Available online at http://scik.org
Eng. Math. Lett. 2014, 2014:21
ISSN: 2049-9337

EXACT SOLUTIONS OF BOUSSINESQ EQUATIONS USING DIFFERENTIAL TRANSFORM METHOD COMBINED WITH ADOMIAN POLYNOMIALS

H.R. MARASI*, M. DANESHBASTAM, M. SEDIGHI, M. NOSRATY

Department of Mathematics, University of Bonab, Bonab 5551761167, Iran

Abstract. The problem addressed in this paper is to obtain new exact solitary solutions for the Boussinesq-like $B(m,n)$ equations with fully nonlinear dispersion. The exact solitary wave solutions can be used to specify initial data for the incident waves in the Boussinesq numerical model and for the verification of the associated computed solution. We use differential transform method. The nonlinear terms can be handled by the use of Adomian polynomials. The proposed technique is general and can be easily modified to solve a wide range of Boussinesq-like equations in coastal engineering.

Keywords: Exact solution; Boussinesq-like $B(m,n)$equation; Solitary solution; dispersion.

2010 AMS Subject Classification: 35A15, 35A20.

1. Introduction

Study of wave propagation in fluids has become very important since long time ago and there is a large number of researches in this field. Shallow water waves have been expressed as a couple of equations by Whitham(1967). Many researchers have been continued studies in this subject by Whitham's shallow water equations shall immediately produce a coupled form of

*Corresponding author

Received May 27, 2014

Boussinesq equation. Since the model is a wave propagation in a shallow water, one would expect soliton solutions for such equations. Several members of Boussinesq system have been studied up to know. Yan [4] introduced a class of fully Boussinesq equations $B(m,n)$

$$u_{tt} = (u^n)_{xx} + (u^m)_{xxxx}, \quad m,n \in R, \tag{1}$$

and presented some of its compacton solutions when $m = n$. Zhu [5, 6] studied Boussinesq-like $B(m,n)$ equations

$$u_{tt} + (u^n)_{xx} - (u^m)_{xxxx} = 0, \quad m,n > 1, \tag{2}$$

$$u_{tt} - (u^n)_{xx} + (u^m)_{xxxx} = 0, \quad m,n > 1, \tag{3}$$

and

$$u_{tt} + (u^{2n})_{xx} + (u^{2n})_{xxxx} = 0, \quad n > 1, \tag{4}$$

by using the extended decomposition method. Yildirim [7] studied the Boussinesq-like equations with fully nonlinear dispersion $B(m,n)$ equations which exhibit solutions with solitary patterns and found new exact solitary solutions of the equations using homotopy perturbation method. In [8] Fernandez commented on some analytical solutions obtained in [7] and derived more general results by means of travelling waves and argue that a curious superposition principle may not be of any mathematical or physical significance. Physical phenomena are generally modeled as functional equations and for most of these equations, exact solutions are very rear. So, there are some analytic techniques to address such issues, which are based on either perturbation techniques [10], or traditional non-perturbation methods. Perturbation method is one of the well-known methods for solving nonlinear problems analytically. It is based on the existence of small/large parameters, the socalled perturbation quantities. However, many nonlinear problems do not contain such kind of perturbation quantities. In general, the perturbation method is valid only for weakly nonlinear problems. To overcome the restrictions of perturbation techniques, some non-perturbation techniques are proposed. Differential Transform Method (DTM) is one of the analytical methods for differential equations. The basic idea was initially introduced by Zhou [11] in 1986. Its main application therein is to solve both linear and nonlinear initial value problems in electrical circuit analysis. This method develops a solution in the form of a polynomial. Though it is based on Taylor series, still it is totally different from the traditional higher order Taylor series method. This technique has been employed to solve a large variety of linear and nonlinear problems. In this paper, this method combined with Adomian polynomials is applied to solving the Boussinesq equations. The paper is organized as follows. In Sec. 2, we present the Differential Transform method(DTM) and its modification(MDTM) by Adomian polynomials to fix notation and provide a convenient reference. In sec. 3, we extend the application of the method to construct analytical solutions of Boussinesq-like equation. Finally, some conclusions are given in Section 4.

2. Modified Differential Transform Method (DTM)

The goal of this section is to recall notations, definitions and some theorems of the DTM and modified differential transform method(MDTM) that will be used in this paper. If function $u(x,t)$ is analytic and differentiated continuously with respect to time t and space x in the domain of interest, then let

$$U_k(x) = \frac{1}{k!}[\frac{\partial^k}{\partial t^k}u(x,t)]_{t=0}, \tag{5}$$

and the inverse transform of $U_k(x)$ is defined as

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k, \tag{6}$$

then combining (5) and (6), we obtain

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}[\frac{\partial^k}{\partial t^k}u(x,t)]_{t=0}t^k. \tag{7}$$

If we consider the expressions (5),(6) and (7), it's clearly shown that the concept of the differential transform is derived from the power series expansion. Here we present some basic properties of the DTM.

Let $u(x,t)$, $v(x,t)$ and $w(x,t)$ be functions of time t and space x and $U_k(x)$, $V_k(x)$ and $W_k(x)$ are their corresponding differential transform. Then

i: If $u(x,t) = v(x,t) \pm w(x,t)$, then $U_k(x) = V_k(x) \pm W_k(x)$.

ii: If $u(x,t) = cv(x,t)$, then $U_k(x) = cV_k(x)$.

iii: if $u(x,t) = x^m t^n$, then $U_k(x) = x^m \delta(k-n)$.

iv: if $u(x,t) = x^m t^n v(x,t)$, then $U_k(x) = x^m V_{k-n}(x)$.

v: if $u(x,t) = v(x,t)w(x,t)$, then $U_k(x) = \sum_{r=0}^{k} V_r(x)W_{k-r}(x) = \sum_{r=0}^{k} V_r(x)W_{k-r}(x)$.

vi: If $u(x,t) = \frac{\partial^r}{\partial t^r} v(x,t)$, then $U_k(x) = (k+1)...(k+r)V_{k+r}(x)$.

vii: if $u(x,t) = \frac{\partial^r}{\partial x^r} v(x,t)$, then $U_k(x) = \frac{d^r}{dx^r} V_k(x)$.

We consider the case of a nonlinear function $f(u)$ that is approximated by the series

$$f(u) = \sum_{n=0}^{\infty} A_n,$$

where the A_n are the Adomian polynomials determined by the definitional formula[1, 2]

$$A_n = \frac{1}{n!}[\frac{d^n}{d\lambda^n}[f(\sum_{i=0}^{\infty} \lambda^i u_i)]]_{\lambda=0}, \qquad n = 0, 1, \cdots.$$

The Adomian polynomials of $f(u(x,t))$ are arranged in the form[2]

$A_0 = f(u_0)$,

$A_1 = u_1 f^{(1)}(u_0)$,

$A_2 = u_2 f^{(1)}(u_0) + \frac{1}{2!} u_1^2 f^{(2)}(u_0)$,

$A_3 = u_3 f^{(1)}(u_0) + u_1 u_2 f^{(2)}(u_0) + \frac{1}{3!} u_1^3 f^{(3)}(u_0)$,

$A_4 = u_4 f^{(1)}(u_0) + (u_1 u_3 + \frac{1}{2!} u_2^2) f^{(2)}(u_0) + \frac{1}{2!} u_1^2 u_2 f^{(3)}(u_0) + \frac{1}{4!} u_1^4 f^{(4)}(u_0)$,

$A_5 = u_5 f^{(1)}(u_0) + (u_2 u_3 + u_1 u_4) f^{(2)}(u_0) + \frac{1}{2!}(u_1^2 u_3 + u_1 u_2^2) f^{(3)}(u_0)$

$+ \frac{1}{3!} u_1^3 u_2 f^{(4)}(u_0) + \frac{1}{5!} u_1^5 f^{(5)}(u_0)$,

$\vdots$

In the following, the differential transform components of $f(u)$ are computed and by using their properties, they can be written in the following form[3]

$\tilde{A}_0(x) = f(u(x,0)) = f(U_0(x))$,

$\tilde{A}_1(x) = \frac{\partial}{\partial t} f(u(x,t))|_{t=0} = \frac{\partial}{\partial t} u(x,0) f^{(1)}(u(x,0)) = U_1(x) f^{(1)}(U_0(x))$,

$\tilde{A}_2(x) = \frac{1}{2!}(\frac{\partial^2}{\partial t^2} u(x,0) f^{(1)}(u(x,0)) + (\frac{\partial}{\partial t} u(x,0))^2 f^{(2)}(u(x,0))$

$= U_2(x) f^{(1)}(U_0(x)) + \frac{1}{2!}(U_1(x))^2 f^{(2)}(U_0(x))$,

$\tilde{A}_3(x) = U_3(x) f^{(1)}(U_0(x)) + U_1(x) U_2(x) f^{(2)}(U_0(x)) + \frac{1}{3!} U_1(x)^3 f^{(3)}(U_0(x))$,

$\tilde{A}_4(x) = U_4(x) f^{(1)}(U_0(x)) + (U_1(x) U_3(x) + \frac{1}{2!}(U_2(x))^2) f^{(2)}(U_0(x))$

$+ \frac{1}{2!}(U_1(x))^2 U_2(x) f^{(3)}(U_0(x)) + \frac{1}{4!}(U_1(x))^4 f^{(4)}(U_0(x))$,

$\tilde{A}_5(x) = U_5(x)f^{(1)}(U_0(x)) + (U_2(x)U_3(x) + U_1(x)U_4(x))f^{(2)}(U_0(x)) + \frac{1}{2!}((U_1(x))^2U_3(x) + U_1(x)(U_2(x))^2)f^{(3)}(U_0(x)) + \frac{1}{3!}(U_1(x))^3U_2(x)f^{(4)}(U_0(x)) + \frac{1}{5!}(U_1(x))^5f^{(5)}(U_0(x)),$

$\vdots$

So, $\tilde{A}_k(x)$ is obtained from the Adomian polynomials of f(u(x,t)) by replacing each u_k by $U_k(x)$. Now, consider the nonlinear differential equation of the form

$$\frac{\partial^2}{\partial t^2}u = f(u(x,t)), \tag{8}$$

where $f(u(x,t))$ denotes a nonlinear function. Therefore, taking differential transform on both sides of (8), we have the following recurrence scheme

$$(k+1)(k+2)U_{k+2}(x) = \tilde{A}_k(x). \tag{9}$$

3. Numerical Applications

In order to illustrate the advantages and the accuracy of the MDTM for solving Boussinesq equations, we have applied the method to three different examples.

Example 1. Consider the Boussinesq-like equation $B(2,2)$, of the form

$$u_{tt} + (u^2)_{xx} - (u^2)_{xxxx} = 0, \tag{10}$$

subject to two initial conditions

$$u(x,0) = \frac{4}{3}v^2 sinh^2(\frac{x}{4}), \qquad u_t(x,0) = -\frac{1}{3}v^3 sinh(\frac{x}{2}). \tag{11}$$

Applying the Modified Differential Transform to (10-11), the recurrence scheme for this problem is given by

$$\begin{cases} (k+1)(k+2)U_{k+2}(x) + \frac{\partial^2}{\partial x^2}(\tilde{A}_k) - \frac{\partial^4}{\partial x^4}(\tilde{A}_k) = 0, \\ U_0(x) = \frac{4}{3}v^2 sinh^2(\frac{x}{4}), \quad U_1(x) = \frac{-1}{3}v^3 sinh(\frac{x}{2}), \end{cases} \tag{12}$$

where the $\tilde{A}_k$ are obtained from the Adomian polynomials for the nonlinearity u^2 as follows

$A_0 = (u_0)^2,$ $\qquad\qquad$ $\tilde{A}_0 = (U_0(x))^2,$

$A_1 = 2(u_0)u_1,$ $\qquad\qquad$ $\tilde{A}_1 = 2U_0(x)U_1(x),$

$A_2 = 2u_0u_2 + (u_1)^2,$ $\qquad\qquad$ $\tilde{A}_2 = 2U_0(x)U_1(x)^2,$

$A_3 = 2u_0u_3 + 2u_1u_2,$ $\tilde{A}_3 = 2U_0(x)U_3(x) + 2U_1(x)U_2(x),$

$\vdots$

So, the following differential transform components are obtained

$$U_2(x) = \frac{1}{12}v^4 cosh(\frac{x}{2}), \quad U_3(x) = -\frac{1}{72}v^5 sinh(\frac{x}{2}), \quad U_4(x) = \frac{1}{576}v^6 cosh(\frac{x}{2}),$$

$$U_5(x) = -\frac{1}{5760}v^6 sinh(\frac{x}{2}), \ldots.$$

By applying the inverse differential transformation, we obtain the series solution as

$$\begin{aligned} u(x,t) = \tfrac{4}{3}v^2 sinh^2(\tfrac{x}{4}) - \tfrac{1}{3}v^3 sinh(\tfrac{x}{2})t + \tfrac{1}{12}v^4 cosh(\tfrac{x}{2})t^2 - \tfrac{1}{72}v^5 sinh(\tfrac{x}{2})t^3 \\ + \tfrac{1}{576}v^6 cosh(\tfrac{x}{2})t^4 - \tfrac{1}{5760}v^6 sinh(\tfrac{x}{2})t^5 + \ldots. \end{aligned} \tag{13}$$

Using Taylor series into (13), we find the closed form solution

$$u(x,t) = \frac{4}{3}v^2 \sinh^2(\frac{x - vt}{4}). \tag{14}$$

In addition as in [7], we can develop another exact solution for the $B(2,2)$ equation. Now we consider another initial value problem of $B(2,2)$ equation

$$u_{tt} + (u^2)_{xx} - (u^2)_{xxxx} = 0, \tag{15}$$

subject to two initial conditions

$$u(x,0) = -\frac{4}{3}v^2 sinh^2(\frac{x}{4}), \qquad u_t(x,0) = \frac{1}{3}v^3 sinh(\frac{x}{2}). \tag{16}$$

Using the manner as discussed above, we obtain another exact solution given by

$$u(x,t) = -\frac{4}{3}v^2 \cosh^2(\frac{x - vt}{4}). \tag{17}$$

Therefore, by combining the two results, we will find that

$$u(x,t) = \frac{4}{3}Kv^2 \sinh^2(\frac{x - vt}{4}) - \frac{4}{3}Lv^2 \cosh^2(\frac{x - vt}{4}), \tag{18}$$

is solution of $B(2,2)$ equation if $K = L$ or $K = L - 1$. When $K = L$ the new exact solution is trivial solution

$$u(x,t) = -\frac{4}{3}Lv^2. \tag{19}$$

And if $K = L-1$ the new exact solution is of the form

$$u(x,t) = \frac{4}{3}(1-L)v^2\sinh^2(\frac{x-vt}{4}) - \frac{4}{3}Lv^2\cosh^2(\frac{x-vt}{4}). \tag{20}$$

Moreover, adding a constant to the arguments in (14) and (17) will exhibit more exact solutions. In other words, we have the exact solutions

$$u(x,t) = \frac{4}{3}v^2\sinh^2(\frac{x-vt}{4}+c), \tag{21}$$

and

$$u(x,t) = -\frac{4}{3}Lv^2\cosh^2(\frac{x-vt}{4}+c), \tag{22}$$

where c is a constant.

Example 2. Consider the initial value problem $B(3,3)$,

$$u_{tt} + (u^3)_{xx} - (u^3)_{xxxx} = 0, \tag{23}$$

$$u(x,0) = \frac{\sqrt{6}}{2}v\, sinh(\frac{x}{3}), \qquad u_t(x,0) = -\frac{\sqrt{6}}{6}v^2 cosh(\frac{x}{3}), \tag{24}$$

where v is an arbitrary constant. Applying the Modified Differential Transform to (23-24), the recurrence scheme for this problem is given by

$$\begin{cases} (k+1)(k+2)U_{k+2}(x) + \frac{\partial^2}{\partial x^2}(\tilde{A}_k) - \frac{\partial^4}{\partial x^4}(\tilde{A}_k) = 0, \\ U_0(x) = \frac{\sqrt{6}}{2}v\, sinh(\frac{x}{3}), \quad U_1(x) = -\frac{\sqrt{6}}{6}v^2 cosh(\frac{x}{3}), \end{cases} \tag{25}$$

where the $\tilde{A}_k$ are obtained from the Adomian polynomials for the nonlinearity u^3 as follows

$A_0 = (u_0)^3,$ $\qquad \tilde{A}_0 = (U_0(x))^3,$

$A_1 = 3(u_0)^2 u_1,$ $\qquad \tilde{A}_1 = 3(U_0(x))^2 U_1(x),$

$A_2 = 3u_0^2 u_2 + 3u_o(u_1)^2,$ $\qquad \tilde{A}_2 = 3(U_0(x))^2(U_2(x)) + 3U_0(x)(U_1(x))^2,$

$A_3 = 3u_0^2 u_3 + 6u_0 u_1 u_2 + u_1^3,$ $\qquad \tilde{A}_3 = 3(U_0(x))^2 U_3(x) + 6U_0(x)U_1(x)U_2(x) + (U_1(x))^3,$

$\vdots$

So, the following differential transform components are obtained

$$U_2(x) = \frac{\sqrt{6}}{36}v^3 sinh(\frac{x}{3}), \quad U_3(x) = -\frac{\sqrt{6}}{324}v^4 cosh(\frac{x}{3}), \quad U_4(x) = \frac{\sqrt{6}}{3888}v^5 sinh(\frac{x}{3}), \ldots.$$

By applying the inverse differential transformation, we obtain the series solution as

$$u(x,t) = \frac{\sqrt{6}}{2}v\ sinh(\tfrac{x}{3}) - \frac{\sqrt{6}}{6}v^2 cosh(\tfrac{x}{3})t + \frac{\sqrt{6}}{36}v^3 sinh(\tfrac{x}{3})t^2 - \frac{\sqrt{6}}{324}v^4 cosh(\tfrac{x}{3})t^3 + \frac{\sqrt{6}}{3888}v^5 sinh(\tfrac{x}{3})t^4 + \ldots. \tag{26}$$

Using Taylor series into (26), we find the closed form solution

$$u(x,t) = \frac{\sqrt{6}}{2}v\sinh(\frac{x-vt}{3}). \tag{27}$$

In addition, we can develop another exact solution for the $B(3,3)$ equation. Now we consider another initial value problem of $B(3,3)$ equation

$$u_{tt} + (u^3)_{xx} - (u^3)_{xxxx} = 0, \tag{28}$$

subject to two initial conditions

$$u(x,0) = \frac{\sqrt{6}}{2}v\ sinh(\frac{x}{3}), \qquad u_t(x,0) = -\frac{\sqrt{6}}{6}v^2 cosh(\frac{x}{3}). \tag{29}$$

According to the similar steps as discussed above, we have another exact solution given by

$$u(x,t) = -\frac{\sqrt{6}}{2}v\sinh(\frac{x-vt}{3}). \tag{30}$$

As the previous example a new exact solution can be obtain by combining the two above results and found that

$$u(x,t) = \frac{\sqrt{6}}{2}vK\sinh(\frac{x-vt}{3}) - \frac{\sqrt{6}}{2}vL\sinh(\frac{x-vt}{3}), \tag{31}$$

satisfies the $B(3,3)$ equation when $K = L,\ K = 1+L$ or $L = -1+L$.

Moreover, similar to the previous example, adding a constant to the arguments in (27) and (30) will exhibit more exact solutions

$$u(x,t) = \frac{\sqrt{6}}{2}v\sinh(\frac{x-vt}{3} + c), \tag{32}$$

and

$$u(x,t) = -\frac{\sqrt{6}}{2}v\sinh(\frac{x-vt}{3} + c), \tag{33}$$

where c is a constant.

Example 3. We consider the following Boussinesq equation

$$u_{tt} - u_{xx} - u_{xxxx} + (u^2)_{xx} = 0, \tag{34}$$

with the initial conditions

$$u(x,0) = \frac{6}{x^2} \qquad u_t(x,0) = -\frac{12}{x^3}. \tag{35}$$

Applying the Modified Differential Transform to (34-35), the recurrence scheme for this problem is given by

$$\begin{cases} (k+1)(k+2)U_{k+2}(x) + \frac{\partial^2}{\partial x^2}U_k(x) - \frac{\partial^4}{\partial x^4}U_k(x) + \frac{\partial^2}{\partial x^2}(\tilde{A}_k) = 0, \\ U_0(x) = \frac{6}{x^2}, \qquad U_1(x) = -\frac{12}{x^2}, \end{cases} \tag{36}$$

where the $\tilde{A}_k$ are obtained from the Adomian polynomials for the nonlinearity u^2. So we have

$$U_2(x) = \frac{18}{x^4}, \quad U_3(x) = -\frac{24}{x^5}, \quad U_4(x) = \frac{30}{x^6}, \ldots.$$

By applying the inverse differential transformation, we obtain the series solution as

$$u(x,t) = \frac{6}{x^2} - \frac{12}{x^3}t + \frac{18}{x^4}t^2 - \frac{24}{x^5}t^3 + \frac{30}{x^6}t^4 + \ldots. \tag{37}$$

This will, in the limit of infinitely many terms, yields the closed-form solution

$$u(x,t) = \frac{6}{(x+t)^2}, \tag{38}$$

which is the exact solution of the equation.

4. Conclusions

In this research, differential transform method combined with Adomian polynomials has been applied to solving the Boussinesq equations. The method is applied in a direct way without using linearization, transformation, discretization or restrictive assumptions. Three different examples were tested and the results were in excellent agreement with the exact solution.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer Academic. Dordrecht. (1994).

[2] H.R. Marasi, M. Nikbakht, Adomian decompositiom method for boundary value problems, Aus. J. Basic. Appl. Sci. 5(2011), 2106-2111.

[3] A. Elsaid, Fractional differential transform method combined with the Adomian polynomials, Apll. Math. Comput. 218 (2012), 6899–6911.

[4] Z. Yan, New families of solitons with compact support for Boussinesq-like B(m, n) equations with fully nonlinear dispersion, Chaos Solitons and Fractals. 14 (2002), 1151-1158.

[5] Y. Zhu, Construction of exact solitary solutions for Boussinesq-like equations with fully nonlinear dispersion by the decomposition method, Chaos Solitons and Fractals. 26 (2005), 897-903.

[6] Y. Zhu, C. Lu, New solitary solutions with compact support for Boussinesq-like B(2n, 2n) equations with fully nonlinear dispersion, Chaos Solitons and Fractals. 32 (2007), 768-772.

[7] A. Yildirim, Homotopy perturbation method to obtain exact special solutions with solitary patterns for Boussinesq-like B (m, n) equations with fully nonlinear dispersion, J. Math. hys. 50(2009), Article ID 023510.

[8] M. Fernandez, On the homotopy perturbation method for Boussinesq-like equations, 230 (2014), 208–210.

[9] Q, Wang, Numerical solution of fractional Boussinesq equation, Commun. Theor. Phys. 47 (2007), 413–420.

[10] A.H. Nayfeh, Perturbation Methods, John Wiley, Sons. New York. (2000).

[11] J. K. Zhou, Deferential Transformation and its application for electrical circuits, Huazhong University Press. Wuhan China. (1986).