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WEAKLY *C*-CONTRACTIVE MAPPINGS IN CONE METRIC SPACES

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Abstract. In this article, we introduce the class of weakly c-contractive mappings in cone metric spaces. A fixed point theorem is established in the framework of cone metric spaces.

Keywords: cone metric space; C-contractive mappings; fixed point.

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1. Introduction

Recently, Huang and Zhang introduced the concept of cone metric spaces by replacing the set of real numbers with an ordered Banach space, for more details; see [4] and the references therein. Subsequently, many fixed point results concerning self mappings in such spaces have been investigated; see [2, 3, 5, 6, 7, 9, 10] and the references therein. In this article, we extent some results in [1] to the framework of cone metric spaces. In this paper, the cones are strongly minihedral and normal to endow the cone metric spaces with an appropriate topology; see [11].

The aim of this paper is to investigate fixed point problems of C-contractive mappings. A fixed point theorem is established in the framework of cone metric spaces.

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The organization of this paper is as follows. In Section 2, we provide some necessary preliminaries. In Section 3, a fixed point theorem is established in the framework of cone metric spaces. The result presented in this paper mainly generalizes the result of Binayak [1].

2. Preliminaries

We first recall some known definitions, notations and results concerning cones in Banach spaces.

Definition 2.1. Let E be a real Banach space with norm $\|.\|$ and let P be a subset of E. Then P is called a cone if and only if

(1) P is closed, nonempty and $P \neq \{\theta\}$, where θ is the zero vector in E;
(2) for any $a,b \geq 0$ (nonnegative real numbers), and $x,y \in P$, we have
$ax+by \in P$;
(3) for $x \in P$, if $-x \in P$, then $x = \theta$.

Given a cone P in a Banach space E, we define on E a partial order $\preceq$ with respect to P by

$$x \preceq y \Longleftrightarrow y - x \in P.$$

We also write $x \prec y$ whenever $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in Int(P)$, where $Int(P)$ stand for the interior of P.

The cone P is called normal if there is a real number $K > 0$, such that for all $x,y \in E$, we have

$$\theta \preceq x \preceq y \Longrightarrow \|x\| \leq K\|y\|.$$

The least positive number satisfying this inequality is called the normal constant of P. Therefore, we shall say that P is a K-normal cone to indicate the fact that the normal constant is K.

The cone is said to be regular if every increasing sequence which is bounded from above is convergent. That is, if (x_n) is a sequence such that $x_n \preceq x_2 \preceq \cdots \preceq y$ for some $y \in E$, then there exists $x^* \in E$ such that $\lim_{n\to\infty} \|x_n - x^*\| = 0$.

Lemma 2.1. [13, 15] *Every regular cone is normal. The cone P is regular if every decreasing sequence which is bounded from below is convergent.*

Definition 2.2. *Let X be a non empty set. A function $d : X \times X \to E$ is called a cone metric on X if:*

(d1) $\theta \preceq d(x,y)\ \ \forall\, x \in X$ *and* $d(x,y) = \theta$ *if and only if* $x = y$*;*

(d2) $d(x,y) = d(y,x) \quad \forall\, x,y \in X$*;*

(d3) $d(x,z) \preceq d(x,y) + d(y,z) \quad \forall\, x,y,z \in X$.

The pair (X,d) is called a cone metric space.

From the definition of the order given by a cone P, it is obvious that $x \in P \Longleftrightarrow \theta \preceq x$. Hence, we can define a concept of positivity on a Banach space as follow.

Definition 2.3. *Let E be a real Banach space. Let P be a cone on E and $\preceq$ the partial order with respect to P. An element $x \in E$ is said to be a nonnegative vector if $\theta \preceq x$ and positive vector if $\theta \prec x$. Hence P is the set of all nonnegative elements. We shall use the following notations:*

- $[\theta, \longrightarrow [\ := P = \{x \in E : \theta \preceq x\}$*;*
- $]\theta, \longrightarrow [\ := \{x \in E : \theta \prec x\}$.

Definition 2.4. *A subset A of E is said to be bounded from above with respect to P (or upper bounded) if there exists $x_0 \in E$ such that $a \preceq x_0$ for all $a \in A$. A subset A of E is said to be bounded from below with respect to P (or lower bounded) if there exists $x_0 \in E$ such that $x_0 \preceq a$ for all $a \in A$.*

Definition 2.5. *A cone P is said to be minihedral if $x \vee y := \sup\{x,y\}$ exists for all $x,y \in E$ and strongly minihedral if every subset of E which is bounded from above has a supremum.*

We also recall the following lemma, which we take from [11] and give the proof as it is there.

Lemma 2.6. *Let (X,d) be a quasi-cone metric space. Then for each $c \in E,\ c \gg \theta$, there exists $\sigma > 0$ such that $x \ll c$ whenever $\|x\| < \sigma,\ x \in E$.*

Proof. Since $c \gg \theta$, we ahve $c \in Int(P)$. Hence, we find $\sigma > 0$ such that $\{x \in E : \|x - c\| < \sigma\} \subset Int(P)$. If $\|x\| < \sigma$, then $\|(c-x) - c\| = \|-x\| = \|x\| < \sigma$ and hence $(c - x) \in Int(P)$.

Lemma 2.7. *Let (X,d) be a cone metric space over a cone K-normal cone P. Then one has*

a) $Int(P)+Int(P) \subset Int(P)$ *and* $\lambda Int(P) \subset Int(P)$ *for any positive real number* λ.

b) *For any given* $c \gg \theta$ *and* $c_0 \gg \theta$, *there exists* $n_0 \in \mathbb{N}$ *such that* $\frac{c_0}{n_0} \ll c$.

c) *If* (a_n) *and* (b_n) *are sequences in* E *such that* $a_n \to a$, $b_n \to b$ *and* $a_n \preceq b_n$ *for all* $n \geq 1$, *then* $a \preceq b$.

Proposition 2.8. *Let* (X,q) *be a cone metric space over a cone* P. *If* $a \preceq \lambda a$, *where* $0 \leq \lambda < 1$, *then* $a = \theta$.

Definition 2.9. *Let* (x_n) *be a sequence in a cone metric space* (X,d).

(a) (x_n) *is convergent to* $x \in X$ *and we denote* $\lim_{n\to\infty} x_n = x$, *if for every* $c \in E$ *with* $c \gg \theta$, *there exists* $n_0 \in \mathbb{N}$ *such that*

$$\forall\, n,m \geq n_0 \quad d(x_n,x) \ll c;$$

(b) (x_n) *is called Cauchy if for every* $c \in E$ *with* $c \gg \theta$, *there exists* $n_0 \in \mathbb{N}$ *such that*

$$\forall\, n,m \geq n_0 \quad d(x_n,x_m) \ll c.$$

Definition 2.10. *A cone metric space* (X,d) *is said to be complete if every Cauchy sequence in* X *is convergent in* X.

Lemma 2.11. [4] *Let* (X,d) *be a cone metric space over a cone* K*-normal cone* P. *The sequence* (x_n) *converges to* $x \in X$ *if and only if* $\lim_{n\to\infty} d(x_n,x) = \theta$. *The sequence* (x_n) *is Cauchy if and only if* $\lim_{n,m\to\infty} d(x_n,x_m) = \theta$.

Throughout this paper, we shall assume that the cones are strongly minihedral and K-normal, hence regular. Except otherwise stated, the notation $\preceq$ designates the partial order with respect to P. Furthermore, we shall assume that $Int(P) \neq \emptyset$.

We conclude this section by the following proposition.

Proposition 2.12. [11] *Every cone metric space* (X,d) *is a topological space.*

3. Main results

In [1], Binayak proved the following result.

Theorem B. *Let $T : X \to X$, where (X,d) is a complete metric space, be a weak C-contraction. Then T has a fixed point.*

We generalize this result in the setting of cone metric spaces in this sectioin.

Definition 3.1. *A mapping $T : X \to X$, where (X,d) is a complete cone metric space, is said to be a weakly C-contractive or a weak C-contraction if for all $x,y \in X$,*

$$d(Tx,Ty) \preceq \frac{1}{2}[d(x,Ty)+d(y,Tx)] - \psi(d(x,Ty),d(y,Tx)), \tag{0.1}$$

where $\psi : P \times P \to P$ is a continuous mapping such that $\psi(x,y) = \theta$ if and only if $x = y = \theta$.

Lemma 3.1. *Let (X,d) be a cone metric space over a K-normal cone P. Then for any $c \in P$ and any $a \in E$, $a - c \preceq a$.*

Proof. Indeed, we have

$$\theta \preceq c \Longleftrightarrow c \in P \Longleftrightarrow a-(a-c) \in P \Longleftrightarrow a-c \preceq a.$$

Theorem 3.3. *Let $T : X \to X$, where (X,d) is a complete cone metric space, be a weak C-contraction. Then T has a fixed point.*

Proof. Let (x_n) be a sequence generated in the iteration $x_{n+1} = Tx_n$. If $x_n = x_{n+1} = Tx_n$, then x_n is a fixed point of T. Next, we assume $x_n \neq x_{n+1}$. Using (0.1), we have

$$\begin{aligned} d(x_n,x_{n+1}) &= d(Tx_{n-1},Tx_n) \\ &\preceq \frac{1}{2}[d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})] - \psi(d(x_{n-1},Tx_n),d(x_n,Tx_{n-1})) \\ &= \frac{1}{2}d(x_{n-1},x_{n+1}) - \psi(d(x_{n-1},x_{n+1}),\theta) \end{aligned}$$

$$\preceq \frac{1}{2}(d(x_{n-1},x_n)+d(x_n,x_{n+1})) - \psi(d(x_{n-1},x_{n+1}),\theta). \tag{0.2}$$

Using (0.2), we find from Lemma 3.1 that

$$d(x_n,x_{n+1}) \preceq d(x_{n-1},x_n). \tag{0.3}$$

Thus $(d(x_n,x_{n+1}))$ is a monotone decreasing sequence in E. Moreover, this sequence is bounded below by θ and since P is regular, the sequence $(d(x_n,x_{n+1}))$ is convergent. Let $d(x_n,x_{n+1}) \to r$

as $n \to \infty$. Next we prove that $r = \theta$.

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\preceq \frac{1}{2}(d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})) - \psi(d(x_{n-1}, x_{n+1}), d(x_n, x_n)) \\ &\preceq \frac{1}{2} d(x_{n-1}, x_{n+1}) \\ &\preceq \frac{1}{2}(d(x_{n-1}, x_n) + d(x_n, x_{n+1})). \end{aligned}$$

Letting $n \to \infty$, we see that

$$r \preceq \lim_{n\to\infty} \frac{1}{2} d(x_{n-1}, x_{n+1}) \preceq \frac{1}{2}r + \frac{1}{2}r,$$

or

$$\lim_{n\to\infty} d(x_{n-1}, x_{n+1}) = 2r. \tag{0.4}$$

Letting $n \to \infty$ in (0.2) and using (0.4) and the continuity of ψ, we have

$$r \preceq r - \psi(2r, \theta)$$

or

$$-\psi(2r, \theta) \in P,$$

which is a contradiction unless $r = \theta$. Thus we have established that

$$d(x_n, x_{n+1}) \to \theta \text{ as } n \to \infty. \tag{0.5}$$

Next we show that (x_n) is a Cauchy sequence. If otherwise, then there exists $\varepsilon \gg \theta$ and increasing sequences of integers $(m(k))$ and $(n(k))$ such that for all integers k, $n(k) > m(k)$, $d(x_{m(k)}, x_{n(k)}) \succeq \varepsilon$ and $d(x_{m(k)}, x_{n(k)-1}) \ll \varepsilon$. Then,

$$\begin{aligned} \varepsilon &\preceq d(x_{m(k)}, x_{n(k)}) \\ &= d(Tx_{m(k)-1}, Tx_{n(k)-1}) \\ &\preceq \frac{1}{2}(d(x_{m(k)-1}, Tx_{n(k)-1}) + d(x_{n(k)-1}, Tx_{m(k)-1})) \\ &- \psi(d(x_{m(k)-1}, Tx_{n(k)-1}), d(x_{n(k)-1}, Tx_{m(k)-1})) \text{ by (0.1)} \end{aligned}$$

$$= \frac{1}{2}(d(x_{m(k)-1}, x_{n(k)}) + d(x_{n(k)-1}, x_{m(k)})) - \psi(d(x_{m(k)-1}, x_{n(k)}), d(x_{n(k)-1}, x_{m(k)})). \tag{0.6}$$

Again, we have

$$\begin{aligned}\varepsilon &\preceq d(x_{m(k)},x_{n(k)})\\ &\preceq d(x_{m(k)},x_{n(k)-1})+d(x_{n(k)-1},x_{n(k)})\\ &\preceq \varepsilon+d(x_{n(k)-1},x_{n(k)}).\end{aligned}$$

Letting $k\to\infty$ in the above inequality and using (0.5), we obtain

$$\lim_{k\to\infty} d(x_{m(k)},x_{n(k)})=\varepsilon \tag{0.7}$$

and

$$\lim_{k\to\infty} d(x_{m(k)},x_{n(k)-1})=\varepsilon. \tag{0.8}$$

Indeed, we also have

$$\begin{aligned}d(x_{m(k)},x_{n(k)-1}) &\preceq d(x_{m(k)},x_{m(k)-1})\\ &+ d(x_{m(k)-1},x_{n(k)})\\ &+ d(x_{n(k)},x_{n(k)-1}).\end{aligned}$$

Note that

$$d(x_{m(k)-1},x_{n(k)})\preceq d(x_{m(k)-1},x_{m(k)})+d(x_{m(k)},x_{n(k)}).$$

Letting $k\to\infty$ in the above two inequalities and using (0.5), (0.7) and (0.8) we get

$$\lim_{k\to\infty} d(x_{m(k)-1},x_{n(k)})=\varepsilon. \tag{0.9}$$

Next, letting $k\to\infty$ in (0.6) and using (0.5), (0.8) and (0.9) we obtain

$$\varepsilon\preceq\frac{1}{2}(\varepsilon+\varepsilon)-\psi(\varepsilon,\varepsilon).$$

Or $\psi(\varepsilon,\varepsilon)\preceq\theta$, which is a contradiction since $\varepsilon\gg\theta$. Hence (x_n) is a Cauchy sequence and therefore is convergent in the complete cone metric space (X,d). Let $x_n\to z$ as $n\to\infty$. We prove that z is a fixed point for T. Indeed, we have

$$\begin{aligned} d(z,Tz) &\preceq d(z,x_{n+1})+d(x_{n+1},Tz) \\ &\preceq d(z,x_{n+1})+d(Tx_n,Tz) \\ &\preceq d(z,x_{n+1})+\frac{1}{2}(d(z,Tx_n))-\psi(d(z,x_n),d(x_n,Tz)) \\ &= d(z,x_{n+1})+\frac{1}{2}(d(z,x_{n+1})+d(x_n,Tz))-\psi(d(z,x_{n+1}),d(x_n,Tz)). \end{aligned}$$

Letting $n \to \infty$, using the continuity of ψ, we obtain

$$d(z,Tz) \preceq \frac{1}{2}d(z,Tz)-\psi(\theta,d(z,Tz)) \preceq \frac{1}{2}d(z,Tz),$$

which is a contradiction unless $d(z,Tz)=\theta$. Hence $z=Tz$.

Next we establish that the fixed point z is unique. If z_1 and z_2 are two fixed points of T, then

$$d(z_1,z_2)=d(Tz_1,Tz_2) \preceq \frac{1}{2}(d(z_1,Tz_2)+d(z_2,Tz_1))-\psi(d(z_1,Tz_2),d(z_2,Tz_1)).$$

That is,

$$d(z_1,z_2) \preceq d(z_1,z_2)-\psi(d(z_1,z_2),d(z_1,z_2)) \prec d(z_1,z_2),$$

which by property of ψ is a contradiction unless $d(z_1,z_2)=\theta$, that is, $z_1=z_2$. This completes the proof.

Conflict of Interests

The author declares that there is no conflict of interests.

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