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I- CONVERGENT SEQUENCE SPACES OF FUZZY REAL NUMBERS DEFINED BY SEQUENCE OF MODULUS FUNCTIONS

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Abstract: In this article our aim to introduce some new *I*- convergent sequence spaces of fuzzy real numbers defined by sequence of modulus functions and studies some topological and algebraic properties. Also we establish some inclusion relations.

Keywords: Fuzzy real number, *I*- convergence, modulus function, sequence of fuzzy real numbers.

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1. INTRODUCTION

The notion of fuzzy sets was introduced by Zadeh [16]. After that many authors have studied and generalized this notion in many ways, due to the potential of the introduced notion. Also it has wide range of applications in almost all the branches of studied in particular science, where mathematics is used. It attracted many workers to introduce different types of fuzzy sequence spaces.

Bounded and convergent sequences of fuzzy numbers were studied by Matloka [8]. Later on sequences of fuzzy numbers have been studied by Kaleva and Seikkala [2], Tripathy and Sarma ([13], [14]) and many others.

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I-convergence of real valued sequence was studied at the initial stage by Kostyrko, Šalát and Wilczyński [4] which generalizes and unifies different notions of convergence of sequences. The notion was further studied by Šalát, Tripathy and Ziman [9].

Let X be a non-empty set, then a non-void class $I \subseteq 2^X$ (power set of X) is called an *ideal* if I is additive (i.e. $A, B \in I \Rightarrow A \cup B \in I$) and hereditary (i.e. $A \in I$ and $B \subseteq A \Rightarrow B \in I$). An ideal $I \subseteq 2^X$ is said to be non-trivial if $I \neq 2^X$. A non-trivial ideal I is said to be *admissible* if I contains every finite subset of N. A non-trivial ideal I is said to be *maximal* if there does not exist any non-trivial ideal $J \neq I$ containing I as a subset.

Let X be a non-empty set, then a non-void class F $\subseteq 2^X$ is said to be a filter in X if $\phi \notin F$; $A, B \in F \Rightarrow A \cap B \in F$ and $A \in F$, $A \subseteq B \Rightarrow B \in F$. For any ideal I, there is a filter $\Psi(I)$ corresponding to I, given by

$$\Psi(I) = \{K \subseteq N : N \setminus K \in I\}.$$

A modulus function f is a function from $[0,\infty)$ to $[0,\infty)$ such that :

(i) $f(x) = 0 \ iff \ x = 0$

(ii) $f(x+y) \leq f(x) + f(y)$ for all $x, y \geq 0$.

(iii) f is increasing.

(iv) f is continous from the right at 0 .

It follows that f must be continous everywhere on $[0,\infty)$ and a modulus function may be bounded or not bounded .

Let X be a linear metric space. A function $p : X \to R$ is called paranorm if

(1) $p(x) \geq 0$ for all $x \in X$

(2) $p(-x) = p(x)$ for all $x \in X$

(3) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$

(4) If (λ_n) be a sequence of scalars such that $\lambda_n \to 0$ as $n \to \infty$ and (x_n) be a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0 \Rightarrow x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space.

2. Definitions and Background

Let D denote the set of all closed and bounded intervals $X = [a_1, b_1]$ on the real line R. For $X = [a_1, b_1] \in D$ and $Y = [a_2, b_2] \in D$, define $d(X, Y)$ by

$$d(X, Y) = \max(|a_1 - b_1|, |a_2 - b_2|).$$

It is known that (D, d) is a complete metric space.

A fuzzy real number X is a fuzzy set on R *i.e.* a mapping $X : R \to L(= [0,1])$ associating each real number t with its grade of membership $X(t)$.

The α- level set $[X]^{\alpha}$ set of a fuzzy real number X for $0 < \alpha \leq 1$, defined as

$X^{\alpha} = \{ t \in R : X(t) \geq \alpha\}$.

A fuzzy real number X is called *convex*, if $X(t) \geq X(s) \wedge X(r) = \min(X(s), X(r))$, where $s < t < r$.

If there exists $t_0 \in R$ such that $X(t_0) = 1$, then the fuzzy real number X is called *normal*.

A fuzzy real number X is said to be *upper semi- continuous* if for each $\varepsilon > 0$, $X^{-1}([0, a + \varepsilon))$, for all $a \in L$ is open in the usual topology of R.

The set of all upper semi-continuous, normal, convex fuzzy number is denoted by $L(R)$.

The absolute value $|X|$ of $X \in L(R)$ is defined as (see for instance Kaleva and Seikkala [2])

$$|X|(t) = \begin{cases} \max\{X(t), X(-t)\}, & \text{if } t \geq 0 \\ 0, & \text{if } t < 0. \end{cases}$$

Let $\bar{d} : L(R) \times L(R) \to R$ be defined by

$$\bar{d}(X, Y) = \sup_{0 \leq \alpha \leq 1} d(X^{\alpha}, Y^{\alpha}).$$

Then $\bar{d}$ defines a metric on $L(R)$.

A sequence $X = (X_k)$ of fuzzy numbers is a function X from the set N of all positive integers into $L(R)$. The fuzzy number X_k denotes the value of the function at $k \in N$ and is called the k-th term or general term of the sequence. The set of all sequences of fuzzy numbers is denoted by w^F.

A sequence (X_k) of fuzzy real numbers is said to be convergent to the fuzzy real number X_0, if for every ε>0, there exists $k_0 \in N$ such that $\bar{d}\,(X_k, X_0) < \varepsilon$ for all $k \geq k_0$.

A sequence space E^F is said to be *symmetric* if $(X_{\pi(k)}) \in E^F$, whenever $(X_k) \in E^F$, π is a permutation on N.

A sequence $X = (X_k)$ of fuzzy numbers is said to be I- convergent if there exists a fuzzy number X_0 such that for all ε>0, the set $\{k \in N : \bar{d}\,(X_k, X_0) \geq \varepsilon\} \in I$. We write I-lim $X_k = X_0$.

A sequence (X_k) of fuzzy numbers is said to be I- bounded if there exists a real number μ such that the set $\{k \in N : \bar{d}\,(X_k, \bar{0}) > \mu\} \in I$.

If $I = I_f$, then I_f convergence coincides with the usual convergence of fuzzy sequences. If $I = I_d\,(I_\delta)$, then $I_d\,(I_\delta)$ convergence coincides with statistical convergence (logarithmic convergence) of fuzzy sequences. If $I = I_u$, I_u convergence is said to be uniform convergence of fuzzy sequences.

Throughout $c^{I(F)}$, $c_0^{I(F)}$ and $\ell_\infty^{I(F)}$ denote the spaces of fuzzy real-valued *I- convergent, I-null* and *I- bounded* sequences respectively.

It is clear from the definitions that $c_0^{I(F)} \subset c^{I(F)} \subset \ell_\infty^{I(F)}$ and the inclusions are proper.

It can be easily shown that $\ell_\infty^{I(F)}$ is complete with respect to the metric ρ defined by f $(X, Y) = \sup_n \bar{d}\,(X_k, Y_k)$, where $X = (X_k)$, $Y = (Y_k) \in \ell_\infty^{I(F)}$.

Lemma 2.1:

(a) The condition $\sup_k f_k(t) < \infty, t > 0$ hold iff there exists $t_0 > 0$ such that $\sup_k f_k(t_0) < \infty$.

(b) The condition $\inf_k f_k(t) > \infty, t > 0$ hold iff there exists $t_0 > 0$ such that $\inf_k f_k(t_0) > \infty$.

Lemma 2.2: Let (α_k) and (β_k) be sequences of real or complex numbers and (p_k) be a bounded sequence of positive real numbers , then

$$|\alpha_k + \beta_k|^{p_k} \leq D(|\alpha_k|^{p_k} + |\beta_k|^{p_k})$$

and $$|\lambda|^{p_k} \leq max(1, |\lambda|^H)$$

where D = $max(1, |\lambda|^{H-1}), H = supp_k$, λ is any real or complex number.

Let $\boldsymbol{F} = (f_k)$ be a sequence of modulus functions, $p = (p_k)$ be a bounded sequence of strictly positive real numbers and $v = (v_k)$ a sequence of positive real numbers. We define the following new sequence spaces as:

$$\left(c^I\right)^F (\boldsymbol{F}, p, v) = \left\{X = (X_k) \in w^F : I - \lim f_k([\overline{d}(v_k X_k, X_0)]^{p_k}) = 0,\ for X_0 \in L(R)\right\} \in I$$

$$\left(c_0^I\right)^F (\boldsymbol{F}, p, v) = \left\{X = (X_k) \in w^F : I - \lim f_k([\overline{d}(v_k X_k, \overline{0})]^{p_k}) = 0\right\} \in I$$

$$\left(l_\infty^I\right)^F (\boldsymbol{F}, p, v) = \left\{X = (X_k) \in w^F : I - \sup_k f_k([\overline{d}(v_k X_k, \overline{0})]^{p_k}) < \infty\right\} \in I$$

Some special cases:

a. If $\boldsymbol{F} = f_k(x) = x$ for all k, then we have

$$\left(c^I\right)^F (p, v) = \left\{X = (X_k) \in w^F : I - \lim[\overline{d}(v_k X_k, X_0)]^{p_k} = 0,\ for X_0 \in L(R)\right\} \in I$$

$$\left(c_0^I\right)^F (p, v) = \left\{X = (X_k) \in w^F : I - \lim[\overline{d}(v_k X_k, \overline{0})]^{p_k} = 0\right\} \in I$$

$$\left(l_\infty^I\right)^F (p, v) = \left\{X = (X_k) \in w^F : I - \sup_k[\overline{d}(v_k X_k, \overline{0})]^{p_k} < \infty\right\} \in I$$

b. If $(p_k) = 1$ and $(v_k) = 1$ for all $k \in N$, we have

$$\left(c^I\right)^F (\boldsymbol{F}) = \left\{X = (X_k) \in w^F : I - \lim f_k([\overline{d}(X_k, X_0)]) = 0,\ for X_0 \in L(R)\right\} \in I$$

$$\left(c_0^I\right)^F (\boldsymbol{F}) = \left\{X = (X_k) \in w^F : I - \lim f_k([\overline{d}(X_k, \overline{0})]) = 0\right\} \in I$$

$$\left(l_\infty^I\right)^F (\boldsymbol{F}) = \left\{X = (X_k) \in w^F : I - \sup_k f_k([\overline{d}(X_k, \overline{0})]) < \infty\right\} \in I$$

c. If $\boldsymbol{F} = f_k(x) = x$ and $(p_k) = 1$ for all $k \in N$, then

$$\left(c^I\right)^F (v) = \left\{X = (X_k) \in w^F : I - \lim[\overline{d}(v_k X_k, X_0)] = 0,\ for X_0 \in L(R)\right\} \in I$$

$$\left(c_0^I\right)^F (v) = \left\{X = (X_k) \in w^F : I - \lim[\overline{d}(v_k X_k, \overline{0})] = 0\right\} \in I$$

$$\left(l_\infty^I\right)^F (v) = \left\{X = (X_k) \in w^F : I - \sup_k[\overline{d}(v_k X_k, \overline{0})] < \infty\right\} \in I$$

d. If $\boldsymbol{F} = f_k(x) = x$, $(p_k) = 1$ and $(v_k) = 1$ for all $k \in N$, then

$$\left(c^I\right)^F = \left\{X = (X_k) \in w^F : I - \lim[\overline{d}(X_k, X_0)] = 0,\ for X_0 \in L(R)\right\} \in I$$

$$\left(c_0^I\right)^F = \left\{X = (X_k) \in w^F : I - \lim[\bar{d}(X_k, \bar{0})] = 0\right\} \in I$$

$$\left(l_\infty^I\right)^F = \left\{X = (X_k) \in w^F : I - \sup_k[\bar{d}(X_k, \bar{0})] < \infty\right\} \in I$$

e. If $(v_k) = 1$ for all $k \in N$, then-

$$\left(c^I\right)^F (\boldsymbol{F}, p) = \left\{X = (X_k) \in w^F : I - \lim f_k([\bar{d}(X_k, X_0)]^{p_k}) = 0, \ for X_0 \in L(R)\right\} \in I$$

$$\left(c_0^I\right)^F (\boldsymbol{F}, p) = \left\{X = (X_k) \in w^F : I - \lim f_k([\bar{d}(X_k, \bar{0})]^{p_k}) = 0\right\} \in I$$

$$\left(l_\infty^I\right)^F (\boldsymbol{F}, p) = \left\{X = (X_k) \in w^F : I - \sup_k f_k([\bar{d}(X_k, \bar{0})]^{p_k}) < \infty\right\} \in I$$

3. Main results

Theorem 3.1: Let $\boldsymbol{F} = (f_k)$ be a sequence of modulus functions, then $\left(c^I\right)^F (\boldsymbol{F}, p, v)$, $\left(c_0^I\right)^F (\boldsymbol{F}, p, v)$ and $\left(l_\infty^I\right)^F (\boldsymbol{F}, p, v)$ are linear spaces.

Proof: We will prove the result for $\left(c_0^I\right)^F (\boldsymbol{F}, p, v)$.

Let, $X = (X_k)$ and $Y = (Y_k) \in \left(c_0^I\right)^F (\boldsymbol{F}, p, v)$. For scalars $\alpha, \beta \in C$, there exist integers a_α and b_β such that $|\alpha| \le a_\alpha$ and $|\beta| \le b_\beta$. Since $\boldsymbol{F} = (f_k)$ be a sequence of modulus functions, we have –

$$f_k([\bar{d}(v_k(\alpha X_k + \beta Y_k), \bar{0})]^{p_k}) \le D(a_\alpha)^H f_k([\bar{d}(v_k X_k, \bar{0})]^{p_k}) + D(b_\beta)^H f_k([\bar{d}(v_k Y_k, \bar{0})]^{p_k})$$

$$\to 0 \text{ as } k \to \infty .$$

Therefore, $\alpha X_k + \beta Y_k \in \left(c_0^I\right)^F (\boldsymbol{F}, p, v)$. This completes the proof.

Theorem 3.2: Let $\boldsymbol{F} = (f_k)$ be a sequence of modulus functions, then

$\left(l_\infty^I\right)^F (p, v) \subset \left(l_\infty^I\right)^F (\boldsymbol{F}, p, v).$

Proof: Let $X = (X_k) \in \left(l_\infty^I\right)^F (p, v)$, then we have $I - \sup_k f_k([\bar{d}(v_k X_k, \bar{0})]^{p_k}) < \infty.$ Let $\varepsilon > 0$ and choose a $\delta > 0$ with $0 < \delta < 1$ such that $f_k(t) < \varepsilon$ for $0 \le \delta \le 1$. Thus

$$I-\sup_k f_k([\bar{d}(v_k X_k,\bar{0})]^{p_k}) = I-\sup_{k,\bar{d}(X_k,\bar{0})\le\delta} f_k([\bar{d}(v_k X_k,\bar{0})]^{p_k}) + I-\sup_{k,\bar{d}(X_k,\bar{0})>\delta} f_k([\bar{d}(v_k X_k,\bar{0})]^{p_k})$$

$$\le \varepsilon + \frac{M}{\delta}\sup_k\left[\left(\bar{d}(v_k X_k,\bar{0})\right)^{p_k}\right] \quad \text{by properties of modulus function.}$$

$$< \infty$$

Hence $X=(X_k)\in\left(l_\infty^I\right)^F(\boldsymbol{F},p,v)$. This completes the proof.

Theorem 3.3: Let $\boldsymbol{F}=(f_k)$ be a sequence of modulus functions and $\alpha=\lim_{t\to\infty}\frac{f_k(t)}{t}>0$, then $\left(l_\infty^I\right)^F(\boldsymbol{F},p,v)\subset\left(l_\infty^I\right)^F(p,v)$.

Proof: Let $X=(X_k)\in\left(l_\infty^I\right)^F(\boldsymbol{F},p,v)$. By definition of α , we have $f_k(t)\ge\alpha.t$ for all $t\ge 0$.

Since $\alpha>0$, we have $t\le\frac{f_k(t)}{\alpha}$.

Thus,

$$I-\sup_k([\bar{d}(v_k X_k,\bar{0})]^{p_k}) \le I-\frac{1}{\alpha}\sup_k f_k([\bar{d}(v_k X_k,\bar{0})]^{p_k})$$

$$< \infty$$

This follows that $X=(X_k)\in\left(l_\infty^I\right)^F(p,v)$.

Theorem 3.4: Let $\boldsymbol{F}=(f_k)$ be a sequence of modulus functions, then

$\left(l_\infty^I\right)^F(v)\subset\left(c_0^I\right)^F(\boldsymbol{F},p,v)$ if $\lim_{t\to\infty} f_k(t)=0$ for $t>0$.

Proof: It is easy to prove, so omitted.

Theorem 3.5: Let $\boldsymbol{F}=(f_k)$ be a sequence of modulus functions and if $\lim_{t\to\infty} f_k(t)=\infty$ for $t>0$ then $\left(l_\infty^I\right)^F(\boldsymbol{F},p,v)\subset\left(c_0^I\right)^F(v)$.

Proof: Let $\lim_{t\to\infty} f_k(t)=\infty$ for $t>0$. If $X=(X_k)\in\left(l_\infty^I\right)^F(\boldsymbol{F},p,v)$.

Then,

$$f_k([\bar{d}(v_k X_k,\bar{0})]^{p_k})\le M<\infty \text{ for all } k.$$

If possible let $X=(X_k)\notin\left(c_0^I\right)^F(v)$. Then for some $\varepsilon>0$ there exists a positive integer k_0 such that $\bar{d}(v_k X_k,\bar{0})<\varepsilon$ for $k\ge k_0$.

Therefore,

$$f_k(\varepsilon) \geq f_k([\overline{d}(v_k X_k, \overline{0})]^{p_k}) \leq M \text{ for } k \geq k_0.$$

This contradicts to our assumption that $\lim_{t\to\infty} f_k(t) = \infty$ for $t > 0$ and hence

$X = (X_k) \in \left(c_0^I\right)^F (v)$.

Theorem 3.6: Let $F = (f_k)$ be a sequence of modulus functions then $\left(c_0^I\right)^F (\boldsymbol{F}, p, v)$ and $\left(l_\infty^I\right)^F (\boldsymbol{F}, p, v)$ are paranormed spaces with the paranorm

$$h(X) = \sup_k \left\{ f_k[\overline{d}(v_k X_k, \overline{0})]^{p_k} \right\}^{\frac{1}{M}}$$

Where $M = \max\left\{1, \sup_k p_k\right\}$

Proof: Obviously $h(X) = h(-X)$ for all $X \in \left(c_0^I\right)^F (\boldsymbol{F}, p, v)$

It is trivial that $v_k X_k = \overline{0}$ for $X = \overline{0}$.

Since $\frac{p_k}{M} \leq 1$, since $\overline{d}$ is translation invariant and by using Minkowski's inequality , we have

$$\left\{ f_k[\overline{d}(v_k (X_k + Y_k), \overline{0})]^{p_k} \right\}^{\frac{1}{M}} \leq \left\{ f_k[\overline{d}(v_k X_k, \overline{0})]^{p_k} \right\}^{\frac{1}{M}} + \left\{ f_k[\overline{d}(v_k Y_k, \overline{0})]^{p_k} \right\}^{\frac{1}{M}}$$

Hence,

$$h(X + Y) \leq h(X) + h(Y)$$

Finally to check the continuity of scalar multiplication, let λ be any scalar, by definition we have

$$h(\lambda X) = \sup_k \left\{ f_k[\overline{d}(v_k \lambda X_k, \overline{0})]^{p_k} \right\}^{\frac{1}{M}} \leq K_\lambda^{\frac{H}{M}} h(X)$$

where $H = \sup_k p_k < \infty$.

Where K_λ is positive integer such that $|\lambda| \leq K_\lambda$. Let $\lambda \to 0$ for any fixed X with $h(X) = 0$.

By definition for $|\lambda| \leq 1$, we have

$$\sup_k \left\{ f_k[\overline{d}(v_k \lambda X_k, \overline{0})]^{p_k} \right\} \leq \varepsilon \text{ for } n > N(\varepsilon).$$

Also for $1 \le n \le N$ by taking λ small enough, since f_k is continuous, we get

$$\sup_k \left\{ f_k[\overline{d}(v_k \lambda X_k, \overline{0})]^{p_k} \right\} \le \varepsilon.$$

Implies that $h(\lambda X) \to 0$ as $\lambda \to 0$. This completes the proof.

Theorem 3.7: If I is an admissible ideal then the spaces $\left(c^I\right)^F (\boldsymbol{F}, p)$, $\left(c_0^I\right)^F (\boldsymbol{F}, p)$ and $\left(l_\infty^I\right)^F (\boldsymbol{F}, p)$ are complete metric spaces under the metric –

$$h(X,Y) = \sup_k \left\{ f_k[\overline{d}(X_k, Y_k)]^{p_k} \right\}^{\frac{1}{M}}$$

Where $M = \max\left\{1, \sup_k p_k\right\}$

Proof: It is easy to see that h is a metric on $\left(c^I\right)^F (\boldsymbol{F}, p)$. To show completeness.

Let $\left(X^i\right)$ be a Cauchy sequence in $\left(c^I\right)^F (\boldsymbol{F}, p)$ where $\left(X^i\right) = \left(X_k^i\right)$.

Therefore for each $\varepsilon > 0$ there exists $i_0 \in N$ such that

$$h(X^i, X^j) < \varepsilon \quad \text{for all } i, j \ge i_0.$$

i.e

$$\sup_k \left\{ f_k[\overline{d}(X_k^i, X_k^j)]^{p_k} \right\}^{\frac{1}{M}} < \varepsilon \quad \text{for all } i, j \ge i_0.$$

This means

$$\sup_k (f_k[\overline{d}(X_k^i, X_k^j)]^{p_k}) < \varepsilon \quad \text{for all } i, j \ge i_0.$$

Since f is modulus function, so choosing suitable $\varepsilon_1 > 0$ and we obtain

$$\overline{d}(X_k^i, X_k^j) < \varepsilon_1 \quad \text{for all } i, j \ge i_0 \quad \text{and for each } k.$$

i.e

$\left(X_k^i\right)$ is a Cauchy sequence in $L(R)$ for each k.

Keeping i fixed and letting $j \to \infty$, one can find that –

$$\sup_k (f_k[\overline{d}(X_k^i, X_k)]^{p_k}) < \varepsilon \qquad \text{for all } i \ge i_0 .$$

That means,

$$h\left(X^i, X\right) < \varepsilon \qquad \text{for all } i \ge i_0 .$$

Next to show $X \in \left(C^I\right)^F (\boldsymbol{F}, p)$, for which the proof as follows:

Since $\left(X_k^i\right) \in \left(C^I\right)^F (\boldsymbol{F}, p)$ for $i \in N$, so for i, j, there exist $L^i, L^j \in L(R)$ and $k_i, k_j \in N$

Such that

$$\sup_k (f_k[\overline{d}(X_k^i, L^i)]^{p_k}) < \varepsilon \qquad \text{for all } k \ge k_i .$$

And

$$\sup_k (f_k[\overline{d}(X_k^j, L^j)]^{p_k}) < \varepsilon \qquad \text{for all } k \ge k_j .$$

Now let $k_0 = \max\left(k_i, k_j\right)$ and $i, j \ge i_0$, we have

$$\begin{aligned}
\sup_k (f_k[\overline{d}(L^i, L^j)]^{p_k}) &\le C \sup_k (f_k[\overline{d}(L^i, X_k^i)]^{p_k}) \\
&+ C \sup_k (f_k[\overline{d}(X_k^i, X_k^j)]^{p_k}) \\
&+ C \sup_k (f_k[\overline{d}(X_k^j, L^j)]^{p_k}) \\
&< 3C\varepsilon \quad \text{for all } i, j \ge i_0 \text{ and } k \ge k_0 .
\end{aligned}$$

Hence $\left(L^i\right)$ is a Cauchy sequence in $L\left(R\right)$. So there exists $L \in L(R)$ such that

$$L^i \to L \text{ as } i \to \infty$$

Now keeping i fixed and letting $j \to \infty$, once can find

$$\sup_k (f_k[\overline{d}(L^i, L)]^{p_k}) < 3C\varepsilon \text{ for all } i \ge i_0 \text{ and } k \ge k_0 .$$

Therefore,

$$\begin{aligned}
\sup_k (f_k[\overline{d}(X_k, L)]^{p_k}) &\le C \sup_k (f_k[\overline{d}(X_k, X_k^{i_0})]^{p_k}) \\
&+ \sup_k (f_k[\overline{d}(X_k^i, L^i)]^{p_k})
\end{aligned}$$

$$+\sup_k (f_k[\overline{d}(L^{i_0}, L)]^{p_k})$$

$$< 2C\varepsilon + 3C^2\varepsilon \cong \varepsilon_1 \quad \text{for all } k \geq k_0 .$$

This implies that $X = (X_k) \in (c^I)^F (\boldsymbol{F}, p)$. This completes the proof.

Conflict of Interests

The author declares that there is no conflict of interests.

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