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SOME APPLICATIONS OF FIXED POINT THEOREMS

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Abstract. In this paper, it is shown that the fixed point theory yields result of best approximation and best approximation yields the variational inequality result. The variational inequality yields fixed point theory. It is also shown that the fixed point theory is equivalent to maximal elements in mathematical economics. In the end, a couple of results are proved extending earlier ones.

Keywords: fixed points; variational inequality; best approximation; maximal elements; metric projection.

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1. Introduction

Fixed point theory is quite useful in the existence theory of differential, integral, partial differential and functional equations. This is a basic mathematical tool used in showing the existence of solutions in game theory and mathematical economics. Applications in best approximation theory, optimization problems, variational inequalities, complementarity problems and eigenvalue problems are well known. Fixed point theory is a very important tool used in nonlinear problems arising is diverse field like mathematical economics, game theory, biology, engineering and physics, and is backbone of the nonlinear analysis.

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2. Main results

Recall that if C is a closed bounded convex subset of R^n and $f : C \to C$ a continuous function, then f has a fixed point, that is, $fx = x$ has a solution.

We need the following preliminaries from [10].

Definition 2.1. *Let X be a normed linear space and let C a nonempty subset of X with $x \in X$ and $x \notin C$. An element $y \in C$ is called an element of best approximation of x by elements of C if*

$$\| x - y \| = d(x,C) = \inf\{\| x - z \| : z \in C\}.$$

Definition 2.2. *Let $Px = \{y \in C : \| x - y \| = d(x,C)\}$ denote the set of all points in C closest to x. The set C is said to be proximinal if Px is nonempty for each $x \in X, x \notin C$. A compact set C is a proximinal set and so are a closed convex subset of a Hilbert space and a closed convex subset of a reflexive Banach space.*

If Px is a singleton for each $x \in X$, then C is said to be Chebyshev.

Definition 2.3. *The set valued map $P : X \to 2^C$ is called the metric projection onto C. P is also said to be best approximation operator.*

In the case that C is a compact set then the problem of best approximation is considered as an optimization problem. The metric projection P onto C is continuous and attains its minimum on C. A set C is an existence set if and only if for each $x \in X$ the metric projection P attains its minimum on C.

If C is a compact convex set, then each point not in C has a unique nearest point. In R^n the metric projection is a continuous function [10].

Now we state the following, known as the best approximation result [6, 10].

Theorem 2.4. *If C is a closed bounded convex subset of R^n and $f : C \to R^n$ is a continuous function, then there is a $y \in C$ such that*

$$\| fy - y \| = d(fy,C) = \inf\{\| fy - z \| : z \in C\}. \quad (2.1)$$

Proof. Given that $f : C \to R^n$ is a continuous function, and $P : R^n \to C$ is continuous best approximation operator. Then $Pof : C \to C$ is a continuous function since f and P both are continuous. Thus $Pof : C \to C$ has a fixed point by Brouwer fixed point theorem [10]. Let $Pofy = y$ for $y \in C$. This gives that

$$\| fy - y \| = d(fy, C).$$

In case $\| y - x \| = d(x, C), x \notin C, y \in C$, then we write $< x - y, y - z > \geq 0$ for all $z \in C$.

Remark 2.5. (2.1) has a solution if and only if Pof has a fixed point, where P is a metric projection onto C.

Remark 2.6. It is easy to see that if $f : C \to C$ is a continuous function (a self map) in Theorem 2.4, then f has a fixed point.

In this case we use Theorem 2.4 to get $< fy - y, y - x > \geq 0$ for all $x \in C$. Since $f : C \to C$, so for $y \in C, fy \in C$ and $d(fy, C) = 0$. Hence by (2.1) we get $fy = y$, that is, f has a fixed point.

The following is a well known result due to Hartman and Stampacchia [8] in variational inequalities.

Theorem 2.7. *Let C be a closed bounded convex subset of R^n and $f : C \to R^n$ a continuous function. Then there is a $y \in C$ such that*

$$< fy, x - y > \geq 0, \tag{2.2}$$

for all $x \in C$.

Proof. Let $g = I - f$, where I is an identity function. Then $g : C \to R^n$ is a continuous function. Using Theorem 2.1, we get that there is a $y \in C$ such that $\| gy - y \| = d(gy, C) =< gy - y, y - x > \geq 0$ for all $x \in C$. Let $gy = (I - f)y$. Then $< (I - f)y - y, y - x > \geq 0$ for all $x \in C$. On simplification one gets $< -fy, y - x > \geq 0$, that is, $< fy, x - y > \geq 0$ for all $x \in C$.

Remark 2.8. (2.2) has a solution if and only if $P(I - f)$ has a fixed point in C, where P is a metric projection onto C.

Theorem 2.7 yields the following fixed point theorem [1,10].

Corollary 2.9 *If $g : C \to C$ is a continuous function, where C is a closed bounded convex subset of R^n, then g has a fixed point in C.*

Proof. Let $f = I - g$. Then $f : C \to R^n$ is a continuous function. Using Theorem 2.4, we get $< fy, x-y > \geq 0$ for all $x \in C$. Replace fy by $(y - gy)$ to get $< y-gy, x-y > \geq 0$. Letting $gy = x$ we get $< y-gy, gy-y > \geq 0$. Hence, $gy = y$, that is, g has a fixed point.

The above results illustrate that the fixed point theory, variational inequality and the best approximation results are equivalent.

Here it is shown that the maximal element in economics is equivalent to fixed point theory; for details, see [2, 3, 5, 10, 11].

Recall that a binary relation F on a set C is a subset of $C \times C$ or a mapping of C into itself. It is written yFx or $y \in Fx$ to mean that y stands in relation F to x. A maximal element of F is a point x such that no point y satisfies $y \in Fx$, that is, $Fx = \phi$; see 2,3,7,11]

Thus the set of maximal elements is $\{x \in C / Fx = \phi\} = \bigcap (C - F^{-1}x)$ where $F^{-1}x = \{y \in C / x \in Fy\}$.

The following is known as the Ky Fan's lemma [6].

Lemma 2.10. *Let C be a nonempty compact convex subset of R^n and $F : C \to 2^C$ a multifunction such that*
(i) Fx is convex for each $x \in C$ and $x \notin Fx$ for each x,
(ii) F has a open graph.
Then F has a maximal element.

Here we are going to show that the existence of the maximal element if and only if fixed point theory and then this will complete the following list. Maximal element if and only if fixed point theory if and only if best approximation if and only if variational inequality.
The following fixed point theorem is derived by using the above Lemma 2.10.

Theorem 2.11 *Let $f : C \to C$ be a continuous function, C a closed bounded convex subset of R^n. Then f has a fixed point.*

Proof. Define $Fx = \{y \in C : \| y - fx \| < \| fx - x \|\}$ for each $x \in C$. Then for each $x \in C$, Fx is convex, $x \notin Fx$, and F has an open graph, so by Ky Fan's Lemma [6], F has a maximal element, that is, $Fx_0 = \phi$ for $x_0 \in C$. This means that $\| fx_0 - y \| \geq \| x_0 - fx_0 \|$ for all $x \in C$. Since $f : C \to C$, so we take $y = fx_0$ to get a fixed point.

Remark 2.12. In case $x_0 = fx_0$, then $Fx = (y \in C / \| y - fx_0 \| < \| fx_0 - x_0 \|)$, that is, $Fx = \{y \in C / \| y - fx_0 \| < 0\} = \phi$ and F has a maximal element.

Let X be a Banach space and C a non-empty subset of X. If $f : C \to C$, then f is said to be a non-expansive map provided that $\| fx - fy \| \leq \| x - y \|$ for all $x, y \in C$. In case $\| fx - fy \| \leq k \| x - y \|$ for all $x, y \in C$ and $0 < k < 1$, then f is called a contraction map. If $f : C \to H$, then f is said to be a non-self map. For example, $f : [0,1] \to R$ defined by $fx = x + p$ for all $x \in C$ and $p \in R$, a translation map, is not a self map. If $f : C \to C$, then f is a self map.

The following result is proved in [1].

Let $f : C \to X$ ba a non-expansive map and C a closed bounded convex subset of a uniformly convex Banach space X, and $0 \in C$.
Then either
(i) f has a fixed point, or
(ii) there exists an $r \in (0,1)$ such that $u = rfu$, for $u \in \partial C$ (boundary of C).

Here we prove the result where C is not necessarily a bounded set.

Theorem 2.13. *Let C be a closed convex subset of a uniformly convex Banach space X with $0 \in C$. Let $f : C \to X$ a nonexpansive map with $f(C)$ bounded. Then one of the following holds.*
(i) f has a fixed point or
(ii) there is a $t \in (0,1)$ such that $u = tfu$ for $u \in \partial C$ (boundary of C).

Proof. Take a sequence $\{r_i\}, 0 < r_i < 1$, such that $r_i \to 1$ as $i \to \infty$. Define $f_i(x) = (1 - r_i)fx$. Then f_i is contraction and one of the following holds for each i. Either (i) f_i has a unique fixed point or (ii) there is a $t \in (0,1)$ such that $x_i = tf_ix_i$ for $x \in \partial C$. In case (i) holds, then $x_i = f_ix_i$. Let $B = coclf(C)$. Then B is a closed bounded and convex set of a uniformly convex Banach space and is weakly closed. Consequently each sequence has a weakly convergent subsequence say $\{x_n\}$ converges weakly to z.

$$\| (I - f)x_n \| = \frac{1}{n} \| (fx_n) \| < \frac{1}{n}(\| fx_n - f0 \| + \| f0 \|) < \frac{1}{n}(\| x_n \| + \| f0 \|),$$

and $\{(I-f)x_n\}$ converges strongly to 0. Since $I-f$ is demiclosed therefore , $0=(I-f)z$, employing that $z=fz$.

Recall that $f:C\to C$ is demiclosed if $x_n\to y$ weakly and $fx_n\to z$ strongly, then $z=fy$.

The following examples illustrate where C is not bounded and f is not a self map.

Example 2.14. *Let $f:[-1,1]\to R$ be given by $fx=1-x$. Then $f(-1)=2\notin[-1,1]$, that is, f is not a self map. But f does have a fixed point.*

Example 2.15. *Let $f:[0,\infty)\to R$ be given by $fx=1/(1+x)$. Then f is a non-expansive map and $[0,\infty)$ is not bounded. However, f does have a fixed point.*

Let C be a closed bounded convex subset of a Hilbert space H. Let $f:C\to H$ be a generalized inward non-expansive map. Then f has a fixed point.

Definition 2.16. *A map $f:D\subset C:\to H$ is said to be weakly inward relative to C if $fx\in cl(I_C(x))$ for $x\in D.$, where $cl(I_C(x))$ is the closure of the inward set $I_C(x)=\{x+r(z-x)/z\in C$ and $r\geq 1\}$.*

Definition 2.17. *A map f is said to be generalized inward map on D relative to C if $d(fx,C)<\| x-fx\|$ for $x\in D$ with $fx\notin C$.*

It is known that a weakly inward map is generalized inward but the converse is not true. Let $P:H\to C$ be a metric projection. Then P is a non-expansive map.

The following Lemma is due to Lan and Wu [9].

Lemma 2.18. *Let $f:D\subset C:\to H$ be a generalized inward map on D relative to C. Let $x\in D$. Then x is a fixed point of f if and only if x is a fixed point of Pof. (P composition f).*

Note that for generalized inward map $x=fx$ if and only if $x=Pofx$.

Theorem 2.19. *Let C be a closed convex subset of a Hilbert space H. Let $f:C\to H$ be a non-expansive, generalized inward map with $f(C)$ bounded. Then f has a fixed point in C.*

Proof. Since $f:C\to H$ and $P:H\to C$, both are non-expansive maps so the map $Pof:C\to C$ is also a non-expansive map. Let $B=cocl(Pof(C))$. Then $Pof:B\to B$ is a non-expansive map and the set B is a closed bounded convex set in a Hilbert space. Hence by a well known theorem of Browder [4] we get that there is an $x \in C$ such that $Pofx = x$. This gives that $\| fx - x \| = d(fx, C)$.

Now by using Lemma 2.1 we derive that $fx = x$, that is, f has a fixed point.

The result due to Lan and Wu [9] follows as a corollary where C is a closed, bounded and convex subset of H.

The following corollaries are easy to derive.

Corollary 2.20. *If $f : C \to H$ is a nonexpansive map and C is a closed convex set with $f(C)$ bounded, with $f(\partial C)$ is contained in C, then f has a fixed point.*

Corollary 2.21. *If $f : C \to C$ is a nonexpansive map, where C is a closed bounded convex subset of H, then f has a fixed point [4].*

Conflict of Interests

The author declares that there is no conflict of interests.

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