NONLINEAR-DAMPED DUFFING OSCILLATORS HAVING FINITE TIME DYNAMICS

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Abstract. A class of modified Duffing oscillator differential equations, having nonlinear damping forces, are shown to have finite time dynamics, i.e., the solutions oscillate with only a finite number of cycles, and, thereafter, the motion is zero. The relevance of this feature is briefly discussed in relationship to the mathematical modeling, analysis, and estimation of parameters for the vibrations of carbon nano-tubes and graphene sheets, and macroscopic beams and plates.

1. Introduction

The primary purpose of this paper is to demonstrate the existence of a class of damping forces, such that when incorporated into the unforced Duffing differential equation [1], the resulting oscillatory motion ends in a finite time. This is in stark contrast with the conclusions reached

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using damping forces which are linear combinations of terms, each of which are proportional to a positive integer power of the velocity [1], [2], [3]. For this latter case, in general for large times after the initiation of the oscillation, the amplitude of the oscillation decreases in time either exponentially or by a power law [2], [3]. An example of a very general form for the undamped Duffing equation with a nonlinear damping is given by[4]

$$\ddot{x} + \Omega^2 x + \varepsilon \beta x^3 = -\varepsilon \left[d_0 + d_1 x^2 + d_2 \dot{x}^2 \right] \dot{x}, \tag{1.1}$$

where, for our purposes, all the parameters, $\left(\Omega^2, \varepsilon, \beta, d_0, d_1, d_2 \right)$ are taken to be non-negative. Note that the left-side is the standard expression for the Duffing equation, while the right-side is a damping force which depends on x and $\dot{x}$. Also observe that this more extended functional form for the damping force is even in x, but odd in $\dot{x}$. Thus, in some sufficiently small neighborhood of the origin, in the (x,y) phase-space, where $y = \dot{x}$, the dominant part of the damping force is its linear term, and, as stated above, the oscillations decrease exponentially. Further, energy arguments [5], [6] may be used to show that all solutions of Eq.(1.1), decrease, with increase of time, to zero, i.e.,

$$\lim_{t \to \infty} (x(t), y(t)) = (0, 0). \tag{1.2}$$

If $0 < \varepsilon \ll 1$, an analytical approximation to the damped oscillatory solution of Eq.(1.1) may be calculated using the method of first-order averaging [2], [3]. Again, the amplitude decreases monotonically and its magnitude goes to zero only as $t \to \infty$, in an exponential fashion.

The results to be presented in the remainder of this paper are of direct relevance to the investigation and analysis of several important nonlinear systems modeled by the Duffing equation. In particular, various forms of this differential equation appear in the study of macroscopic vibrations of beams and plates [7] as well as the microscopic oscillations of carbon nanotubes and sheets of graphene [8], [9], [10].

In the next section, a new type of nonlinear damping force is introduced and its properties are examined. It has the novel feature that it contains a term for which the velocity is raised to a fractional power, i.e., $f(\dot{x}) \propto \left| \dot{x} \right|^\alpha$, where $0 < \alpha < 1$. In Section 3, a number of general mathematical comments are discussed. They directly impact the mathematical computations of this paper and provide clarification as to exactly what is the proper interpretation which should

be given to the existence-uniqueness theorem in the theory of ordinary differential equations. Section 4 gives the results of an order ε analysis of Eq.(2.3) using the method of averaging. Section 5, discusses possible application of our results and looks at various limiting cases for the solution to the amplitude equation. In Section 6, general mathematical arguments are presented to show that the Duffing equation has finite time dynamics. Finally, Section 7 gives a concise summary and possible extension of the work reported in this paper.

2. A nonlinear damping force

Consider a Duffing equation having the following nonlinear damping force

$$f(\dot{x}) = -\varepsilon \left[c_1 \dot{x} + c_2 \mathrm{sgn}(\dot{x}) \left| \dot{x} \right|^{\alpha} \right], \tag{2.1}$$

where sgn(z) is the "sign" function, i.e.,

$$\mathrm{sgn}(z) = \begin{cases} 1, & z > 0, \\ -1 & z < 0, \end{cases} \tag{2.2}$$

and (ε, c_1, c_2) are non-negative parameters, with $0 < \alpha < 1$. In normalized form, the corresponding Duffing equation is

$$\ddot{x} + x + \varepsilon \beta x^3 = -\varepsilon \left[c_1 \dot{x} + c_2 \mathrm{sgn}(\dot{x}) \left| \dot{x} \right|^{\alpha} \right], \tag{2.3}$$

where, for our purposes, β may be selected to be non-negative.

Given the assumed properties of the various parameters, the function $f(v)$, where $v = \dot{x}$, has the following features:

(i) $f(v) < 0$, for $v > 0$;

(ii) $f(0) = 0$;

(iii) $f(-v) = -f(v)$;

(iv) in a neighborhood of the origin, i.e., $v = 0, f(v)$ is dominated by the $|v|^{\alpha}$ term.

In summary, f(v) is a negative, odd, continuous, nonlinear function, having the limiting behaviors

$$f(v) = \begin{cases} -\varepsilon c_1 v, & v \text{ large}; \\ -\varepsilon c_2 sgn(v) |v|^{\alpha} & v \text{ small}. \end{cases} \tag{2.4}$$

It should be noted that without a fundamental theory to calculate α, the following argument could be used to restrict the "representation" of α:

(a) Since $f_\alpha(v) = -\varepsilon c_2 \mathrm{sgn}(\dot{x}) \left| \dot{x} \right|^\alpha$ must be odd and continuous, and further, since it is more "aesthetically pleasing" not to have the sign-function appear, then what are the possibilities for α, if it is required that $0 < \alpha < 1$?

(b) Observe that if α is written as

$$\begin{cases} \alpha = \dfrac{2m+1}{2n+1}; & \text{n, m are non-negative integers;} \\ 0 \le m < n, \end{cases} \tag{2.5}$$

then $0 < \alpha < 1$.

(c) Call this set of $\alpha-$values, $\alpha(n,m)$. Then it follows that

$$\mathrm{sgn}(v) \left|v\right|^{\alpha(n,m)} = v^{\alpha(n,m)}, \tag{2.6}$$

and $v^{\alpha(n,m)}$ is, for v real, an odd function of v. However, for arbitrary non-negative real values of α, then this term of the damping force must be written with both the sign-function and the absolute value operator.

3. Mathematical comments

Equation (2.3) can be rewritten as a system of two coupled differential equations, i.e.,

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x - \varepsilon \beta x^3 - \varepsilon c_1 y - \varepsilon c_2 sgn(y) \left|y\right|^\alpha. \tag{3.1}$$

In the 2-dimensional (x,y) phase-plane [3], [6], [11], this system has a fixed-point at $(\bar{x}, \bar{y}) = (0,0)$,i.e., the origin.

A technique to study and analyze the dynamics in the neighborhood of a fixed-point is to use the method of dominant balance [12], [13]. This general methodology allows the determination of the asymptotic behavior of the solution by retaining only terms in the relevant equations which are dominant in the neighborhood of the fixed-point. Since

$$\left|\varepsilon \beta x^3\right| \ll \left|x\right|, \quad \left|c_1 y\right| \ll \left|y\right|^\alpha, \tag{3.2}$$

for sufficiently small x and y, the dynamics of Eq. (2.3) near the fixed-point $(\bar{x}, \bar{y}) = (0,0)$ is determined by the equations

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x - \varepsilon c_2 sgn(y) |y|^{\alpha} \tag{3.3}$$

or

$$\ddot{x} + x = -\varepsilon c_2 sgn(\dot{x}) |\dot{x}|^{\alpha}. \tag{3.4}$$

Another issue to resolve is whether the differential equation

$$\frac{du}{dt} = -\lambda u^{\alpha}, \quad 0 < \alpha < 1, \ \lambda > 0, \ u(0) > 0, \tag{3.5}$$

has a unique solution. In spite of the fact that the function on the right-side of Eq. (2.1) does not satisfy the Lipschitz condition [11], [14], Eq. (3.5) has a well defined piece-wise, continuous solution given by the expression

$$u(t) = \begin{cases} \left[u_0^{(1-\alpha)} - \lambda (1-\alpha)t \right]^{\left(\frac{1}{1-\alpha} \right)}, & 0 < t < t^*, \\ 0, & t > t^*, \end{cases} \tag{3.6}$$

where

$$u_0 = u(0) > 0, \quad t^* = \frac{u_0^{(1-\alpha)}}{\lambda (1-\alpha)}. \tag{3.7}$$

Examining in detail the standard existence and uniqueness theorem, it states that if the Lipschitz condition holds, then a unique solution exists. However, a differential equation not satisfying a Lipschitz condition may or may not have a unique solution. For the case of Eq. (3.5), a unique solution exists:

i) For $\lambda > 0$ and $u(0) > 0$, the derivative is negative except when $u = 0$.

ii) $u(t) = 0$ is a fixed-point or constant solution of Eq. (3.5).

iii) This result implies that the solutions either decrease, as a function of t, or remain at $u(t) = 0$ for $t > 0$.

iv) Thus, the unique solution to Eq. (3.5) decreases from $u(0) > 0$ to zero, at $t = t^*$. At $t = t^*$, this solution joins with $u(t) = 0, t > t^*$ to form a continuous solution Equation (3.6), with condition Eq. (3.7). This is a piecewise continuous solution to Eq. (3.5).

A very concise and clear discussion of these issues is given by Liu [14]. Also, additional insights on the issue of uniqueness may be gotten from the book of Kaplan [15].

4. Order of ε and ε^2 analysis of Eq. (2.3)

An approximation to the oscillatory solution of Eq. (2.3) can be determined using the method of first-order averaging (see Mickens [3], section3.2). This method takes for the exact solution of the form

$$x(t,\varepsilon) = a(t,\varepsilon)\cos[t + \phi(t,\varepsilon)], \tag{4.1}$$

and under the condition

$$0 < \varepsilon \ll 1, \tag{4.2}$$

gives the following relations for the first-order differential equations for the amplitude, $a(t,\varepsilon)$, and the phase, $\psi(t,\varepsilon) = t + \phi(t,\varepsilon)$:

$$\frac{da}{dt} = -\left(\frac{\varepsilon}{2\pi}\right)\int_0^{2\pi} F(a\cos\psi, -a\sin\psi)\sin\psi\,d\psi, \tag{4.3a}$$

$$\frac{d\phi}{dt} = -\left(\frac{\varepsilon}{2\pi a}\right)\int_0^{2\pi} F(a\cos\psi, -a\sin\psi)\cos\psi\,d\psi, \tag{4.3b}$$

where for Eq. (2.3)

$$F(a\cos\psi, -a\sin\psi) = (ac_1)\sin\psi + (a^\alpha)c_2\left[\text{sgn}(\sin\psi)\right]|\sin\psi|^\alpha - \beta a^3(\cos\psi)^3, \tag{4.4}$$

and the initial conditions are

$$a(0,\varepsilon) = A > 0, \quad \phi(0,\varepsilon) = 0. \tag{4.5}$$

Note that use has been made of the constraint that the amplitude function can always be determined in such a way that it is real and non-negative, i.e.,

$$a(0,\varepsilon) > 0 \Longrightarrow a(t,\varepsilon) \geqslant 0, \ t > 0. \tag{4.6}$$

Substitution of Eq. (4,4) into Eq. (4.3), and using the Fourier expansion relations contained in Kovacic [16], permits the integration of the expressions appearing on the right-sides of Eq. (4.3):

$$\frac{da}{dt} = -\left(\frac{\varepsilon}{2}\right)[c_1 a + (b_1 c_2)a^\alpha], \tag{4.7a}$$

$$\frac{d\phi}{dt} = \left(\frac{3\varepsilon\beta}{8}\right)a^2, \tag{4.7b}$$

where b_1 is [16]

$$b_1 = \left(\frac{2}{\sqrt{\pi}}\right) \frac{\Gamma\left(\frac{2+\alpha}{2}\right)}{\Gamma\left(\frac{3+\alpha}{2}\right)}. \tag{4.8}$$

Inspection of Eq.(4.7a) shows that it is a Bernoulli differential equation [11] and methods exist to determine its exact solution [11]. If these techniques are applied, then after a straightforward, but lengthy calculation and further application of careful mathematical analysis, the following expression is obtained for the amplitude function

$$a(t,\varepsilon) = \left\{\left[A^{(1-\alpha)} + \frac{b_1 c_2}{c_1}\right] \exp\left[-\frac{\varepsilon c_1 (1-\alpha)t}{2}\right] - \left(\frac{b_1 c_2}{c_1}\right)\right\}^{\frac{1}{1-\alpha}}, \tag{4.9a}$$

for $0 \le t \le t^*$, and

$$a(t,\varepsilon) = 0, \quad t > t^*, \tag{4.9b}$$

where

$$t^* = \left[\frac{2}{\varepsilon c_1 (1-\alpha)}\right] Ln\left[1 + \frac{c_1 A^{(1-\alpha)}}{b_1 c_2}\right]. \tag{4.10}$$

Note that $a(t,\varepsilon)$ is a piecewise, continuous, function of t, and this is true also for its derivative.

The $\phi(t,\varepsilon)$ may be determined by substituting Eqs. (4.9) into Eq.(4.7b) to obtain

$$\phi(t,\varepsilon) = \begin{cases} \left(\frac{3\varepsilon\beta}{8}\right) \int_0^t a(z,\varepsilon)^2 dz, & 0 \le t \le t^* \\ \left(\frac{3\varepsilon\beta}{8}\right) \int_0^{t^*} a(z,\varepsilon)^2 dz, & t > t^*. \end{cases} \tag{4.11}$$

However, since the main focus of this paper is on the properties of the amplitude functions, no calculation of $\phi(t,\varepsilon)$ will be presented. Using the fact that $a(t,\varepsilon)$ is bounded, for all t, it follows from Eqs. (4.11) that

$$\phi(t,\varepsilon) = O(\varepsilon), \quad t > 0. \tag{4.12}$$

Inspection of the argument of the cosine function in Eq. (4.1) and the use of the result given in Eq. (4.12), implies that the period is

$$T = 2\pi + O(\varepsilon), \quad 0 < \varepsilon \ll 1. \tag{4.13}$$

Therefore an estimate of the number of the oscillations, before the amplitude becomes zero, is

$$N = \frac{t^*}{T}. \tag{4.14}$$

Substituting Eqs. (4.10) and (4.13) into Eq. (4.14) gives

$$N = \left[\frac{1}{\varepsilon \pi c_1 (1-\alpha)}\right] Ln \left[1 + \frac{c_1 A^{(1-\alpha)}}{b_1 c_2}\right]. \tag{4.15}$$

Before ending this section, several comments are needed to clarify the results given in Eqs. (4.9) and (4.10):

i) The first order, nonlinear differential equation, given in Eq. (4.7a), is a Bernoulli type equation and has the exact solution given by the expression listed in Eq. (4.9a); see Ross [11], pp. 44-46.

ii) Observe that the right-side of Eq. (4.7a) is negative, since only $a \geqslant 0$ is physical, and that this function does not satisfy the Lipschitz condition because of the term a^α, where $0 < \alpha < 1$. Never the less, a solution exists, as given by Eq. (4.9a), and it is unique.

iii) The uniqueness may be shown by the following argument. Let $a(0) = A > 0$. From Eq.(4.7a), it follows that $a(t)$ is monotonic never increasing function. Further, $a(t) = 0$ is a separate (equilibrium) solution. If $a(t)$ is to be real, then the expression inside the outer brackets, in Eq. (4.9a), must be non-negative. Since this expression becomes zero at $t = t^*$, where t^* is given by Eq. (4.10), the solution to our problem is a piecewise, continuous function, defined by Eqs. (4.9), i.e., starting at $t = 0$, with the value $a(0) = A > 0$, the amplitude function decreases smoothly and monotonically, until at $t = t^*$, where it becomes zero; for $t > t^*$, both $a(t^*) = 0$ and $\frac{da(t^*)}{dt} = 0$, and as a consequence, the complete solution for $a(t)$ is continuous, as well as its derivative.

Finally, we make several comments on the use of the first-order averaging method as applied to the issues of this paper. The use of dominant balance reduces Eq. (2.3) to the form given by Eq. (3.4), i.e.,

$$\ddot{x} + x = -\varepsilon c_2 sgn(\dot{x}) |\dot{x}|^\alpha, \tag{4.16}$$

where, as before, ε and c_2 are positive parameters, $0 < \varepsilon \ll 1$, and $0 < \alpha < 1$. The $O(\varepsilon)$ and $O(\varepsilon^2)$ expressions for the amplitude differential equation are given, respectively, by (see Mickens [3], sections 3.4 and 3.5)

$$\frac{da}{dt} = \varepsilon A_1(a), \tag{4.17a}$$

$$\frac{da}{dt} = \varepsilon A_1(a) + \varepsilon^2 A_2(a). \tag{4.17b}$$

The function $A_1(a)$ was calculated above and shown to be (see the second term on the right-side of Eq. (4.7a))

$$A_1(a) = -\varepsilon \left(\frac{b_1 c_2}{2} \right) a^\alpha. \tag{4.18}$$

In a similar manner, although the calculations are very intense with complicated algebraic and trigonometric manipulations, the function $A_2(a)$ can be determined. It turns out that $A_2(a)$ is identically zero, i.e.,

$$A_2(a) = 0. \tag{4.19}$$

Thus, both the first- and second-order results for $a(t, \varepsilon)$ are exactly the same. This means that in a neighborhood of the two-dimensional (x, y) phase-space, all the trajectories (of our nonlinear oscillating system) go to zero to terms of order ε^2, according to the averaging method.

Note that the time for the oscillations to completely stop is $t^* = O(\varepsilon^{-1})$, while the validity of the second-order calculations extend to times $O(\varepsilon^{-2})$. This result confirms the conclusion that the finite-time dynamics is an actual feature of the system.

5. General Comments

It has been shown, for the particular case of the Duffing differential equation, that there exists damping forces for which only a finite number of oscillations take place once the motions are initiated. While the damping force, given in this paper, consists of a linear velocity term and a second one proportional to the velocity raised to a fractional power, the results are of general applicability. It is to be expected that the presence of any fractional power term in the damping will lead to oscillations for which the amplitude goes to zero within a finite time interval.

In many areas of the natural and engineering services, mathematical models of the relevant phenomena give rise to Duffing type differential equations. Any analysis of these models, in conjunction with experimental data, often allows the estimation of various parameters which are directly related to physical properties and the dynamics of these systems. However, previous mathematical models have generally assumed that the damping forces are either linear or consists of a combination of terms, each proportional to a positive, odd-integer power of the velocity [3], [4], [9], [10]. In this case, with no external forcing, oscillations are found either

analytically or by numerical methods to exist for all times [2], [3], [16], i.e., the oscillations take place with decreasing amplitude for all $t > 0$. One, implicit, implication of the results present-ed here is that the presence of fractional power damping terms may yield estimates of system parameter values which differ from the situation of integer-valued power damping forces. Also this same result might be expected for systems with external forcing terms. For this latter case, the powerful mathematical techniques created by I. Kovacic [16], [17] and A. K. Mallik [18] offer important possibilities for carrying out the necessary analysis.

It should be noted that one of the important results obtained here is the determination of an estimate for the finite duration of the oscillations for the particular damping force given on the right-side of Eq.(2.3). This estimate is given in Eq.(4.15) and inspection of its mathematical structure leads to the following conclusions:

i) the number of oscillations depends on all of the parameters appearing in the damping force; these are $(\varepsilon, c_1, c_2, \alpha)$ and the initial value of the amplitude, A.

ii) When ε goes to zero, i.e., there is no damping force, then the number of oscillations becomes unbounded, and this is exactly what is to be expected of the undamped harmonic oscillator [2], [3].

iii) If the limit is taken, where c_2 goes to zero, then $N \to \infty$, a result expected for the case of linear damping. Also, in this limit, the amplitude function, see, Eq.(4.9a), takes the expected form

$$a(t, \varepsilon) = A e^{-\frac{\varepsilon c_1 t}{2}}. \tag{5.1}$$

iv) If c_1 is taken to be zero, then using the limit $c_1 \to 0$, Eqs. (4.9a), (4.10), and (4.15) reduce to the following expressions:

$$a(t, \varepsilon) = \left\{ A^{(1-\alpha)} - \left[\frac{\varepsilon b_1 c_2 (1-\alpha)}{2} \right] t \right\}^{\left(\frac{1}{1-\alpha} \right)} \text{ for } 0 \le t \le t^*, \tag{5.2a}$$

and

$$a(t, \varepsilon) = 0, \ t > t^*, \tag{5.2b}$$

where, for this case

$$t^* = \frac{2A^{(1-\alpha)}}{\varepsilon b_1 c_2 (1-\alpha)}, \tag{5.3}$$

and, the estimate for the number of oscillations is

$$N = \frac{A^{(1-\alpha)}}{\varepsilon \pi b_1 c_2 (1-\alpha)}.$$ (5.4)

One point should be clearly understood within the framework of oscillations as they occur for actual physical systems: In practice, oscillating physical systems only execute a finite number of oscillations before the amplitudes of such motions effectively become zero. Thus, a consequence of this fact should be the imposition on any mathematical model of such phenomena, the condition that the involved forces should be of a nature that the solutions to the mathematical equations also have this property. The work presented here strongly indicates that the inclusion in the damping force of a term having the velocity raised to a fractional power will give this desired result.

6. Mathematical consideration of finite time dynamics

The main purpose of the discussion in this section is to provide a more rigorous demonstration that Eq. (2.3) has finite dynamics. In part, several of the previous arguments and equations will be listed again.

To begin, consider the Duffing equation with only fractional power damping, i.e.,

$$\ddot{x} + \varepsilon c_2 \left[sgn(\dot{x}) \right] \left| \dot{x} \right|^{\alpha} + x + \varepsilon \beta x^3 = 0,$$ (6.1)

where, $0 < \alpha < 1$, $c_2 > 0$, $\beta \geqslant 0$, and $\varepsilon > 0$. Note: the parameter ε is not *a priori* required to be small; it may take any finite positive value.

This equation has the fixed-point $(\bar{x}, \bar{y}) = (0,0)$, where $y = \dot{x}$, and in a sufficiently small neighborhood of this fixed-point, the method of dominant balance [12], [13] allows consideration only on the following equation

$$\ddot{x} + \varepsilon c_2 \left[sgn(\dot{x}) \right] \left| \dot{x} \right|^{\alpha} + x = 0,$$ (6.2)

and this differential equation can be written as a system of two first-order equations

$$\dot{x} = y, \quad \dot{y} = -x - \varepsilon c_2 \left[sgn(\dot{x}) \right] |y|^{\alpha}.$$ (6.3)

If a polar representation is used for $x(t)$ and $y(t)$, i.e.,

$$x(t) = r(t)\cos\theta(t), \quad y(t) = r(t)\sin\theta(t), \tag{6.4}$$

then $r(t)$ and $\theta(t)$ satisfy the following first-order differential equations

$$\dot{r} = -\varepsilon c_2 r^\alpha |\sin\theta|^{\alpha+1} \tag{6.5a}$$

$$r\dot{\theta} = -r - \varepsilon c_2 r^\alpha (\cos\theta)[sgn(\sin\theta)]|\sin\theta|^\alpha, \tag{6.5b}$$

where $r(t)$ and $\theta(t)$, are, respectively, the amplitude and phase functions. Our interest is primarily centered on the amplitude function, $r(t)$. This function has the following properties:

(i) In the polar representation, $r(t)$ is a non-negative function, i.e., $r(t) \geqslant 0$.

(ii) From Eq. (6.5a), it follows that $r(t)$ decreases, since, in general, its derivative is negative.

(iii) Using energy arguments [5], [6], it follows that

$$\underset{t \longrightarrow \infty}{Lim}\, r(t) = 0. \tag{6.6}$$

See Eq. (1.2).

(iv) It is also true, from Eq. (6.5a), that $\bar{r}(t) = 0$, is a nontrivial solution of the nonlinear system of differential equations given by Eqs. (6.5).

Note that independent of the actual time dependence of the phase function, $\theta(t)$, the following result holds

$$0 \leq |\sin\theta|^{1+\alpha} \leq 1; \tag{6.7}$$

and this is the basis for the comment (ii) above. Therefore, the integration of Eq. (6.5a) gives the result

$$[r(t)]^{1-\alpha} = (r_0)^{1-\alpha} - \varepsilon H(t), \tag{6.8a}$$

where

$$H(t) \equiv c_2 \int_0^t |\sin\theta(z)|^{1+\alpha}\, dz. \tag{6.8b}$$

From Eq. (6.7), it follows that $H(t)$ is a monotonic increasing function [19] with

$$H(0) = 0, \quad H(t) > 0, \quad t > 0, \tag{6.9}$$

since $c_2 > 0$. Now there are two possibilities for $H(t)$ [19]:

$$\text{(A)} \quad \underset{t \longrightarrow \infty}{Lim} \ H(t) = H^* > 0, \tag{6.10}$$

$$\text{(B)} \quad \underset{t \longrightarrow \infty}{Lim} \ H(t) = \infty. \tag{6.11}$$

For (A), it follows that

$$[r(\infty)]^{1-\alpha} = (r_0)^{1-\alpha} - \varepsilon H^*; \quad r_0 > 0, \text{ given.} \tag{6.12}$$

But this possibility will, in general not hold. The argument is as follows. From the energy method, we know that $r(\infty) = 0$, since this is a dissipative system and the fixed point, $(\bar{x}, \bar{y}) = (0,0)$, implies that $r(\infty) = 0$. However, regardless of the values r_0 and H^*, a non-zero value of ε can be chosen such that $r(\infty) > 0$, in Eq. (6.12). Thus, as a consequence, possibility (A) can not hold.

For (B), given any fixed values of $r_0 > 0$ and $\varepsilon > 0$, there exists a value of t, call it t^*, such that

$$(r_0)^{1-\alpha} = \varepsilon H(t^*), \tag{6.13}$$

since $H(t)$ is a monotonic increasing function of t, having the property stated in Eq. (6.11). Thus, $r(t)$ must have the following behavior as a function of time: At $t = 0$, $r(0) = r_0 > 0$. From Eqs. (6.5a) and (6.8), $r(t)$ decreases to zero and achieves the value zero at $t = t^*$. For $t > t^*$, $r(t)$ is zero. This argument demonstrates that the amplitude function, $r(t)$, is a piece-wise continuous function, with a continuous first-derivative.

Observe that the above arguments did not depend on having an explicit knowledge of $\theta(t)$ as a function of time.

7. Summary

We have given arguments in support of the proposition that the use of nonlinear damping forces, proportional to a fractional power of the first-derivative, gives rise to dynamics for which the oscillations of Duffing type equations end in a finite time. This is in contrast to the use of

integer-valued powers of the velocity (first-derivative) which produce damping dynamics over an arbitrarily large, i.e., infinite time interval. Using the method of averaging for ε small, it was shown how to estimate the magnitude of this time interval. We also provided mathematical arguments to demonstrate that finite time dynamics are a general feature of one-dimensional, nonlinear oscillating systems which contain a damping force having the property that it is proportional to the velocity (i.e., first derivative) raised to a fractional power α.

An extension of the work presented here would be to investigate the detailed properties for the solutions in the case where an external forcing is included [17]. A particular example of such an equation is

$$\ddot{x} + \Omega^2 x + \varepsilon \beta x^3 = -\varepsilon \left[c_1 \dot{x} + c_2 \mathrm{sgn}(\dot{x}) \left| \dot{x} \right|^\alpha + F_0 \cos(\omega t) \right]. \tag{7.1}$$

It should be pointed out that preliminary results on this issue has already been done in the fundamental paper by Kovacic [16]; however, the possibility of finite time dynamics for the unforced, conservative oscillations with fractional damping was not examined.

Conflict of Interests

The authors declares that there is no conflict of interests.

REFERENCES

[1] I. Kovacic, M. J. Brennan, The Duffing Equations, Wiley, Chichester, 2011.

[2] A.H. Nayfeh, Problems in Perturbation, Wiley-Interscience, New York, 1985.

[3] R.E. Mickens, Oscillations in Planar Dynamic Systems, World Scientific, Singapore, 1996.

[4] G. Schmidt, A. Toondl, Non-Linear Vibrations, Cambridge University Press. New York, 1986.

[5] R.E. Mickens, Truly Nonlinear Oscillators, World Scientific, London, 2010.

[6] D. W. Jordan, P. Smith, Nonlinear Ordinary Differential equations, 2nd ed., Clarendon Press, Oxford, 1987.

[7] A. H. Nayfeh, D.T, Mook, Nonlinear Oscillations, Wiley-Interscience, New York, 1979.

[8] A. Eichler, J. Mason, J. Chaste, M. Zdrojck, I. Wilson-Rae, A. Bachtold, Nonlinear damping in Mechanical resonators made from carbon nanotubes and graphene, Nature Nano-technology 6 (2011), 339-342.

[9] R. Lifshitz, M. C. Crass, Nonlinear dynamics of nanomechanical and micromechanical resonators, pps. 1-52, in H. G. Schuster (editor), Review of Nonlinear Dynamics and Complexity, Vol. 1 Wiley-VCH, New York, 2008.

[10] S. Zaitsev, O. Shtempluck, E. Buks, O. Gottlieb, Nonlinear damping in a micromechanical oscillator, Nonlinear Dynam. 67 (2012), 859-883.

[11] S. L. Ross, Differential Equations, Blairdell, Waltham, MA, 1964.

[12] C.M. Bender, S.A. Orszag, Advanced Mathematical Methods for Scientists and Engineers, McGraw-Hill, New York, 1978.

[13] W. Paulsen, Asymptotic Analysis and Perturbation Theory, CRC Press, Boco Raton, FL. 2014.

[14] J.H. Liu, Qualitative Theory of Differential Equations, Prentice Hall, Upper Saddle River, NJ, 2003.

[15] W. Kaplan, Ordinary Differential Equations, Addison-Wesley, London, 1958.

[16] I. Kovacic, The method of multiple scales for forced oscillators with some real-power nonlinearities in the stiffness and damping force, Chaos, Solitons Fractals 44 (2011), 891-901.

[17] I. Kovacic, Forced vibrations of oscillators with a purely nonlinear power-form restoring force, J. Sound Vibration 330 (2011), 4313-4317.

[18] A. K. Mallik, Forced harmonic vibration of a Duffing oscillator with different damping mechanics, Chapter 6 in, I. Kovacic, M. J. Brennan (editors), The Duffing Equations, Wiley, Chichester, 2011.

[19] http://en.wikipedia.org/wiki/Monotonic_function