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A NOTE ON OSCILLATION CRITERIA FOR SOME PERTURBED HALF-LINEAR ELLIPTIC EQUATIONS

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Abstract. In this paper, we show if the half-linear part of an equation is oscillatory, so would be some of its related perturbed equations. For one-dimentional cases, it can be resumed as the following: if the half-linear equation $P(y) := \{a(t)\phi(y')\}' + c(t)\phi(y) = 0$ is oscillatory then any of its perturbed equations $P(z) + Q'(t)h(y, y') = 0$ will also be oscillatory whenever $Q \in C^1(\mathbb{R})$ and $h \in C(\mathbb{R}^2, \mathbb{R})$.

Keywords: oscillation criteria; half-linear; elliptic equation; solution.

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1. Introduction

This work is somehow an addendum to our earlier result in [4]. For a $T > 0$, define accordingly $\Omega_T := \{x \in \mathbb{R}^n \mid ||x|| > T, \quad 1 < n \in \mathbb{N}\}$ or $\Omega_T := (T, \infty) \subset \mathbb{R}$. We investigate some oscillation criteria for equations of the type

$$\begin{cases} (i) \quad \Big\{a(t)\phi(y')\Big\}' + c(t)\phi(y) + g(t)f(y, y') = 0, \quad t \in \Omega_T \text{ or} \\ (ii) \quad \nabla \cdot \Big\{A(x)\Phi(\nabla v)\Big\} + C(x)\phi(v) + H(x) \cdot F(v, \nabla v) = 0, \quad x \in \Omega_T, \end{cases} \tag{1.1}$$

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where $a,\ c,\ g \in C(\Omega_T, \mathbb{R})$; $f \in C(\mathbb{R}^2,\ \mathbb{R})$; $H \in C(\Omega_T,\ \mathbb{R}^n)$, $F \in C(\mathbb{R}\times\mathbb{R}^n,\ \mathbb{R}^n)$; the dot denotes the scalar product in $\mathbb{R}^n$. For some $\alpha > 0$, $\phi(S) := |S|^{\alpha-1}S$ for $S \in \mathbb{R}$ and $\Phi(\zeta) := |\zeta|^{\alpha-1}\zeta$, $\zeta \in \mathbb{R}^n$. They have the following properties: $\forall t, s \in \mathbb{R}$ and $\zeta \in \mathbb{R}^n$

$\phi(t)\phi(s) = \phi(ts);\quad t\phi'(t) = \alpha\phi(t);\quad t\phi(t) = |t|^{\alpha+1};$

$\phi(s)\Phi(\zeta) = \Phi(s\zeta);\quad \zeta\Phi(\zeta) = |\zeta|^{\alpha+1}.$

Definition 1.1. Let $u \in C(\mathbb{R},\ \mathbb{R})$ (or $C(\mathbb{R}^n,\ \mathbb{R})$).

(1) A nodal set of u is any open and connected $D(u) \neq \emptyset$ such that $u \neq 0$ in $D(u)$ and $u|_{\partial D(u)} = 0$.

(2) u is said to be oscillatory (strongly oscillatory) if it has a zero in any Ω_R, $R > 0$ (in any nodal set $D(u) \subset \Omega_R$).

(3) An equation will be said to be oscillatory if any of its non-trivial and bounded solutions is oscillatory.

In the sequel the general hypotheses are: for some $T, m > 0$,

(H1): $a \in C^1(\Omega_T,\ (m,\ \infty))$ is non decreasing: $A \in C^1(\Omega_T,\ (m,\ \infty))$;

$g \in C(\Omega_T,\ \mathbb{R})$;

(H2): $c, C \in C(\Omega_T,\ (m,\ \infty))$ eventually; H, f, F are as stated above;

(H3): On any compact $E \subset \Omega_T$, $\exists k > 0$ such that

(i) $|g(t)f(S,w)| \leq k|w|^\alpha + \phi(S)$ if $|w| < 1$ and $S > 0$ for the (1.1)(i) case;

(ii) $|H(x)\cdot F(S,\zeta)| \leq k|\zeta|^\alpha + \phi(S)$ if $|\zeta| < 1$ and $S > 0$ for the (1.1)(ii) case.

(The condition (H3) is to ensure that non-trivial solutions are not compact-supported (see [2]).

Oscillation criteria for the equations (1.1)(i) will be obtained through some comparison methods, using some Picone-type identity.Some oscillation criteria for the half-linear equations

$$\begin{aligned} &(i) \quad \Big\{a(t)\phi(y')\Big\}' + c(t)\phi(y) = 0, \quad t \in \Omega_T \quad \text{and} \\ &(ii) \quad \nabla\cdot\Big\{A(x)\Phi(\nabla v)\Big\} + C(x)\phi(v) = 0, \quad x \in \Omega_T \end{aligned} \tag{1.2}$$

are well known; see *e.g.* [1,3,4] and references therein. For any $w \in C(\mathbb{R}^n,\ \mathbb{R}^+)$ define $W^+(r) := r^{n-1}\max_{|x|=r} w(x)$ and $W^-(r) := r^{n-1}\min_{|x|=r} w(x)$. The equations in (1.2) are oscillatory if

(i) a satisfies (H1) and c satisfies (H2) or $t \mapsto \int_T^t c(s)ds$ diverges to infinity for (1.2) (i); (Theorem 1.5 of [3])

(ii) $a := A^-$ and $c := C^+$ satisfy the conditions displayed in (i) above for (1.2) (ii). (Theorem 5.1 of [4])

The criteria for (1.2) (ii) are obtained from those of (1.2) (i) using some rightaway transformations and some Picone-type identities; see [1] [3] and the references therein.

2. Picone-type formulae for the equations in (1.1)

If y is a non-trivial C^2-solution , non zero in some $D \subset \Omega_T$ of (1.1) (i) and z such a solution for (1.2) (i) then

$$\begin{aligned}
&(a) \quad \text{if } \exists G \in C^1(\Omega_T, \mathbb{R}) \text{ such that } G'(t) = g(t) \text{ in } \Omega_T, \\
&(b) \quad \left\{ a(t)z\phi(z') - a(t)z\phi(\frac{z}{y}y') - z\phi(\frac{z}{y})G(t)f(y,y') \right\}' \\
&= a(t)\zeta_\alpha(z,y) - G(t)\left\{ z\phi(\frac{z}{y})f(y,y') \right\}',
\end{aligned} \tag{2.1}$$

where, $\forall \gamma > 0$, the two-form function ζ_γ is defined $\forall u, v \in C^1(\mathbb{R}, \mathbb{R})$ by

$$(\mathbf{Z1}): \qquad \zeta_\gamma(u,v) \begin{cases} = |u'|^{\gamma+1} - (\gamma+1)u'\phi_\gamma(\frac{u}{v}v') + \gamma v'\frac{u}{v}\phi_\gamma(\frac{u}{v}v') \\ = |u'|^{\gamma+1} - (\gamma+1)u'\phi_\gamma(\frac{u}{v}v') + \gamma|\frac{u}{v}v'|^{\gamma+1} \end{cases}$$

is strictly positive for non null $u \neq v$ and is null only if $u = \lambda v$ for some $\lambda \in \mathbb{R}$. Similarly, if $v \in C^2(\Omega_T, \mathbb{R})$ is a non-trivial solution for (1.1) (ii) and u such a solution of (1.2) (ii) then wherever $v \neq 0$

$$
\begin{aligned}
&(a) \quad \text{if } \exists h \in C^1(\Omega_T,\, \mathbb{R}) \text{ such that } \nabla h = H(x) \text{ in } \Omega_T,\\
&(b) \quad \nabla \cdot \Big\{ A(x) u \Phi(\nabla u) - A(x) u \Phi(\frac{u}{v} \nabla v) - u \phi(\frac{u}{v}) h(t) F(v, \nabla v) \Big\}\\
&= A(x) Z_\alpha(u,v) - h(t) \nabla \cdot \Big\{ u \phi(\frac{u}{v}) F(v, \nabla v) \Big\},\\
&\text{where} \quad \forall \gamma > 0, \quad \forall u, v \in C^1(\mathbb{R}^n).\\
&\textbf{(Z2)}: \quad Z_\gamma(u,v) := |\nabla u|^{\gamma+1} - (\gamma+1) \Phi_\gamma(\frac{u}{v} \nabla v) \cdot \nabla u + \gamma |\frac{u}{v} \nabla v|^{\gamma+1}\\
&= |\nabla u|^{\gamma+1} - (\gamma+1) |\frac{u}{v} \nabla v|^{\gamma-1} \, \frac{u}{v} \nabla v \cdot \nabla u + \gamma |\frac{u}{v} \nabla v|^{\gamma+1}.
\end{aligned} \tag{2.2}
$$

We recall that $\forall \gamma > 0$ the two-form $Z_\gamma(u,v) > 0$ for distinct non null u, v and is null only if $\exists k \in \mathbb{R}$; $u = kv$; see [1].

3. Main results

Theorem 3.1. *Assume that a, c, g and f satisfy (H1) to (H3). Then*

$$
\begin{aligned}
&(i) \quad \Big\{ a(t) \phi(y') \Big\}' + c(t) \phi(y) = 0,\ t > T \quad \textit{is oscillatory,}\\
&(ii) \quad \textit{so is} \ \Big\{ a(t) \phi(y') \Big\}' + c(t) \phi(y) + g(t) f(y, y') = 0, \quad t \in \Omega_T
\end{aligned} \tag{3.1}
$$

provided that $\exists G \in C^1(\Omega_T)$; $\quad G'(t) = g(t)$.

Theorem 3.2. *Assume that A, C ,F, with $a := A^-$, $c := C^+$ and H satisfy (H1) to (H3). Then*

$$
\nabla \cdot \Big\{ A(x) \Phi(\nabla v) \Big\} + C(x) \phi(v) + H(x) \cdot F(v, \nabla v) = 0, \quad x \in \Omega_T \tag{3.2}
$$

is oscillatory provided that $\exists h \in C^1(\Omega_T,\, \mathbb{R})$; $\quad \nabla h(x) = H(x)$.

Since the proofs of the two theorems are similar, we prove only the first one.

Proof of Theorem 3.1. In equation (3.1) (ii) $g(t)$ can be replaced by $G'_\mu(t) := [G(t) + \mu]'$, $\forall \mu \in \mathbb{R}$. With that replacement, if y is a non-trivial solution of (3.1)(ii) with $y > 0$ in an Ω_R, then for any oscillatory solution z of (3.1) (i), for any nodal set $D(z) \subset \Omega_R$

$$\begin{aligned} 0 &= \int_{D(z)} \left[a(t)\zeta_\alpha(z,y) \right] dt \\ &- \int_{D(z)} (\, G(t) + \mu \,) \left\{ z\phi(\frac{z}{y}) f(y,y') \right\}' dt \quad \forall \mu \in \mathbb{R}. \end{aligned} \tag{3.3}$$

For $\mu = 0$ we get $0 = \int_{D(z)} \left[a(t)\zeta_\alpha(z,y) \right] dt - \int_{D(z)} G(t) \left\{ z\phi(\frac{z}{y}) f(y,y') \right\}' dt$ whence $\mu [z\phi(\frac{z}{y}) f(y,y')] \equiv 0$ and so is $\zeta_\alpha(z,y)$ in any such a $D(z)$. Therefore no such a solution y can be non-zero in any Ω_R; it has to have a zero in any $D(z) \subset \Omega_R$.

Remark 3.3. Following the processes similar to those in the proofs of Theorem 3.4 and Theorem 5.1 of [4], the hypotheses on G and H can be weakened to

$$\exists k \in C(\Omega_T, \, \mathbb{R}) \ \text{ and } K \in C(\Omega_T, \, \mathbb{R}^n)$$

bounded in Ω_T such that the functions G and h above satisfy

$$G'(t) = g(t) + k(t) \quad \text{ and } \ \nabla h(x) = H(x) + K(x). \tag{3.4}$$

But the penality to pay is that the corresponding solutions y will be oscillatory unless $\liminf_{t \nearrow \infty} |y(t)| = 0 \quad (\liminf_{|x| \nearrow \infty} |y(x)| = 0)$.

Conflict of Interests

The author declares that there is no conflict of interests.

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