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STRONG CONVERGENCE THEOREMS FOR FIXED POINTS OF NONLINEAR MAPPINGS

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Abstract. In this article, we investigate an iteration for nonexpansive-type mappings. Strong convergence theorems are established in the framework of Banach spaces.

Keywords: nonexpansive mapping; fixed point; iteration.

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1. Introduction-Preliminaries

Let E be a smooth Banach space. Consider the functional defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \text{for } x,y \in E. \tag{1.1}$$

Observe that, in a Hilbert space H, (1.1) is reduced to $\phi(x,y) = \|x-y\|^2$, $x,y \in H$. The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x,y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x},x) = \min_{y\in C} \phi(y,x)$$

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existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x,y)$ and strict monotonicity of the mapping J. In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that

$$(\|y\| - \|x\|)^2 \leq \phi(y,x) \leq (\|y\| + \|x\|)^2, \quad \forall x, y \in E.$$

Let E be a Banach space with the dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. Let $U_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided that $\lim\limits_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in U_E$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in U_E$. It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. It is also well known that if E is uniformly smooth if and only if E^* is uniformly convex.

Recall that a Banach space E has the Kadec-Klee property if for any sequence $\{x_n\} \subset E$ and $x \in E$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$ as $n \to \infty$ for more details on Kadec-Klee property. It is well known that if E is a uniformly convex Banach spaces, then E enjoys the Kadec-Klee property.

Let C be a nonempty closed and convex subset of a Banach space E and $T : C \to C$ a mapping. The mapping T is said to be closed if for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, then $Tx_0 = y_0$. A point $x \in C$ is a fixed point of T provided $Tx = x$. In this paper, we use $F(T)$ to denote the fixed point set of T and use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively.

Recall that the mapping T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

It is well known that if C is a nonempty bounded closed and convex subset of a uniformly convex Banach space E, then every nonexpansive self-mapping T on C has a fixed point. Further, the fixed point set of T is closed and convex.

As we all know that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Let C be a nonempty closed convex subset of E and T a mapping from C into itself. A point p in C is said to be an asymptotic fixed point of T [20] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$. A mapping T from C into itself is said to be relatively nonexpansive if $\widetilde{F}(T) = F(T) \neq \emptyset$ and $\phi(p,Tx) \leq \phi(p,x)$ for all $x \in C$ and $p \in F(T)$. The mapping T is said to be quasi-ϕ-nonexpansive if $F(T) \neq \emptyset$ and $\phi(p,Tx) \leq \phi(p,x)$ for all $x \in C$ and $p \in F(T)$.

Recently, fixed point iterations of relatively nonexpansive mappings and quasi-ϕ-nonexpansive mappings have been considered by many authors; see, for example [1-12] and the references therein. In 2005, Matsushita and Takahashi [12] considered fixed point problems of a single relatively nonexpansive mapping in a Banach space. To be more precise, They proved the following theorem:

Theorem MT. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let T be a relatively nonexpansive mapping from C into itself, and let $\{\alpha_n\}$ be a sequence of real numbers such that* $0 \leq \alpha_n < 1$ *and* $\limsup_{n\to\infty} \alpha_n < 1$.

Suppose that $\{x_n\}$ *is given by*

$$\begin{cases} x_0 = x \in C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ H_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x_0, \quad n = 0, 1, 2, \ldots, \end{cases}$$

where J *is the duality mapping on* E. *If* $F(T)$ *is nonempty, then* $\{x_n\}$ *converges strongly to* $P_{F(T)}x$, *where* $P_{F(T)}$ *is the generalized projection from* C *onto* $F(T)$.

In this paper, motivated by Theorem MT, we investigate an iteration for quasi-ϕ-nonexpansive-type mappings. Strong convergence theorems are established in the framework of Banach s-paces.

Lemma 1.1 [1] *Let* C *be a nonempty closed convex subset of a smooth Banach space* E *and* $x \in E$. *Then,* $x_0 = \Pi_C x$ *if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0 \quad \forall y \in C.$$

Lemma 1.2 [1] *Let* E *be a reflexive, strictly convex and smooth Banach space,* C *a nonempty closed convex subset of* E *and* $x \in E$. *Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x) \quad \forall y \in C.$$

Lemma 1.3 [12] *Let* E *be a strictly convex and smooth Banach space,* C *a nonempty closed convex subset of* E *and* $T : C \to C$ *a hemi-relatively nonexpansive mapping. Then* $F(T)$ *is a closed convex subset of* C.

Lemma 1.4 *Let* E *be a uniformly convex Banach space and* $B_r(0)$ *be a closed ball of* E. *Then there exists a continuous strictly increasing convex function* $g : [0, \infty) \to [0, \infty)$ *with* $g(0) = 0$ *such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda \|x\|^2 + \mu \|y\|^2 + \gamma \|z\|^2 - \lambda \mu g(\|x - y\|)$$

for all $x, y, z \in B_r(0)$ *and* $\lambda, \mu, \gamma \in [0,1]$ *with* $\lambda + \mu + \gamma = 1$.

2. Main results

Theorem 2.1. *Let E be a uniformly smooth and uniformly convex Banach space and C a nonempty closed and convex subset of E. Let $T : C \to C$ and $S : C \to C$ be two closed and quasi-ϕ-nonexpansive mappings such that $\mathscr{F} = F(T) \cap F(S)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\beta_{n,0} J x_n + \beta_{n,1} J T x_n + \beta_{n,2} J S x_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \le \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \forall n \ge 0, \end{cases}$$

where $\{\beta_{n,0}\}$, $\{\beta_{n,1}\}$ and $\{\beta_{n,2}\}$ are real sequences in $[0,1]$ satisfying the following restrictions:

(a) $\beta_{n,0} + \beta_{n,1} + \beta_{n,2} = 1$;
(b) $\liminf_{n\to\infty} \beta_{n,0}\beta_{n,1} > 0$ *and* $\liminf_{n\to\infty} \beta_{n,0}\beta_{n,2} > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_{\mathscr{F}} x_0$, where $\Pi_{\mathscr{F}}$ is the generalized projection from E onto $\mathscr{F}$.

Proof. First, we show that C_n is closed and convex for each $n \ge 1$. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_h is closed and convex for some h. For $z \in C_h$, we see that $\phi(z, y_h) \le \phi(z, x_h)$ is equivalent to $2\langle z, Jx_h - Jy_h \rangle \le \|x_h\|^2 - \|y_h\|^2$. It is to see that C_{h+1} is closed and convex. Then, for each $n \ge 1$, C_n is closed and convex. Now, we are in a position to show that $\mathscr{F} \subset C_n$ for each $n \ge 1$. Indeed, $\mathscr{F} \subset C_1 = C$ is obvious. Suppose that $\mathscr{F} \subset C_h$ for some h. Then, for $\forall w \in \mathscr{F} \subset C_h$, we have

$$
\begin{aligned}
\phi(w, y_h) &\leq \|w\|^2 - 2\beta_{h,0}\langle w, Jx_h\rangle - 2\beta_{h,1}\langle w, JTx_h\rangle - 2\beta_{h,2}\langle w, JSx_h\rangle \\
&\quad + \beta_{h,0}\|x_h\|^2 + \beta_{h,1}\|Tx_h\|^2 + \beta_{h,2}\|sx_h\|^2 \\
&= \beta_{h,0}\phi(w, x_h) + \beta_{h,1}\phi(w, Tx_h) + \beta_{h,2}\phi(w, Sx_h) \\
&\leq \beta_{h,0}\phi(w, x_h) + \beta_{h,1}\phi(w, x_h) + \beta_{h,2}\phi(w, x_h) \\
&= \phi(w, x_h).
\end{aligned} \tag{2.1}
$$

which shows that $w \in C_{h+1}$. This implies that $\mathscr{F} \subset C_n$ for each $n \geq 1$. On the other hand, we obtain from Lemma 1.2 that $\phi(x_n, x_0) = \phi(\Pi_{C_n} x_0, x_0) \leq \phi(w, x_0) - \phi(w, x_n) \leq \phi(w, x_0)$, for each $w \in \mathscr{F} \subset C_n$ and for each $n \geq 1$. This shows that the sequence $\phi(x_n, x_0)$ is bounded. We see that the sequence $\{x_n\}$ is also bounded. Since the space is reflexive, we may, without loss of generality, assume that $x_n \rightharpoonup \bar{x}$. Note that C_n is closed and convex for each $n \geq 1$. It is easy to see that $\bar{x} \in C_n$ for each $n \geq 1$. Note that $\phi(x_n, x_0) \leq \phi(\bar{x}, x_0)$. It follows that $\phi(\bar{x}, x_0) \leq \liminf_{n\to\infty} \phi(x_n, x_0) \leq \limsup_{n\to\infty} \phi(x_n, x_0) \leq \phi(\bar{x}, x_0)$. This implies that $\lim_{n\to\infty} \phi(x_n, x_0) = \phi(\bar{x}, x_0)$. Hence, we have $\|x_n\| \to \|\bar{x}\|$ as $n \to \infty$. In view of the Kadec-Klee property of E, we obtain that $x_n \to \bar{x}$ as $n \to \infty$.

Next, we show that $\bar{x} \in F(T)$. By the construction of C_n, we have that $C_{n+1} \subset C_n$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_n$. It follows that $\phi(x_{n+1}, x_n) \leq \phi(x_{n+1}, x_0) - \phi(x_n, x_0)$. Letting $n \to \infty$, we obtain that $\phi(x_{n+1}, x_n) \to 0$. In view of $x_{n+1} \in C_{n+1}$, we arrive at $\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n)$. It follows that $\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0$. It follows that $\lim_{n\to\infty} \|x_n - y_n\| = 0$. Since J is uniformly norm-to-norm continuous on any bounded sets, we have $\lim_{n\to\infty} \|Jx_n - Jy_n\| = 0$. Let

$r = \max\{\sup_{n\geq 1}\{\|x_n\|\}, \sup_{n\geq 1}\{\|Tx_n\|\}, \sup_{n\geq 1}\{\|Sx_n\|\}\}$. Fixing $q \in \mathscr{F}$, we have from Lemma 1.6 that

$$\begin{aligned}
\phi(q,y_n) &= \phi\big(q, J^{-1}(\beta_{n,0}Jx_n + \beta_{n,1}JTx_n + \beta_{n,2}JSx_n)\big) \\
&= \|q\|^2 - 2\langle q, \beta_{n,0}Jx_n + \beta_{n,1}JTx_n + \beta_{n,2}JSx_n)\rangle \\
&\quad + \|\beta_{n,0}Jx_n + \beta_{n,1}JTx_n + \beta_{n,2}JSx_n\|^2 \\
&\leq \|q\|^2 - 2\beta_{n,0}\langle q, Jx_n\rangle - 2\beta_{n,1}\langle q, JTx_n\rangle - 2\beta_{n,2}\langle q, JSx_n\rangle \\
&\quad + \beta_{n,0}\|Jx_n\|^2 + \beta_{n,1}\|JTx_n\|^2 + \beta_{n,2}\|JSx_n\|^2 - \beta_{n,0}\beta_{n,1}g(\|Jx_n - JTx_n\|) \\
&= \beta_{n,0}\phi(q,x_n) + \beta_{n,1}\phi(q,Tx_n) + \beta_{n,2}\phi(q,Sx_n) - \beta_{n,0}\beta_{n,1}g(\|Jx_n - JTx_n\|) \\
&\leq \beta_{n,0}\phi(q,x_n) + \beta_{n,1}\phi(q,x_n) + \beta_{n,2}\phi(q,x_n) - \beta_{n,0}\beta_{n,1}g(\|Jx_n - JTx_n\|) \\
&= \phi(q,x_n) - \beta_{n,0}\beta_{n,1}g(\|Jx_n - JTx_n\|).
\end{aligned}$$

It follows that $\beta_{n,0}\beta_{n,1}g(\|Jx_n - JTx_n\|) \leq \phi(q,x_n) - \phi(q,y_n)$. $\lim_{n\to\infty} g(\|Jx_n - JTx_n\|) = 0$. This implies that $\lim_{n\to\infty}\|JTx_n - J\bar{x}\| = 0$. That is, $\lim_{n\to\infty}\|Tx_n - \bar{x}\| = 0$. It follows from the closedness of T that $T\bar{x} = \bar{x}$. This shows that $\bar{x} \in \mathscr{F}$.

Finally, we show that $\bar{x} = \Pi_{\mathscr{F}}x_0$. From $x_n = \Pi_{C_n}x_0$, we have $\langle x_n - w, Jx_0 - Jx_n\rangle \geq 0, \forall w \in \mathscr{F} \subset C_n$. Taking the limit as $n \to \infty$, we obtain that $\langle \bar{x} - w, Jx_0 - J\bar{x}\rangle \geq 0, \forall w \in \mathscr{F}$, and hence $\bar{x} = \Pi_{F(T)}x_0$ by Lemma 1.1. This completes the proof.

Conflict of Interests

The author declares that there is no conflict of interests.

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