Available online at http://scik.org
Eng. Math. Lett. 2015, 2015:1
ISSN: 2049-9337

A COMMON FIXED POINT THEOREM FOR FOUR WEAKLY COMPATIBLE MAPPINGS IN COMPLETE METRIC SPACE USING RATIONAL INEQUALITY

ARVIND KUMAR SHARMA[1,*], V. H. BADSHAH[2] AND V. K. GUPTA[3]

[1]Department of Applied Mathematics, Mahakal Group of Institutes, Ujjain (M.P.) India

[2]School of Studies in Mathematics, Vikram University, Ujjain (M.P.) India

[3]Department of Mathematics, Govt. Madhav Science College, Ujjain (M.P.) India

Abstract: The purpose of this paper is to present a common unique fixed-point theorem for four continuous mappings in complete metric space using weaker condition such as weakly compatible and associated sequence in place of compatible mappings and completeness of metric space. More over the condition of continuity of any one of mapping is being dropped. .Our result generalizes the results of Sharma, Badshah and Gupta [16], Fisher [4], Lohani and Badshah [13] and Singh and Chouhan [17].

Keywords: complete metric space; compatible mappings; weakly compatible mappings; common fixed point.

2010 Mathematics Subject Classification: 54H25, 47H10.

1. Introduction

Jungck [6], proved a common fixed point theorem for commuting mappings in 1976, which generalizes the Banach's [1] fixed point theorem in complete metric space. This result was generalized and extended in various ways by Iseki and Singh [5], Das and Naik [2], Singh [18], Singh and Singh [19], Fisher [3], Park and Bae [14]. Recently, some common fixed point theorems of three and four commuting mappings were proved by Fisher [3], Khan and Imdad [12], Kang and Kim [10] and Lohani and Badshah [13].The result of Jungck [6] has so many applications but it requires the continuity of mappings. S. Sessa [15], introduced the concept of

*Corresponding author

Received September 23, 2014

weak commutativity and proved a common fixed point theorem for weakly commuting maps. Further Jungck [8], introduced more generalized commutativity; known as compatibility, which is weaker than weakly commuting maps. Since various author proved a common fixed point theorem for compatible mappings satisfying contractive type conditions and continuity of one of the mapping is required.

In 1968, kannan [11] proved that there exists maps that have a discontinuity in the domain but which have fixed points, moreover the maps involved in every case were continuous at the fixed point. In 1998, Jungck and Rhodes [9] introduced the weaker condition; known as weakly compatible and showed that compatible maps are weakly compatible but converse is not true in general.

The purpose of this paper is to generalize some common fixed point theorems, which extend the results of Sharma ,Badshah and Gupta [16], Fisher [4], Jungck [7,8] , Lohani and Badshah [13], and Singh and Chouhan [17] by using a rational inequality and weakly compatible mappings instead of compatible mappings. In support of our main theorems, an example is also given.

2. Preliminaries

Definition 2.1 : Two mapping S and T from a metric space (X, d) into itself, are called commuting on X, if $d(STx, TSx) = 0$ that is $STx = TSx$ for all x in X.

Definition 2.2 : Two mapping S and T from a metric space (X, d) into itself, are called weakly commuting on X, if $d(STx, TSx) \le d(Sx, Tx)$ for all x in X.

Clearly, commuting mappings are weakly commuting, but converse is not necessarily true, given by following example :

Example 2.1

Let $X = [0, 1]$ with the Euclidean metric d. Define S and $T : X \to X$ by

$Sx = \frac{x}{5-x}$ and $Tx = \frac{x}{5}$ for all x in X.

Then for any x in X,

$$d(STx, TSx) = \left| \frac{x}{25-x} - \frac{x}{25-5x} \right|$$

$$= \left| \frac{-4x^2}{(25-x)(25-5x)} \right|$$

$$\leq \frac{x^2}{25-5x}$$

$$= \left| \frac{x}{5-x} - \frac{x}{5} \right|$$

$$= d(Sx, Tx)$$

$d(STx, TSx) \leq d(Sx, Tx)$ for all x in X.

Thus S and T are weakly commuting mappings on X, but they are not commuting on X, because

$STx = \frac{x}{25-x} < \frac{x}{25-5x} = TSx$ for any $x \neq 0$ in X. Hence $STx < TSx$ for any $x \neq 0$ in X.

Definition 2.3. If Two mapping S and T from a metric space (X, d) into itself, are called compatible mappings on X, if $\lim_{m\to\infty} d(STx_m, TSx_m) = 0$, when $\{x_m\}$ is a sequence in X such that $\lim_{m\to\infty} Sx_m = \lim_{m\to\infty} Tx_m = x$ for some x in X.

Clearly Two mapping S and T from a metric space (X, d) into itself, are called compatible mappings on X, then $d(STx, TSx) = 0$ when $d(Sx, Tx) = 0$ for some x in X . Note that weakly commuting mappings are compatible, but the converse is not necessarily true.

Example 2.2 [16]

Let $X = [0, 1]$ with the Euclidean metric d. Define S and $T : X \to X$ by

$Sx = x$ and $Tx = \frac{x}{x+1}$ for all x in X.

Then for any x in X ,

$$STx = S(Tx) = S\left(\frac{x}{x+1}\right) = \frac{x}{x+1}$$

$$TSx = T(Sx) = T(x) = \frac{x}{x+1}$$

$$d\left(Sx,Tx\right)=\left|x-\frac{x}{x+1}\right|=\left|\frac{x^2}{x+1}\right|$$

Thus we have

$$d\left(STx,TSx\right)=\left|\frac{x}{x+1}-\frac{x}{x+1}\right|$$

$$=0\leq\frac{x^2}{x+1}\ \text{for all}\ \ x\ \text{in}\ X.$$

$$=d\left(Sx,Tx\right)$$

. $d\left(STx,TSx\right)\leq d\left(Sx,Tx\right)$ for all x in X.

Thus S and T are weakly commuting mappings on X, and then obviously S and T are compatible mappings on X .

Example 2.3 [16]

Let $X = R$ with the Euclidean metric d. Define S and $T : X \to X$ by

$Sx=x^2$ and $Tx=2x^2$ for all x in X.

Then for any x in X ,

$$STx=S\left(Tx\right)=S\left(2x^2\right)=4x^4$$

$TSx=T\left(Sx\right)=T\left(x^2\right)=2x^4$ are compatible mappings on X , because

$$d\left(Sx,Tx\right)=\left|x^2-2x^2\right|=\left|-x^2\right|\to 0\ \ as\ \ x\to 0$$

Then

$$d\left(STx,TSx\right)=\left|4x^4-2x^4\right|=2\left|x^4\right|\to 0\ \ as\ \ x\to 0$$

But $d\left(STx,TSx\right)\leq d\left(Sx,Tx\right)$ is not true for all x in X.

Thus S and T are not weakly commuting mappings on X.

Hence all weakly commuting mappings are compatible, but converse is not true.

Definition 2.4 If Two mapping S and T from a metric space $\left(X,d\right)$ into itself, are called Weakly compatible mappings on X, if they commute at their coincidence point i.e. if $Su = Tu$ for some u in X ,then $STu = TSu$.

Example 2.4

Let $X = [0, 1]$ with the usual metric $d(x,y)=\left|x-y\right|$.

Define S and $T : X \to X$ by

$$Sx = \begin{cases} x & \text{when } 0 \le x \le \frac{1}{2} \\ 1 & \text{when } \frac{1}{2} < x \le 1 \end{cases}$$

$Tx = 1 - x$ for all x in X.

Then clearly $\frac{1}{2}$ is coincidence point of S and T, because $S\left(\frac{1}{2}\right) = T\left(\frac{1}{2}\right) = \frac{1}{2}$.

Also $ST\left(\frac{1}{2}\right) = S\left\{T\left(\frac{1}{2}\right)\right\} = S\left\{1 - \frac{1}{2}\right\} = S\left(\frac{1}{2}\right) = \frac{1}{2}$

$$TS\left(\frac{1}{2}\right) = T\left\{S\left(\frac{1}{2}\right)\right\} = T\left\{\frac{1}{2}\right\} = 1 - \frac{1}{2} = \frac{1}{2}$$

Hence (S, T) is weakly compatible on X, because they commutes at their coincidence point $\frac{1}{2}$.

But (S, T) is not compatible on X, for this take a sequence $x_n = \frac{1}{2} - \frac{1}{n}$, $n \ge 2$.

Then $\lim_{n\to\infty} Sx_n = \frac{1}{2}$, $\lim_{n\to\infty} Tx_n = \frac{1}{2}$ Also $\lim_{n\to\infty} TSx_n = \lim_{n\to\infty} T\left(\frac{1}{2} - \frac{1}{n}\right) = \lim_{n\to\infty}\left(\frac{1}{2} + \frac{1}{n}\right) = \frac{1}{2}$.

But $\lim_{n\to\infty} STx_n = \lim_{n\to\infty} S\left(\frac{1}{2} + \frac{1}{n}\right) = \lim_{n\to\infty}(1) = 1$ so that $\lim_{n\to\infty}(TSx_n, STx_n) \ne 0$.

Hence (S, T) is not compatible on X. Note that compatible mappings are weakly compatible, but the converse is not necessarily true.

In 1998, Singh and Chouhan [17] proved the following theorem.

Theorem 2.1 Let A, B, S and T be mappings from a complete metric space (X, d) into itself satisfying the following conditions:

$$A(X) \subseteq T(X) \text{ and } B(X) \subseteq S(X) \tag{3.1}$$

One of A, B, S and T is continuous,

$$\begin{aligned}[d(Ax, By)]^2 \le\ & k_1\left[d(Ax, Sx)d(By, Ty) + d(By, Sx)d(Ax, Ty)\right] \\ & + k_2\left[d(Ax, Sx)d(Ax, Ty) + d(By, Ty)d(By, Sx)\right]\end{aligned} \tag{3.2}$$

for all $x, y \in X$, where $k_1, k_2 \ge 0$ and $0 \le k_1 + k_2 < 1$.

The pairs (A, S) and (B, T) are compatible on X, then A, B, S and T have a unique common fixed point in X.

In 2014, Sharma, Badshah and Gupta [16] proved the following theorem.

Theorem 2.2 Let P, Q, S and T *be* mappings from a complete metric space (X, d) into itself satisfying the conditions

$$S(X) \subseteq Q(X), \; T(X) \subseteq P(X) \tag{3.3}$$

$$d(Sx, Ty) \le \left\{ \alpha + \beta \frac{d(Sx, Px)}{1 + d(Px, Qy)} \right\} d(Ty, Qy) \tag{3.4}$$

for all $x, y \in X$, where $\alpha, \beta \ge 0$ and $\alpha + \beta < 1$.

Suppose that

(i) One of P, Q, S and T is continuous,

(ii) Pairs (S, P) and (T, Q) are compatible on X.

Then P, Q, S and T have a unique common fixed point in X.

Now we generalize the theorem using weakly compatible mappings in place of compatible mappings also condition of any one of the mapping is being dropped. First we define the associated sequence.

Associated Sequence: Suppose P, Q, S and T be mappings from a complete metric space (X, d) into itself satisfying the conditions (3.3) and (3.4).

Then for an arbitrary point x_0 in X, by (3.3) we choose a point x_1 in X such that $Sx_0 = Qx_1$ and for this point x_1, there exists a point x_2 in X such that $Tx_1 = Px_2$ and so on. Proceeding in the similar manner, we can define a sequence $\{y_m\}$ in X such that

$$y_{2m+1} = Qx_{2m+1} = Sx_{2m} \text{ for } m \ge 0 \text{ and } y_{2m} = Px_{2m} = Tx_{2m-1} \quad \text{for } m \ge 1 \tag{3.5}$$

we shall call this sequence as an "Associated sequence of x_0" relative to four self mappings P, Q, S and T.

Lemma 2.1 Let P, Q, S and T be mappings from a complete metric space (X, d) into itself satisfying the conditions (3.3) and (3.4). Then the Associated sequence $\{y_m\}$ relative to four self mappings P, Q, S and T defined in (3.5) is a Cauchy sequence in X.

Proof. By definition (3.5) we have

$$d(y_{2m+1}, y_{2m}) = d(Sx_{2m}, Tx_{2m-1})$$

$$\leq \left\{\alpha + \beta \frac{d(Sx_{2m}, Px_{2m})}{1 + d(Px_{2m}, Qx_{2m-1})}\right\} d(Tx_{2m-1}, Qx_{2m-1})$$

$$\leq \left\{\alpha + \beta \frac{d(y_{2m+1}, y_{2m})}{1 + d(y_{2m}, y_{2m-1})}\right\} d(y_{2m}, y_{2m-1})$$

$$\leq \alpha d(y_{2m}, y_{2m-1}) + \beta d(y_{2m+1}, y_{2m})$$

$$d(y_{2m+1}, y_{2m}) \leq \frac{\alpha}{1-\beta} d(y_{2m}, y_{2m-1})$$

Hence $d(y_{2m+1}, y_{2m}) \leq h\, d(y_{2m}, y_{2m-1})$

Where $h = \dfrac{\alpha}{1-\beta} < 1$

Similarly we can show that

$$d(y_{2m+1}, y_{2m}) \leq h^{2m}\, d(y_1, y_0)$$

For k > m , we have

$$d(y_{m+k}, y_m) \leq \sum_{i=1}^{k} d(y_{n+i}, y_{n+i-1})$$

$$\leq \sum_{i=1}^{k} h^{n+i-1} d(y_1, y_0)$$

$$d(y_{m+k}, y_m) \leq \left(\frac{h^n}{1-h}\right) d(y_1, y_0) \to 0 \text{ as } n \to \infty$$

Since $h<1$, $h^n \to 0$ as $n \to \infty$. So that $d(y_m\,, \, y_{m+k}) \to 0$. This Show that the sequence $\{y_m\}$ is a Cauchy's sequence in *X*. and since *X* is a complete metric space, it converges to a limit, say *u* in *X*. The converse of the lemma is not true, that is *P, Q, S* and *T* satisfying (3.3) and (3.4), even if for x_0 in X and the Associated sequence of x_0 converges, the metric space (X, d) need not be complete. The following example establishes this.

Example 2.5

Let *X* = (-1, 1) with usual metric $d(x, y) = |x - y|$

$$Sx = Tx = \begin{cases} \frac{1}{4} & \text{if } -1 < x < \frac{1}{5} \\ \frac{1}{5} & \text{if } \frac{1}{5} \le x < 1 \end{cases}, \text{ and } \quad Px = Qx = \begin{cases} \frac{1}{4} & \text{if } -1 < x < \frac{1}{5} \\ \frac{2}{5} - x & \text{if } \frac{1}{5} \le x < 1 \end{cases}$$

Then $S(X) = T(X) = \left\{\frac{1}{4}, \frac{1}{5}\right\}$ while, $P(X) = Q(X) = \left\{\frac{1}{4} \cup \left[\frac{1}{5} \quad \frac{-3}{5}\right)\right\}$

Clearly $S(X) \subset Q(X)$ and $T(X) \subset P(X)$. Also inequality (3.4) can be easily verified with appropriate values of α and β. Also the sequence $Sx_0, Tx_1, Sx_2, Tx_3, \ldots Sx_{2n}, Tx_{2n+1}, \ldots$ converges to $\frac{1}{4}$ if $-1 < x < \frac{1}{5}$ and $\frac{1}{5}$ if $\frac{1}{5} \le x < 1$. But (X, d) is not a complete metric space.

3. Main Result

Theorem 3.1 Let *P*, *Q*, *S* and *T* be mappings from a complete metric space (X, d) into itself satisfying the conditions (3.3) and (3.4). Suppose that the pairs (S, P) and (T, Q) are weakly compatible on *X*. Further the associated sequence relative to four self mappings *P*, *Q*, *S* and *T* such that Sx_0, Tx_1, Sx_2, $Tx_3, \ldots$, Sx_{2m}, Tx_{2m+1} converges to u in *X* as $n \to \infty$ .Then *P*, *Q*, *S* and *T* have a unique common fixed point *u* in *X*.

Proof. Let $\{y_m\}$ be the associated sequence in *X* defined by (3.5).Then by lemma 2.1, Associated sequence $\{y_m\}$ is a Cauchy sequence in *X* and hence it converges to some point *u* in *X*. Consequently, the subsequences $\{Sx_{2m}\}$, $\{Px_{2m}\}$, $\{Tx_{2m-1}\}$ and $\{Qx_{2m-1}\}$ of $\{y_m\}$ also converges to *u*.

Since $S(X) \subseteq Q(X)$ then there exists $v \in X$ such that $u = Qv$ we prove that $Tv = u$.

By (3.4), we obtain

$$\begin{aligned} d(u, Tv) &= \lim_{m\to\infty} d(Sx_{2m}, Tv) \\ &\le \lim_{m\to\infty} \left\{\alpha + \beta \frac{d(Sx_{2m}, Px_{2m})}{1 + d(Px_{2m}, Qv)}\right\} d(Tv, Qv) \\ &= \left\{\alpha + \beta \frac{d(u,u)}{1 + d(u,u)}\right\} d(Tv, u) \end{aligned}$$

$(1-\alpha)d(Tv,u)\le 0$

So that $u=Tv$. Since (T,Q) is weakly compatible and $Qv=Tv=u$, then $QTv=TQv$ therefore $Qu=Tu$ and $T(X)\subseteq P(X)$ there exists $v'\in X$ such that $u=Pv'$, we prove that $Sv'=Pv'$.

By (3.4), we obtain

$$\begin{aligned} d(Sv',u) &= \lim_{m\to\infty} d(Sv', Tx_{2m-1}) \\ &\le \lim_{m\to\infty}\left\{\alpha+\beta\frac{d(Sv',Pv')}{1+d(Pv',Qx_{2m-1})}\right\}d(Tx_{2m-1},Qx_{2m-1}) \\ &= \left\{\alpha+\beta\frac{d(Sv',u)}{1+d(u,u)}\right\}d(u,u) \end{aligned}$$

$$d(Sv',u)\le 0$$

So that $u=Sv'$. Hence $u=Pv'=Sv'$ and (S,P) is weakly compatible on *X*, then $SPv'=PSv'$ therefore $Su=Pu$.

Now consider

$$\begin{aligned} d(Su,u) &= d(Su,Tv) \\ &\le \left\{\alpha+\beta\frac{d(Su,Pu)}{1+d(Pu,Qv)}\right\}d(Tv,Qv) \\ &= \left\{\alpha+\beta\frac{d(Su,Su)}{1+d(Su,u)}\right\}d(u,u) \end{aligned}$$

$$d(Su,u)\le 0$$

So that $u=Su$. Hence $u=Pu=Su$ therefore u is a common fixed point of *S* and *P*. Now we show that $u=Tu$.

Now consider

$$\begin{aligned} d(u,Tu) &= d(Su,Tu) \\ &\le \left\{\alpha+\beta\frac{d(Su,Pu)}{1+d(Pu,Qu)}\right\}d(Tu,Qu) \\ &= \left\{\alpha+\beta\frac{d(u,u)}{1+d(u,Qu)}\right\}d(Qu,Qu) \end{aligned}$$

$$d(u,Tu)\le 0$$

So that $u = Tu$. Hence $u = Tu = Qu$ therefore u is a common fixed point of T and Q. Thus $u = Tu = Qu = Su = Pu$. Hence u is a common fixed point of P, Q, S and T.

For uniqueness of u, suppose u and z, $u \neq z$, are common fixed points of P, Q, S and T. Then by (3.4), we obtain

$$\begin{aligned} d(u,z) &= d(Su,Tz) \\ &\leq \left\{\alpha + \beta \frac{d(Su,Pu)}{1+d(Pu,Qz)}\right\} d(Tz,Qz) \\ &\leq \left\{\alpha + \beta \frac{d(u,u)}{1+d(u,z)}\right\} d(z,z) \\ &\leq 0 \\ d(u,z) &\leq 0 \end{aligned}$$

which is a contradiction .Hence $u = z$. This completes the proof.

Remark 3.1 From the example 2.4 given above, clearly the pairs (S,P) and (T,Q) are weakly compatible on X as they commute at their coincidence point $\frac{1}{5}$.But the pairs (S,P) and (T,Q) are not compatible on X ,for this take a sequence $x_n = \frac{1}{5} + \frac{1}{n}$, $n \geq 2$.

Then $\lim_{n\to\infty} Sx_n = \frac{1}{5}$, $\lim_{n\to\infty} Tx_n = \frac{1}{5}$ Also $\lim_{n\to\infty} PSx_n = \lim_{n\to\infty} P\left(\frac{1}{5} + \frac{1}{n}\right) = \lim_{n\to\infty} P\left(\frac{1}{5}\right) = \frac{1}{5}$.

But $\lim_{n\to\infty} SPx_n = \lim_{n\to\infty} S\left\{P\left(\frac{1}{5} + \frac{1}{n}\right)\right\} = \lim_{n\to\infty} S\left(\frac{1}{5} - \frac{1}{n}\right) = \frac{1}{4}$, so that $\lim_{n\to\infty}\left(PSx_n, SPx_n\right) \neq 0$.

Hence (S,P) is not compatible on X. Also note that none of the mappings are continuous and the rational inequality holds for appropriate value of α, β with α, $\beta \geq 0$ and $\alpha + \beta < 1$. Clearly $\frac{1}{5}$ is the unique common fixed point of P, Q, S and T.

Conclusion: In this paper that we shown that a unique common fixed point theorem which generalize the result of Sharma, Badshah and Gupta [16] in sense that we using weaker condition weakly compatible, instead of compatible mappings.

Conflict of Interests

The authors declare that there is no conflict of interests.

Acknowledgment: Authors are thankful to the referee for his/her valuable comments for the Improvement of this paper.

REFERENCES

[1] Banach, S., Surles operations dans les ensembles abstraits et leur application aux equations integrals, Fund. Math., 3 (1922), 133-181.

[2] Das, K.M. and Naik, K.V., Common fixed point theorems for commuting maps on a metric space, Proc. Amer. Math. Soc., 77(1979), 369-373.

[3] Fisher, B., Common fixed point of commuting mappings, Bull. Inst. Math. Acad. Scinica, 9 (1981), 399-406.

[4] Fisher, B.: Common fixed points of four mappings, Bull. Inst. Math. Acad. Scinica, 11(1983), 103-113.

[5] Iseki, K. and Singh, Bijendra, On common fixed point theorems of mappings, Math. Sem. Notes, Kobe Univ., 2(1974), 96..

[6] Jungck, G., Commuting mappings and fixed points, Amer. Math. Monthly, 83(1976), 261-263.

[7] Jungck, G., Compatible mappings and common fixed points, Internat. J. Math. and Math. Sci., 9(4) (1986), 771-779.

[8] Jungck, G., Compatible mappings and common fixed points (2), Int. J. Math. and Math. Sci., 11(2) (1988), 285-288.

[9] Jungck, G. and Rhoades, B.E., Fixed points for set valued functions without continuity, Indian J. Pure Appl. Math. 29(1998), 227-238.

[10] Kang, S.M. and Kim, Y.P., Common fixed point theorems, Math. Japon, 37(1992), 1031-1039.

[11] Kannan, R., Some results on fixed points, Bull. Cal. Math. Soc., 60 (1968), 71-76.

[12] Khan, M.S. and Imdad, M., Some common fixed point theorems, Glasnik Mat., 18(38) (1983), 321-326.

[13] Lohani, P.C. and Badshah, V.H., Compatible mappings and common fixed point for four mappings, Bull. Cal. Math. Soc., 90(1998), 301-308.

[14] Park, S. and Bae, J.S., Extensions of common fixed point theorem of Meir and Keeler, Ark. Math., 19(1981), 223-228.

[15] Sessa, S., On a weak commutativity condition in fixed point consideration, Publ. Inst. Math. Beograd, 32(46) (1982), 146-153.

[16] Sharma, A.K., Badshah, V.H. and Gupta, V.K., A Common fixed point theorem for Compatible mappings in complete metric space using rational inequality, Inst. J. Adv. Tech. Engg. And Science, 2(8)(2014), 395-407.

[17] Singh, B. and Chauhan, S., On common fixed points of four mappings, Bull. Cal. Math. Soc., 88(1998), 301-308.

[18] Singh, S.L., Application of a common fixed point theorem, Math. Sem. Notes, 6(1) (1978), 37-40.

[19] Singh, S.L. and Singh, S.P., A fixed point theorem, Indian J. Pure Appl. Math. 11(1980), 1584-1586.