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NUMERICAL TREATMENT FOR FIRST ORDER NEUTRAL DELAY DIFFERENTIAL EQUATIONS USING SPLINE FUNCTIONS

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Abstract. In this article, we introduce a new technique to find an approximate solution for first order neutral delay differential equations. This technique depends on approximate the solution using the spline functions expansion. Special attention is given to study the error estimation and the convergence of the proposed method. Also, the stability of the technique is presented. The numerical results are compared with the conventional approximate method, variational iteration method.

Keywords: neutral delay differential equations; spline functions expansion; stability analysis; error estimation; variational iteration method.

2000 AMS Subject Classification: 65N18.

1. Introduction

In fact, the neutral delay differential equations appear in modelling of the networks containing lossless transmission lines (as in high-speed computers where the lossless transmission lines are used to interconnect switching circuits), in the study of vibrating masses attached to an elastic bar, as the Euler equation in some variational problems, theory of automatic control and in neuromechanical systems in which inertia plays an important role ([3], [4], [5], [9]).

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Consider the following first order neutral delay differential equation:

$$y'(x) = f\left(x, y(x), y(g(x)), y'(g(x))\right), \qquad a \le x \le b, \tag{1.1}$$

with the following initial condition:

$$y(x_0) = y_0, \qquad y(x) = \phi(x), \quad x \in [a^*, a], \tag{1.2}$$

where f is a given function and y is the unknown function to be found in the interval $[a, b]$. The authors ([2]-[8]) have introduced the different methods to the approximate solution of neutral differential equations. Also, the authors ([13], [15], [19]) have studied spline approximation for solving differential equations with deviating argument and some others discussed the numerical treatment of delay differential equations ([14], [17], [18]) and second order Fredholm integro-differential equations ([1], [11], [16]). The introduced method is a one-step method $o(h^{m+\alpha})$ in $y^i(x),\ i = 0, 1$. Assuming that $f \in C[a, b] \times R^3$, $0 < \alpha \le 1$ and m is an arbitrary positive integer number which is the number of iterations used in computing the spline functions defined in the method.

The rest of this paper is organized as follows: Section 2 is assigned to introduce some assumptions and procedure of the proposed method. In section 3, the error estimation and convergence are given. In section 4, the stability of the method is presented. In section 5, a test problem has been solved by the proposed method, to illustrate and show the efficiency of the proposed method. Also, the conclusions and remarks will appear in section 6.

2. Assumptions and procedure solution

We shall consider Eqs.(1.1)-(1.2) in a case, the delay function g is assumed to be continuous in the interval $[a, b]$, $\ \phi \in C[a^*, a]$, $\ a^* \le g(x) \le x,\ x \in [a, b]$.
Suppose that the function $f : [a, b] \times R^3 \to R$ is continuous and satisfies the Lipschitz condition:

$$|f(x, y_1, v_1, z_1) - f(x, y_2, v_2, z_2)| \le L\left[\,|y_1 - y_2| + |v_1 - v_2| + |z_1 - z_2|\,\right], \tag{2.1}$$

and there are two constants c_1 and c_2 such that:

$$\big|v_1 - v_2\big| \le c_1 \,\big|f(x, y_1, v_1, z_1) - f(x, y_2, v_2, z_2)\big|, \tag{2.2}$$

$$\big|z_1 - z_2\big| \le c_2 \,\big|f(x, y_1, v_1, z_1) - f(x, y_2, v_2, z_2)\big|, \tag{2.3}$$

with $L(c_1 + c_2) < 1$ for all (x, y_1, v_1, z_1) and (x, y_2, v_2, z_2) in the domain of definition of the function f. These conditions assure the existence of the unique solution of problem (1.1).

Let $\triangle$ be an uniform partition of the interval $[a, b]$ defined by the nodes

$$\triangle := a = x_0 < x_1 < x_2 < ... < x_k < x_{k+1} < ... < x_n = b,$$

where $x_k = x_0 + kh,\ h = \frac{b-a}{n} < 1$ and $k = 0, 1, ..., n-1$.

We define the spline function approximating the solution $y(x)$ by $S(x)$ where

$$S(x) = \begin{cases} S_\Delta(x), & a \le x \le b; \\ \phi(x), & a^* \le x \le a. \end{cases}$$

Assume that the function y' has a modulus of continuity:

$$w(y', h) = w(h) = o(h^\alpha), \quad 0 < \alpha \le 1.$$

Choosing the required positive integer number m, then for any $[\,x_k, x_{k+1}\,]$, $k = 0, 1, 2, ...n-1$, we define the spline function approximating the solution $y(x)$ by $S_\Delta(x)$ where

$$S_\Delta(x) = S_k^m(x) = S_{k-1}^m(x_k) + \int_{x_k}^{x} f(x, S_k^{m-1}(x), S_k^{m-1}(g(x)), S_k^{\prime\, m-1}(g(x)))dx, \tag{2.4}$$

where $\ S_{-1}^m(x_0) = y_0, \quad S\,_{-1}^{m}(g(x_0)) = \phi(g(x_0)), \quad$ and $\ S'\,_{-1}^{m}(g(x_0)) = \phi'(g(x_0))$.

In Eq.(2.4) we use the following m iterations for $x \in [x_k, x_{k+1}]$, $k = 0, 1, 2, ..., n-1$, $j = 1, 2, ..., m$.

$$S_k^j(x) = S_{k-1}^m(x_k) + \int_{x_k}^{x} f(x, S_k^{j-1}(x), S_k^{j-1}(g(x)), S_k^{\prime\, j-1}(g(x)))dx, \tag{2.5}$$

where

$$S_k^0(x) = S_{k-1}^m(x_k) + \sum_{i=0}^{r} \frac{M_k^i (x - x_k)^{i+1}}{(i+1)!}, \tag{2.6}$$

$$M_k^i = f\left(x_k, S_{k-1}^i(x_k), S_{k-1}^i(g(x_k)), S'\,_{k-1}^{i}(g(x_k))\right), \tag{2.7}$$

it is clear that $S_\Delta(x) \in C[a,b]$ exists and unique.

The Eqs.(2.5)-(2.7) present the main scheme which obtained from the proposed method. From this scheme, we can obtain the approximate solution of the problem (1.1). The error estimate and the convergence of this scheme is studied in the following section.

3. Error estimation and convergence

To estimate the error, it is convenient to represent the exact solution $y(x)$ in various forms as described by the following scheme:

$$y^0(x) = y(x) = y_k + \sum_{i=0}^{r-1} \frac{y_k^{i+1}(x-x_k)^{i+1}}{(i+1)!} + \frac{y^{r+1}(\xi_k)(x-x_k)^{r+1}}{(r+1)!}, \tag{3.1}$$

where $\xi_k \in (x_k, x_{k+1})$, $y_k = y(x_k)$. For $i = 1, 2, ..., m$, we can write

$$y^i(x) = y(x) = y_k + \int_{x_k}^{x} f\left(x, y^{i-1}(x), y^{\,i-1}(g(x)), y'^{\,i-1}(g(x))\right) dx. \tag{3.2}$$

Moreover, we denote to the estimated error of $y^i(x)$ at any point $x \in [a,b]$ where $i = 0, 1$ by:

$$e(x) = \left| y(x) - S_\Delta(x) \right|, \qquad e_k = \left| y_k - S_\Delta(x_k) \right|. \tag{3.3}$$

Lemma 3.1. *Let α and β be non-negative real numbers and $\{A_i\}_{i=0}^m$ be a sequence satisfying $A_i \le \alpha + \beta A_{i+1}$ for $i = 1, 2, ..., m-1$, then:*

$$A_1 \le \beta^{m-1} A_m + \alpha \sum_{i=0}^{m-2} \beta^i.$$

Lemma 3.2. *Let α and β be non-negative real numbers, $\beta \ne 1$ and $\{A_i\}_{i=0}^k$ be a sequence satisfying $A_0 \ge 0$ and $A_{i+1} \le \alpha + \beta A_i$ for $i = 0, 1, ..., k$, then:*

$$A_{k+1} \le \beta^{k+1} A_0 + \alpha \left[\frac{\beta^{k+1} - 1}{\beta - 1} \right].$$

Definition 3.1. *For any $x \in [x_k, x_{k+1}]$, $k = 0, 1, ..., n-1$ and $j = 1, 2, ..., m$, we define the operator $T_{kj}(x)$ by*

$$T_{kj}(x) = \left| y^{m-j}(x) - S_k^{m-j}(x) \right|,$$

whose norm is defined by

$$||T_{kj}|| = \max_{x \in [x_k, x_{k+1}]} \{T_{kj}(x)\}.$$

Lemma 3.3. *For any $x \in [x_k, x_{k+1}]$, $k = 0, 1, ..., n-1$ and $j = 1, 2, ..., m$, then*

$$||T_{km}|| \leq (1 + hd_1)e_k + d_2 h^{r+1} w(h), \tag{3.4}$$

$$||T_{k1}|| \leq d_3 e_k + d_4 h^{r+m} w(h), \tag{3.5}$$

such that

$$d_0 = \frac{L}{1 - L(c_1 + c_2)},\ d_1 = d_0 \sum_{i=0}^{r} \frac{1}{(1+i)!},\ d_2 = \frac{1}{(1+i)!},\ d_3 = \sum_{i=0}^{m-1} d_0^i + d_0^{m-1} d_1,\ \textit{and}\ d_4 = d_0^{m-1} d_2.$$

where the constants L, c_1 and c_2 are defined above in (2.1)-(2.3).

Proof.

Using (2.1), (2.2), (2.3), (2.6), (3.1) and (3.3), we get:

$$\begin{aligned} T_{km}(x) = \left|y^0(x) - S_k^0(x)\right| &\leq \left|y_k - S_{k-1}^m(x_k)\right| + \sum_{i=0}^{r-1} \frac{|y_k^{i+1} - M_k^i|\,|x - x_k|^{i+1}}{(i+1)!} \\ &\quad + \frac{|y^{r+1}(\xi_k) - M_k^r|\,|x - x_k|^{r+1}}{(r+1)!}. \end{aligned} \tag{3.6}$$

Since

$$\begin{aligned} \left|y_k^{i+1} - M_k^i\right| &= \left|f^{(i)}(x_k, y_k, y(g(x_k)), y'(g(x_k))) - f^{(i)}(x_k, S_{k-1}^m(x_k), S_{k-1}^m(g(x_k)), {S'}_{k-1}^{m}(g(x_k)))\right| \\ &\leq \frac{L}{1 - L(c_1 + c_2)} \left|y_k - S_{k-1}^m(x_k)\right| = d_0 e_k, \end{aligned} \tag{3.7}$$

where d_0 defined above. Similarly:

$$|y^{r+1}(\xi_k) - M_k^r| \leq |y^{r+1}(\xi_k) - y_k^{r+1}| + |y_k^{r+1} - M_k^r| \leq w(h) + d_0 e_k.$$

Using (3.7) in (3.6), we get:

$$\begin{aligned} ||T_{km}|| = \max_{x \in [x_k, x_{k+1}]} \{T_{km}(x)\} &\leq e_k + \sum_{i=0}^{r-1} \frac{d_0 e_k h^{i+1}}{(i+1)!} + \frac{h^{r+1}}{(r+1)!}\Big[w(h) + d_0 e_k\Big] \\ &\leq (1 + hd_1)e_k + d_2 h^{r+1} w(h), \end{aligned}$$

where d_1 and d_2 are defined above

To prove (3.5), we compute $||T_{kj}||$ using (2.1), (2.2), (2.3), (2.5), (3.2) and (3.3), we get:

$$T_{kj}(x) = \left|y^{m-j}(x) - S_k^{m-j}(x)\right| \leq e_k + d_0 \int_{x_k}^{x} T_{k(j+1)}(x)dx,$$

$$||T_{kj}|| = \max_{x \in [x_k, x_{k+1}]} \{T_{kj}(x)\} \leq e_k + d_0 h ||T_{k(j+1)}||.$$

Using Lemma 3.1, and the inequality (3.4), we get:

$$\begin{aligned}||T_{k1}|| &\leq (d_0h)^{m-1}||T_{km}|| + \Big[\sum_{i=0}^{m-2}(d_0h)^i\Big]e_k \\ &\leq \Big[\sum_{i=0}^{m-2} d_0^i + d_0^{m-1}d_1\Big]e_k + d_0^{m-1}d_2h^{m+r}w(h) \\ &\leq d_3e_k + d_4h^{m+r}w(h),\end{aligned}$$

where d_3 and d_4 are constants independent of h and defined above.

Lemma 3.4. *Let $e(x)$ be defined as in (3.3), if there exist constants d_5, d_6, independent of h, then the following inequality holds:*

$$e(x) \leq (1 + hd_5)e_k + d_6h^{m+r+1}w(h).$$

Proof.

Using (2.1), (2.2), (2.3), (2.4), (3.2), (3.3) and (3.5), we get:

$$\begin{aligned}e(x) = |y(x) - S_\Delta(x)| &\leq e_k + d_0\int_{x_k}^{x} \max_{x\in[x_k,x_{k+1}]}\{T_{k1}(x)\}dx \leq e_k + hd_0||T_{k1}|| \\ &\leq (1 + hd_5)e_k + d_6h^{m+r+1}w(h),\end{aligned} \tag{3.8}$$

where $d_5 = d_0d_3$ and $d_6 = d_0d_4$ are constants independent of h. The inequality (3.8) holds for any $x \in [a, b]$. Setting $x = x_{k+1}$, we get:

$$e_{k+1} \leq (1 + hd_5)e_k + d_6h^{m+r+1}w(h).$$

Using Lemma 3.2 and noting that $e_0 = 0$, we get

$$e(x) \leq d_7h^{m+r}w(h) = o(h^{m+r+\alpha}), \tag{3.9}$$

where $d_7 = \frac{d_6}{d_5}\Big[e^{d_5(b-a)} - 1\Big]$ is a constant independent of h.

Now, we are going to estimate $|y'(x) - S'_\Delta(x)|$. For this purpose we use (2.1), (2.2), (2.3), (2.4), (3.2), (3.3), (3.5) and (3.9), we get

$$|y'(x) - S'_\Delta(x)| \leq d_0||T_{k1}|| \leq d_0\left[d_3e_k + d_4h^{m+r}w(h)\right] \leq d_8h^{m+r}w(h),$$

where $d_8 = d_0[d_3d_7 + d_4]$.

Hence from above Lemma we have arrive to the following theorem.

Theorem 3.1. *Let $y(x)$ be the exact solution of the problem (1.1), $S_\triangle(x)$ given by (2.4) is the approximate solution for the same problem, $f \in C[a,b] \times R^3$, then there exist a constant p independent of h, such that the following inequalities*

$$\left|y^{(q)}(x) - S_\triangle^{(q)}(x)\right| \le p h^{m+r} w(h),$$

hold for all $x \in [a,b]$ and $q = 0,1$.

4. Stability of the proposed method

To study the stability of the proposed method given by (2.4), we change $S_\triangle(x)$ to $W_\triangle(x)$ where

$$W_\triangle(x) = W_k^m(x) = W_{k-1}^m(x_k) + \int_{x_k}^{x} f(x, W_k^{m-1}(x), W_k^{m-1}(g(x)), W_k'^{m-1}(g(x)))dx, \quad (4.1)$$

where $W_{-1}^m(x_0) = y_0^*$, $W\,{}_{-1}^{m}(g(x_0)) = \phi(g(x_0))$. In Eq.(4.1) we use the following m iterations, i.e., for $x \in [x_k, x_{k+1}]$, $k = 0,1,...,n-1$ and $j = 1,2,...,m$ we obtained

$$W_k^i(x) = W_{k-1}^m(x_k) + \int_{x_k}^{x} f(x, W_k^{m-1}(x), W_k^{m-1}(g(x)), W_k'^{m-1}(g(x)))dx, \quad (4.2)$$

$$W_k^0(x) = W_{k-1}^m(x_k) + \sum_{i=0}^{r} \frac{N_k^i (x-x_k)^{i+1}}{(i+1)!}, \quad (4.3)$$

$$N_k^i = f^{(i)}\left(x_k, W_{k-1}^m(x_k), W_{k-1}^m(g(x_k)), W'\,{}_{k-1}^{m-1}(g(x_k))\right). \quad (4.4)$$

Moreover, we use the following notation:

$$e^*(x) = \left|S_\triangle(x) - W_\triangle(x)\right|, \quad e_k^* = \left|S_\triangle(x_k) - W_\triangle(x_k)\right|. \quad (4.5)$$

Definition 4.1 *For any $x \in [\,x_k, x_{k+1}\,]$, $k = 0,1,...,n-1$ and $j = 1,2,...,m$, we define the operator $T_{kj}^*(x)$ by:*

$$T_{kj}^*(x) = \left|S_k^{m-j}(x) - W_k^{m-j}(x)\right|,$$

whose norm is defined by:

$$||T_{kj}^*|| = \max_{x \in [x_k, x_{k+1}]} \{T_{kj}^*(x)\}.$$

Lemma 4.1. *For any $x \in [\,x_k, x_{k+1}\,]$, $k = 0,1,...,n-1$ and $j = 1,2,...,m$, then:*

$$||T_{km}^*|| \le (1 + hd_1)e_k^*, \quad (4.6)$$

$$||T^*_{k1}|| \le d_3\, e^*_k, \tag{4.7}$$

where d_1 and d_3 are constants defined in Lemma 3.3.

Proof.

To prove (4.6)-(4.7), using (2.1), (2.2), (2.3), (2.6), (4.3) and (4.5). The proof is similar to the proof of Lemma 3.3.

Lemma 4.2.

Let $e^*(x)$, be defined as in (4.5), then the following inequality holds:

$$e^*(x) \le (1 + hd_5)e^*_k,$$

where d_5 is a constant defined as in Lemma 3.4.

Proof.

Using $(2.1), (2.2), (2.3),\ (2.4), (3.8), (4.5)$, and (4.7). The proof is similar to the proof of Lemma 3.4.

Theorem 4.1. *Let $S_\triangle(x)$ given by (2.4) be the approximate solution of the problem (1.1) with the initial condition $y(x_0) = y_0$ and let $W_\triangle(x)$ given by (4.1) be the approximate solution for the same problem with the initial condition $y^*(x_0) = y^*_0$ and $f \in C[a, b] \times R^3$, then the inequalities hold:*

$$\left| S^{(q)}_\triangle - W^{(q)}_\triangle \right| \le d_9\, e^*_0,$$

for all $x \in [a, b],\ q = 0, 1,$ and $e^*_0 = |y_0 - y^*_0|$, where d_9 is a constant independent of h.

5. Numerical example

In this section, we consider the following neutral delay differential equation:

$$y'(x) = \frac{1}{2}y(x) + \frac{1}{2}\, y(x/2).y'(x/2),$$

with the initial conditions, $y(0) = 1$. The exact solution of this problem is $y(x) = e^x$.

Table 1, shows the numerical results of this problem. In this table we compute the first approximate solution (First app. sol.), first absolute error (the difference between the exact and approximate solution before change), the second approximate solution (Second app. sol.) and the second absolute error (the difference between the first and second solutions), where $r = 2$, with different iteration number m at some values of $x = 0.1, 0.2, 0.3, 0.4, 0.5$.

The above simulation proves that the proposed method is a very useful numerical method to get accurate solutions to first order neutral delay differential equations.

Figure 1, presents a comparison between the exact solution, y_{exact}, the solution obtained from the proposed method, y_{spline} and the solution using the variational iteration method, y_{VIM} in the interval $[0, 1]$. From figure 1, we can deduce that the proposed method provides excellent approximations to the solution of related equation to first order neutral delay differential equations. The numerical results showed that this method has very accuracy and reductions of the size of calculations compared with the VIM ([10], [12], [20], [21]).

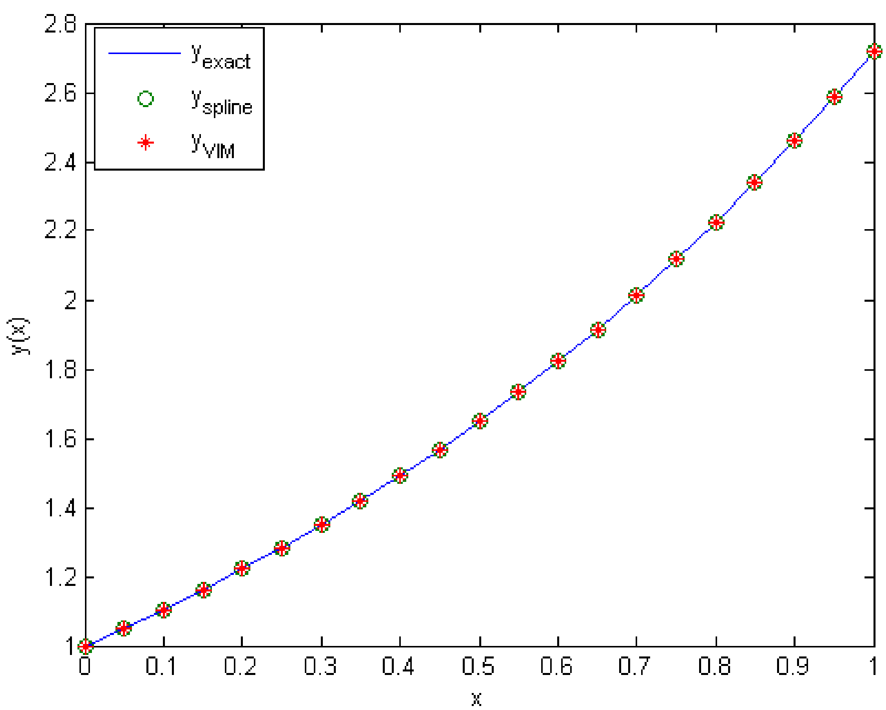

Figure 1. Comparison between the exact solution and the solution obtained from the proposed method with the solution using VIM.

6. Concluding remarks and discussion

This paper centralized to present a new method for solving the first order neutral delay differential equations. From the presented analysis shows that the proposed technique has much impact on the accuracy and efficiency of the solution on the first order neutral delay differential equations. We investigate the error estimation and the stability of the proposed method. The analytical approximation to the solutions is reliable, and confirms the power and ability of the proposed technique as an easy device for computing the solution of such these problems. Also, a comparison with the approximate method, variational iteration method is given. All computations in this paper are done using Matlab 7.1.

x	m	First app. sol.	First absolute error	Second app. sol.	Second absolute error
0.1	1	1.10517091	3.5×10^{-09}	1.10518247	1.1×10^{-5}
	2	1.10517091	1.3×10^{-10}	1.10518222	1.1×10^{-5}
	3	1.10517091	4.4×10^{-12}	1.10518292	1.2×10^{-5}
	4	1.10517091	1.4×10^{-13}	1.10518299	1.2×10^{-5}
	5	1.10517091	4.7×10^{-15}	1.10518303	1.2×10^{-5}
0.2	1	1.22140275	7.8×10^{-9}	1.22141653	1.4×10^{-5}
	2	1.22140273	2.1×10^{-8}	1.22141692	1.4×10^{-5}
	3	1.22140273	2.6×10^{-8}	1.22141710	1.4×10^{-5}
	4	1.22140273	1.9×10^{-8}	1.22141727	1.4×10^{-5}
	5	1.22140274	1.0×10^{-8}	1.22141732	1.4×10^{-5}
0.3	1	1.35774334	7.0×10^{-3}	1.35776052	7.9×10^{-3}
	2	1.35308270	3.2×10^{-3}	1.35309976	3.2×10^{-3}
	3	1.35093385	1.0×10^{-3}	1.35095107	1.0×10^{-3}
	4	1.50144873	2.9×10^{-4}	1.35016221	3.0×10^{-4}
	4	1.34991115	5.2×10^{-5}	1.34992858	6.9×10^{-4}
0.4	1	1.50189327	1.0×10^{-2}	1.50191342	1.0×10^{-2}
	2	1.49656938	4.7×10^{-3}	1.49658973	4.7×10^{-3}
	3	1.49382185	1.9×10^{-3}	1.49384232	2.0×10^{-3}
	4	1.49125616	2.3×10^{-4}	1.49258221	7.6×10^{-4}
	5	1.49205339	2.2×10^{-4}	1.49207150	2.5×10^{-4}
0.5	1	1.66054058	1.1×10^{-2}	1.66056441	1.7×10^{-2}
	2	1.65526313	6.5×10^{-3}	1.65528715	6.5×10^{-3}
	3	1.65195568	3.2×10^{-3}	1.65198099	3.2×10^{-3}
	4	1.65009713	1.3×10^{-3}	1.65012150	1.4×10^{-3}
	5	1.64920214	4.8×10^{-4}	1.64922664	5.0×10^{-4}

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