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FIXED POINT THEOREMS IN PROBABILISTIC CONE METRIC SPACES

MANOJ SHUKLA*, ARCHANA AGRAWAL AND SURENDRA GARG

Govt. Model Science College (autonomous), Jabalpur (M.P.) India

Abstract: We find some fixed point theorems in Probabilistic Cone metric space for weak contraction condition and implicit relation.

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1. INTRODUCTION

The generalizations of metric space is Menger space introduced in 1942 by Menger [2] who used distribution functions instead of nonnegative real numbers as values of the metric. The notion of probabilistic metric space correspond to situations when we do not know exactly the distance between the two points but we know probabilities of possible values of this distance. A probabilistic generalization of metric spaces appears to be interest in the investigation of physical quantities and physiological threshold. It is also a fundamental importance in probabilistic functional analysis. Schweizer and Sklar [4] studied this concept and then the important development of Menger space theory was due to Sehgal and Bharucha-Reid [5].

Huang and Zhang [1] generalized the concept of metric spaces, replacing the set of real numbers by an ordered Banach space, hence they have defined the cone metric spaces. They also described the convergence of sequences and introduced the notion of completeness in cone metric spaces. They have proved some fixed point theorems of contractive mappings on complete cone metric space with the assumption of normality of a cone. Subsequently, various authors have generalized the results of Huang and Zhang and have studied fixed point theorems

*Corresponding author

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for normal and non-normal cones. This principle has been extended kind of contraction mappings by Sumitra, V.R., Uthariaraj,R. Hemavathy [6].

2. PRELIMINARY

Definition 2.1 A probabilistic metric space (FPM space) is an ordered pair (X,*F*) consisting of a nonempty set X and a mapping F from XxX into the collections of all distribution functions .For x, y $\in$ X we denote the distribution function *F*(x,y) by $F_{x,y}$ and $F_{x,y}(u)$ is the value of $F_{x,y}$ at u in R. The functions $F_{x,y}$ assumed to satisfy the following conditions:

2.1.1 $F_{x,y}(u) = 1 \ \forall\ u > 0$ iff x = y,

2.1.2 $F_{x,y}(0) = 0 \ \forall$ x , y in X,

2.1.3 $F_{x,y} = F_{y,x} \ \forall$ x , y in X,

2.1.4 If $F_{x,y}(u) = 1$ and $F_{y,z}(v) = 1$ then $F_{x,z}(u+v) = 1 \ \forall$ x ,y, z in X and u, v>0

Definition 2.2A commutative, associative and non-decreasing mapping t: $[0,1] \times [0,1] \rightarrow [0,1]$ is a t-norm if and only if t(a,1) = a $\forall a \in [0,1]$, t(0,0)=0 and t(c,d) $\geq$ t(a,b) for c $\geq$ a, d $\geq$ b.

Definition 2.3 A Menger space is a triplet (X,*F*,t), where (X,*F*) is a PM-space,t is a t-norm and the generalized triangle inequality for all x, y, z in X u, v> 0 .

$F_{x,y}(u+v) \geq t(F_{x,z}(u), F_{z,y}(v))$

Definition 2.4: Let (E, τ) be a topological vector space and P a subset of E, P is called a cone if

1. P is non-empty and closed, P $\neq \{0\}$,

2. For x, y $\in$ P and a, b $\in$ R $\Longrightarrow$ ax + by $\in$ P where a,b$\geq$ 0

3. If x $\in$ P and –x $\in$ P $\Longrightarrow$x = 0

For a given cone P $\subseteq$ E, a partial ordering $\geq$with respect to P is defined by x $\geq$ y if and only if x − y $\in$ P, x > y if x $\geq$ y and x $\neq$ y, while $x \gg y$ will stand for x − y $\in$ int P, int P denotes the interior of P.

Definition 2.5: Let M be a nonempty set and the mapping d: M $\rightarrow$ X and P$\subset$ X be a cone, satisfies the following conditions:

2.5.1) $F_{x,y}(u) > 1 \forall\ x, y \in X \Leftrightarrow x = y$

2.5.2) $F_{x,y}(u) = F_{y,x}(u) \ \ \forall\, x, y \in X$,

2.5.3) $F_{x,y}(u+v) \geq t(F_{x,z}(u)\ ,\ F_{z,y}(v)) \forall\, x, y \in X$.

2.5.4) $For\ any\ x, y \in X, (x, y) \quad is\ non-increa\sin g\ and\ left\ continuous$.

Definition 2.5 Let (X,F,t) be a Menger cone space. If $x\in X$, $\varepsilon > 0$ and $c \in(0,1)$ then (ε,c) - neighborhood of x, called $U_x(\varepsilon,c)$, is defined by

$$U_x(\varepsilon,c) = \{y\in X: F_{(x,y)}(\varepsilon)>(1-c)\}$$

An (ε,c)-topology in X is the topology induced by the family $\{U_x(\varepsilon,c): x \in X, \varepsilon > 0, c\in(0,1)\}$ of neighborhood.

Remark: If t is continuous, then Menger cone space (X,F,t) is a Housdroff space in (ε,c)-topology.

Let (X,F,t) be a complete Menger cone space and $A\subset E$. Then A is called a bounded set if for u>0 $\lim_{u\to\infty} \inf_{x,y\in A} F_{x,y}(u) =1$.

Definition 2.6 A sequence $\{x_n\}$ in (X,F,t) is said to be convergent to a point x in X if for every $\varepsilon>0$and $c>0$, there exists an integer $N=N(\varepsilon,c)$ such that $x_n \in U_x(\varepsilon,c)$ for all $n \geq N$ or equivalently $F_{xn,x}(\varepsilon) > 1-c$ for all $n \geq N$.

Definition 2.7 A sequence $\{x_n\}$in (X,F,t) is said to be cauchy sequence if for every $\varepsilon > 0$ and $c > 0$, there exists an integer $N=N(\varepsilon,c)$ such that $F_{(xn,xm)}(\varepsilon) > 1-c\ \forall\ n, m \geq N$.

Definition 2.8 A Menger cone space (X,F,t) with the continuous t-norm is said to be complete if every Cauchy sequence in X converges to a point in X .

Lemma 1 Let $\{x_n\}$ be a sequence in a Menger cone space (X,F,t), if there exists a constant k(0,1) such that $\forall p > 0$ and $n\in N$

$$F_{xn,xn+1}(kp) \geq F_{xn-1,xn}(p),$$

then $\{x_n\}$ is cauchy sequence.

Definition 2.10: Implicit Relation

Φ be the family of real continuous function $\phi : (R^+)^4 \to R$ satisfying the properties

(G_h) for every $u \geq 0$, $v \geq 0$ with $\phi(u,v,u,v)\geq 0$ or $\phi(u,v,v,u)\geq 0$ we have $u \geq v$.

(G_u) $\phi(u,u,1,1) \geq 0$ implies that $u \geq 1$

3. MAIN RESULTS

Theorem 3.1 : Let (X,F,t) be a complete Menger cone metric space and let M be a nonempty separable closed subset of menger cone metric space X and let A, B , S and T be continuous mapping defined on M satisfying contraction.

(I) $A(X)\subseteq T(X)$ and $B(X)\subseteq S(X)$;

(II) the pair (A,S) is semi compatible and (B,T) is weak compatible;

(III) one of A or S is continuous;

for some $\phi\in\Phi$, there exist $k\in(0,1)$ such that for all $x, y\in X$ and $p>0$

(IV) $\phi(F_{Ax,By}(kp), F_{Sx,Ty}(p), F_{Ax,Sx}(p), F_{By,Ty}(p)) \geq 0$;

then A, B, S and T have unique common fixed point in X.

Proof: Proof: Let x_0 be any arbitrary point of X, as $A(X)\subseteq T(X)$ and $B(X)\subseteq S(X)$ there exists x_1,x_2 in X such that $Ax_0=Tx_1$, $Bx_1=Sx_2$. Inductively, construct sequences $\{y_n\}$, and $\{x_n\}$ in X such that $y_{2n+1} = Ax_{2n} = Tx_{2n+1}$, $y_{2n+2} = Bx_{2n+1} = Sx_{2n+2}$, for n = 0,1,2,….. .

Now by (IV)

$\phi(F_{Ax2n,Bx2n+1}(kp),F_{Sx2n,Tx2n+1}(p)),F_{Ax2n,Sx2n}(p), F_{Bx2n+1,Tx2n+1}(p)) \geq 0$

$\Rightarrow \phi(F_{y2n+1, y2n+2}(kp), F_{y2n, y2n+1}(p), F_{y2n+1, y2n}(p), F_{y2n+2, y2n+1}(p)) \geq 0$

By implicit relation

$F_{y2n+2, y2n+1}(kp) \geq F_{y2n+1, y2n}(p)$

$\Rightarrow F_{y2n+2, y2n+1}(kp) \geq F_{y2n+1, y2n}(p)$

Again putting $x = x_{2n+2}$ and $y = x_{2n+1}$ in (IV), we have

$F_{y2n+3, y2n+2}(kp) \geq F_{y2n+2, y2n+1}(p)$

Hence by closeness of M and completeness of X. Therefore $\{y_n\}$ converge to u in X. Therefore its subsequences $\{Ax_{2n}\}$, $\{Tx_{2n+1}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n+2}\}$ also converge to u.

Case 1 If S is continuous, we have

$SAx_{2n} \to Su$, $SSx_{2n} \to Su$

So, weak compatibility of the pair (A,S) gives $ASx_{2n} \to Su$ as $n \to \infty$

Step (i) By putting $x = Sx_{2n}$, $y = x_{2n+1}$ in (IV) , we obtain that

$\phi(F_{ASx2n, B x2n+1}(kp), F_{SSx2n, T x2n+1}(p), F_{ASx2n, SSx2n}(p),F_{Bx2n+1, T x2n+1}(p)) \geq 0$

Now letting $n \to \infty$ and by the continuity of the t- norm, we have

$\phi(F_{Su, u}(kp), F_{Su, u}(p),F_{Su, Su}(p), F_{u, u}(p)) \geq 0$

$\Rightarrow \phi(F_{Su, u}(p), F_{Su, u}(p), 1,1) \geq 0$

Now as ϕ is non-decreasing in the first argument, we have

$\Rightarrow \phi(F_{Su, u}(p), F_{Su, u}(p), 1,1) \geq 0$

Using (Gu), we get $F_{Su, u}(p) \geq 1$, for all $p>0$, which gives $F_{Su, u}(p) =1$

$\Rightarrow(Su, u)\in extP$ but $(Su, u)\in P$,therefore Su=u.

$\Rightarrow Su = u$

Step (ii) By putting x = u and y = x_{2n+1} in (IV) , we obtain that

$\phi(F_{Au,\,Bx2n+1}(kp), F_{Su,\,Tx2n+1}(p), F_{Au,Su}(p), F_{B\,x2n+1,T\,x2n+1}(p)) \geq 0$

On taking limit $n \to \infty$ and as Su = u & Bx_{2n+1},T $x_{2n+1} \to$ u, we get

$$\phi(F_{Au,\,u}(p), 1, F_{Au,\,u}(p), 1) \geq 0$$

Now as ϕ is non-decreasing in the first argument, we have

$$\phi(F_{Au,\,u}(p), 1, F_{Au,\,u}(p), 1) \geq 0$$

Using (G_h), we get $F_{Au,\,u}(p) \geq 1$, for all p>0, which gives $F_{Au,\,u}(p) = 1$,

$\Rightarrow$(Au, u)$\in$extP but (Au, u)$\in$P ,therefore Au=u.

$\Rightarrow$ Au = u = Su.

Step (iii) By (I) A(X) $\subseteq$ T(X), there exists w in X such that Au= u=Su =Tw.

By putting x = x_{2n} and y = w in (IV), we obtain that

$\phi(F_{Ax2n,Bw}(kp), F_{Sx2n,Tw}(p), F_{Ax2n,Sx2n}(p), F_{Bw,Tw}(p)) \geq 0$

On taking limit $n \to \infty$ and as Ax_{2n}, $Sx_{2n} \to$ u, we get

$$\phi(F_{u,Bw}(kp), 1, 1, F_{Bw,\,u}(p)) \geq 0$$

By using (G_h), we get $F_{u,Bw}(kp) \geq 1$, for all p > 0, which gives $F_{u,Bw}(p) = 1$,

$\Rightarrow$(u, Bw)$\in$extP but (u, Bw)$\in$P ,therefore u=Bw.

Therefore Bw = Tw = u. Since (B,T) is weak compatible, we get TBw = BTw, it implies Bu = Tu.

Step (iv) Now putting x = u and y = u in (IV) and as Au= u = Su & Bu = Tu

We get that

$\phi(F_{Au,\,Bu}(kp), F_{Su,\,Tu}(p), F_{Au,\,Su}(p), F_{Bu,\,Tu}(p)) \geq 0$

$\phi(F_{Au,\,Bu}(kp), F_{Su,\,Tu}(p), 1, 1) \geq 0$

Now as ϕ is non-decreasing in the first argument, we have

$\Rightarrow \quad \phi(F_{Au,\,Bu}(p), F_{Au,\,Bu}(p), 1,1) \geq 0$

Using (Gu), we get $F_{Au,Bu}(p) \geq 1$, for all p>0, which gives $F_{Au,\,Bu}(p) = 1$

$\Rightarrow$(Au, Bu)$\in$extP but (Au, Bu)$\in$P ,therefore Au=Bu.

Thus u = Au = Su = Bu = Tu.

Case 2 If A is continuous i.e. $ASx_{2n} \to$ Au. Also the pair (A,S) is semi-compatible, therefore $ASx_{2n} \to$ Su . By the uniqueness of the limit Au = Su.

Step (v) By putting x = u and y = x_{2n+1} in (IV), we get

$\phi(F_{Au,\,Bx2n+1}(kp), F_{Su,\,Tx2n+1}(p), F_{Au,\,Su}(p), F_{Bx2n+1,Tx2n+1}(p)) \geq 0$

On taking limit $n \to \infty$ and as Bx_{2n+1}, $Tx_{2n+1} \to$ u, we get

$$\phi(F_{Au,\,u}(kp), 1, F_{Au,\,u}(p), 1) \geq 0.$$

Now as ϕ is non-decreasing in the first argument, we have

$\phi(F_{Au,\,u}(p), 1, F_{Au,\,u}(p), 1) \geq 0$.

Using (G_h), we have $F_{Au,\,u}(p) \geq 1$ for all p>0, which gives (Au,u)∈extP but (Au,u)∈P ,therefore Au=u. The rest of the proof follows from step (iii) onwards of the **case 1**.

Example: Let M=R and P={x∈M:x≥0} Let X = [0,∞) and metric d is defined by

d(x,y) = $\dfrac{|x-y|}{1+|x-y|}$. For each p define F(x, y, p) = $\begin{cases} 1 & for.x = y \\ H(p) & for.x \neq y \end{cases}$,

where H(p) = $\begin{cases} 0 & if \quad p \leq 0 \\ p.d(x,y) & if \quad 0 < p < 1 \\ 1 & if \quad p \geq 1 \end{cases}$

Clearly, (X, F, p) is a complete probabilistic space where t is defined by t(p,p) ≥ p.

The sequence $\{x_n\}$ is defined as $x_n = 2 - \dfrac{1}{2n}$. Tx = $\begin{cases} x & 0 \leq x \leq 1 \\ \dfrac{4-x}{2} & x > 1 \end{cases}$,

we see the all conditions of **Theorem 3.1** are satisfied and hence 1 is the common fixed point in X.

Conflict of Interests

The authors declare that there is no conflict of interests.

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