1-8] and the references therein.

Let C be a nonempty closed and convex subset of a real Hilbert space H and P_C the metric projection of H onto C. Let $A : C \to H$ be a mapping.

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Recall that A is said to be monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

(2) A is said to be r-strongly monotone if there exists a constant $r > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq r\|x - y\|^2, \quad \forall x, y \in C.$$

(3) A is said to be γ-cocoercive if there exists a constant $\gamma > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \gamma\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

(4) A is said to be relaxed γ-cocoercive if there exists a constant $\gamma > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq (-\gamma)\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

(5) A is said to be relaxed (γ, r)-cocoercive if there exist two constants $\gamma, r > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq (-\gamma)\|Ax - Ay\|^2 + r\|x - y\|^2, \quad \forall x, y \in C.$$

Given nonlinear mappings $A : C \to H$ and $B : C \to H$, consider the problem of finding an $x \in C$ such that

$$\langle x - Bx + \lambda Ax, y - x\rangle \geq 0, \quad \forall y \in C, \tag{1.1}$$

where $\lambda > 0$ is a constant. We see that an element $x \in C$ is a solution to the problem (1.1) if and only if $x \in C$ is a fixed point of the mapping $P_C(B - \lambda A)$. This equivalence plays an important role in this work. If $B = I$, the identity mapping, then the problem (1.1) is reduced to the classical variational inequality problem: find an $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.2}$$

In this work, we use $VI(C, A)$ to denote the solution set of the problem (1.2). For given $z \in H$ and $x \in C$, we see that the following inequality holds $\langle x - z, y - x\rangle \geq 0, \forall y \in C$, if and only if $x = P_C z$. Let $A, B, \widehat{A}, \widehat{B} : C \to H$ be four nonlinear mappings. Next, we consider the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle x^* - By^* + \lambda Ay^*, x - x^*\rangle \geq 0, & \forall x \in C, \\ \langle y^* - \widehat{B}x^* + \widehat{\lambda}\widehat{A}x^*, x - y^*\rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.3}$$

where $\widehat{\lambda} > 0$ and $\mu > 0$ are two constants. (1.3) is said to be a general system of variational inequalities involving four nonlinear mappings.

If $B = \widehat{B} = I$, then the problem (1.3) is reduced to the following.

Find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle x^* - y^* + \lambda A y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\ \langle y^* - x^* + \widehat{\lambda} \widehat{A} x^*, x - y^* \rangle \geq 0, \quad \forall x \in C, \end{cases} \tag{1.4}$$

where $\lambda > 0$ and $\widehat{\lambda} > 0$ are two constants. The problem (1.4) was considered by many authors, see, for example, Chang et al. [9], Cho and Qin [10], Huang and Noor [11], Qin et al. [12] and the references therein.

If $B = \widehat{B} = I$ and $A = \widehat{A}$, then the problem (1.3) is reduced to the following.

Find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle x^* - y^* + \lambda A y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\ \langle y^* - x^* + \lambda A x^*, x - y^* \rangle \geq 0, \quad \forall x \in C, \end{cases} \tag{1.5}$$

where $\lambda > 0$ is a constant. The problem (1.5) was introduced by Verma [13] in 1999.

Further, if we add up the requirement that $x^* = y^*$, then problem (1.3) is reduced to the general variational inequality (1.1).

Recall that a mapping $R : C \to C$ is said to be nonexpansive if

$$\|Rx - Ry\| \leq \|x - y\|, \quad \forall x, y \in C.$$

The mapping $R : C \to C$ is said to be strictly pseudo-contractive if there exists a constant $k \in [0, 1)$ such that

$$\|Rx - Ry\|^2 \leq \|x - y\|^2 + k\|(I - R)x - (I - R)y\|^2, \quad \forall x, y \in C.$$

For such a case, R is said to be a k-strict pseudo-contraction. The mapping $R : C \to C$ is said to be pseudo-contractive if

$$\|Rx - Ry\|^2 \leq \|x - y\|^2 + \|(I - R)x - (I - R)y\|^2, \quad \forall x, y \in C.$$

Clearly, the class of strict pseudo-contractions falls into the one between classes of nonexpansive mappings and pseudo-contractions.

Recently, many authors studied the problem of finding a common element of the set of fixed points of nonexpansive mappings and the set of solutions of the variational inequality problem (1.2) in the framework of real Hilbert space.

In this paper, we investigate the problem of finding common elements of the set of solutions of a general system of variational inequalities for relaxed cocoercive mappings and of the set of fixed points of a strict pseudo-contraction based on iterative methods. Strong convergence theorems are established. In order to prove our main results, we need the following lemmas and definitions.

Lemma 1.1 . *For given $x^* \in C$ and $y^* \in C$, (x^*, y^*) is a solution of the problem (1.3) if and only if x^* is a fixed point of the mapping $T : C \to C$ defined by*

$$Tx = P_C\big(BP_C(\widehat{B}x - \widehat{\lambda}\widehat{A}x) - \lambda AP_C(\widehat{B}x - \widehat{\lambda}\widehat{A}x)\big), \quad \forall x \in C.$$

Lemma 1.2 *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a real Hilbert space H and $\{\beta_n\}$ a sequence in $[0,1]$ with*

$$0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1.$$

Suppose that $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \ge 0$ and

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 1.3 *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \le (1-\gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(b) $\limsup_{n\to\infty} \delta_n/\gamma_n \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 1.4 *Let H be a real Hilbert space, C a nonempty closed and convex subset of H and $R : C \to C$ a nonexpansive mapping. Then $I - R$ is demiclosed at zero.*

The following lemma is a corollary of Bruck's result.

Lemma 1.5. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let R_1 and R_2 be two nonexpansive mappings from C into C with a common fixed point. Define a mapping $R : C \to C$ by*

$$Rx = \rho R_1 x + (1-\rho) R_2 x, \quad \forall x \in C,$$

where ρ is a constant in $(0,1)$. Then R is nonexpansive and $F(R) = F(R_1) \cap F(R_2)$.

Lemma 1.6 *Let C be a nonempty closed and convex subset of a real Hilbert space H and $S : C \to C$ a k-strict pseudo-contraction. Define $R : C \to C$ by $Rx = \alpha x + (1-\alpha)Sx$ for each $x \in C$. Then, as $\alpha \in [k,1)$, R is nonexpansive such that $F(R) = F(S)$.*

2. Main results

Theorem 2.1. *Le $A : C \to H$ be a relaxed (γ_a, r_a)-cocoercive and μ_a-Lipschitz continuous mapping, $\widehat{A} : C \to H$ a relaxed $(\widehat{\gamma}_a, \widehat{r}_a)$-cocoercive and $\widehat{\mu}_a$-Lipschitz continuous mapping, $B : C \to H$ a relaxed (γ_b, r_b)-cocoercive and μ_b-Lipschitz continuous mapping, $\widehat{B} : C \to H$ a relaxed $(\widehat{\gamma}_b, \widehat{r}_b)$-cocoercive and $\widehat{\mu}_b$-Lipschitz continuous mapping and $S : C \to C$ a nonexpansive mapping with a fixed point. Assume that $\mathscr{F} := F(S) \cap F(T) \neq \emptyset$, where T is defined as Lemma 1.1. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_1 = u \in C, \\ z_n = P_C(\widehat{B}x_n - \widehat{\lambda}\widehat{A}x_n), \\ y_n = P_C(Bz_n - \lambda A z_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n[\rho S x_n + (1-\rho) y_n], \quad n \geq 1, \end{cases}$$

where $f : C \to C$ be a contractive mapping, $\rho \in (0,1)$, λ and $\widehat{\lambda}$ are positive constants and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are sequences in $(0,1)$. Assume that the following restrictions are satisfied

(a) $\alpha_n + \beta_n + \gamma_n = 1, \forall n \geq 1;$

(b) $\lim_{n\to\infty}\alpha_n = 0, \sum_{n=1}^{\infty}\alpha_n = \infty$;

(c) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$;

(d) $\sqrt{1-2\lambda r_a+\lambda^2\mu_a^2+2\lambda\gamma_a\mu_a^2}+\sqrt{1-2r_b+\mu_b^2+2\gamma_b\mu_b^2}\le 1$;

(e) $\sqrt{1-2\widehat{\lambda}\widehat{r}_a+\widehat{\lambda}^2\widehat{\mu}_a^2+2\widehat{\lambda}\widehat{\gamma}_a\widehat{\mu}_a^2}+\sqrt{1-2\widehat{r}_b+\widehat{\mu}_b^2+2\widehat{\gamma}_b\widehat{\mu}_b^2}\le 1$.

Then the sequence $\{x_n\}$ converges strongly to $\bar{x} = P_{\mathscr{F}}u$ and $(\bar{x},\bar{y})$ is a solution of the problem (1.3), where $\bar{y} = P_C(\widehat{B}-\widehat{\lambda}\widehat{A})\bar{x}$.

Proof. Letting $x^* \in F(S)\cap F(T)$, we find that $Sx^* = x^* = P_C\big(BP_C(\widehat{B}x^*-\widehat{\lambda}\widehat{A}x^*)-\lambda AP_C(\widehat{B}x^*-\widehat{\lambda}\widehat{A}x^*)\big)$. Putting $y^* = P_C(\widehat{B}-\widehat{\lambda}\widehat{A})x^*$, we obtain that $x^* = P_C(B-\lambda A)y^*$.

Next, we show that the mappings $B-\lambda A$ and $\widehat{B}-\widehat{\lambda}\widehat{A}$ are nonexpansive, respectively. Indeed, for any $x,y \in C$, we have

$$\|(B-\lambda A)x-(B-\lambda A)y\| \le \|(x-y)-(Bx-By)\|+\|(x-y)-\lambda(Ax-Ay)\|. \tag{2.1}$$

Note that

$$\|(x-y)-\lambda(Ax-Ay)\| \le \sqrt{1-2\lambda r_a+\lambda^2\mu_a^2+2\lambda\gamma_a\mu_a^2}\,\|x-y\|. \tag{2.2}$$

We also have

$$\|(x-y)-(Bx-By)\| \le \sqrt{1-2r_b+\mu_b^2+2\gamma_b\mu_b^2}\,\|x-y\|. \tag{2.3}$$

Substituting (2.2) and (2.3) into (2.1) yields that

$$\begin{aligned}&\|(B-\lambda A)x-(B-\lambda A)y\|\\ &\le \left(\sqrt{1-2\lambda r_a+\lambda^2\mu_a^2+2\lambda\gamma_a\mu_a^2}+\sqrt{1-2r_b+\mu_b^2+2\gamma_b\mu_b^2}\right)\|x-y\|.\end{aligned}$$

In view of the condition (d), we obtain that $B-\lambda A$ is nonexpansive, so is $\widehat{B}-\widehat{\lambda}\widehat{A}$. On the other hand, we have

$$\|z_n-y^*\| = \|P_C(\widehat{B}x_n-\widehat{\lambda}\widehat{A}x_n)-P_C(\widehat{B}-\widehat{\lambda}\widehat{A})x^*\| \le \|x_n-x^*\|,$$

from which it follows that $\|y_n-x^*\| \le \|z_n-y^*\| \le \|x_n-x^*\|$. Put $t_n = Sx_n+(1-\rho)y_n, \forall n\ge 1$. It follows that

$$\begin{aligned}\|t_n-x^*\| &\le \rho\|Sx_n-x^*\|+(1-\rho)\|y_n-x^*\|\\ &\le \rho\|x_n-x^*\|+(1-\rho)\|x_n-x^*\|\\ &= \|x_n-x^*\|.\end{aligned}$$

From the algorithm, we arrive at

$$\begin{aligned}\|x_{n+1}-x^*\| &\le \alpha_n\|f(x_n)-x^*\|+\beta_n\|x_n-x^*\|+\gamma_n\|t_n-x^*\|\\ &\le \alpha_n\|u-x^*\|+\beta_n\|x_n-x^*\|+\gamma_n\|x_n-x^*\|\\ &\le (1-\alpha_n(1-\alpha)\|x_n-x^*\|+\alpha_n\alpha\|f(x^*)-x^*\|\end{aligned}$$

which implies that the sequence $\{x_n\}$ is bounded. Consequently, $\{y_n\}$, $\{z_n\}$, $\{w_n\}$ and $\{t_n\}$ are bounded. Notice that

$$\begin{aligned}\|y_{n+1}-y_n\| &= \|P_C(Bz_{n+1}-\lambda Az_{n+1})-P_C(Bz_n-\lambda Az_n)\|\\ &\le \|z_{n+1}-z_n\|\\ &= \|P_C(\widehat{B}x_{n+1}-\widehat{\lambda}\widehat{A}x_{n+1})-P_C(\widehat{B}x_n-\widehat{\lambda}\widehat{A}x_n)\|\\ &\le \|x_{n+1}-x_n\|.\end{aligned}$$

It follows from (2.4) and (2.5) that

$$\begin{aligned}\|t_{n+1}-t_n\| &\le \rho\|Sx_{n+1}-Sx_n\|+(1-\rho)\|y_{n+1}-y_n\|\\ &\le \|x_{n+1}-x_n\|.\end{aligned}\tag{2.4}$$

Next, we show that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.\tag{2.5}$$

Put

$$x_{n+1}=(1-\beta_n)l_n+\beta_n x_n,\quad \forall n\ge 1.\tag{2.6}$$

Now, we compute $l_{n+1}-l_n$. Notice that

$$\|l_{n+1}-l_n\|\le\frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})-t_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|t_n-f(x_n)\|+\|t_{n+1}-t_n\|.\tag{2.7}$$

Substituting (2.4) into (2.7), we arrive at

$$\|l_{n+1}-l_n\|-\|x_{n+1}-x_n\|\le\frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})-t_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|t_n-f(x_n)\|.$$

It follows from the conditions (b) and (c) that

$$\limsup_{n\to\infty}\left(\|l_{n+1}-l_n\|-\|x_{n+1}-x_{n+1}\|\right)\le 0.$$

From Lemma 1.2, we obtain that

$$\lim_{n\to\infty}\|l_n-x_n\|=0.\tag{2.8}$$

Thanks to (2.6), we see that $x_{n+1}-x_n=(1-\beta_n)(l_n-x_n)$, which combines with (2.8), we obtain that (2.5) holds. Note that $x_{n+1}-x_n=\alpha_n(f(x_n)-x_n)+\gamma_n(t_n-x_n)$. It follows from (2.5) and the condition (b) and (c) that

$$\lim_{n\to\infty}\|t_n-x_n\|=0. \tag{2.9}$$

Next, we show that

$$\limsup_{n\to\infty}\langle f(\bar{x})-\bar{x},x_n-\bar{x}\rangle\le 0, \tag{2.10}$$

where $\bar{x}=P_{\mathscr{F}}f(\bar{x})$. To show (2.10), we can choose a sequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(\bar{x})-\bar{x},x_n-\bar{x}\rangle=\lim_{i\to\infty}\langle f(\bar{x})-\bar{x},x_{n_i}-\bar{x}\rangle. \tag{2.11}$$

Since $\{x_{n_i}\}$ is bounded, we see that there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to w. Without loss of generality, we may assume that $x_{n_i}\rightharpoonup w$. Next, we show that $w\in F(T)\cap F(S)$. In fact, define a mapping $J:C\to C$ by

$$Qx=\rho Sx+(1-\rho)P_C(B-\lambda A)P_C(\widehat{B}-\widehat{\lambda}\widehat{A})x,\quad \forall x\in C.$$

From Lemma 1.5, we see that Q is a nonexpansive mapping such that

$$F(Q)=F(R)\cap F(P_C(B-\lambda A)P_C(\widehat{B}-\widehat{\lambda}\widehat{A}))=F(S)\cap F(T).$$

From (2.9), we obtain that

$$\lim_{n\to\infty}\|Qx_n-x_n\|=0. \tag{2.10}$$

It follows from Lemma 1.4 that $w\in F(Q)=F(S)\cap F(T)$. In view of (2.11), we arrive at

$$\limsup_{n\to\infty}\langle f(\bar{x})-\bar{x},x_n-\bar{x}\rangle=\langle f(\bar{x})-\bar{x},w-\bar{x}\rangle\le 0. \tag{2.11}$$

In view of Lemma 1.3, it is not hard to draw the desired conclusion easily. This completes the proof.

Conflict of Interests

The author declares that there is no conflict of interests.

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