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PERTURBATION ANALYSIS IN NON-STATIONARY AR(1) TIME SERIES

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Abstract. This paper studies the application of perturbation analysis in approximating non-stationary first order autoregressive models. Asymptotic results using perturbation analysis are given. Error analysis shows that our method works well. Some simulation results are also given.

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1. Introduction.

Perturbation is a numerical technique designed for analyzing problems which cannot be solved exactly but they can be characterized by adding a small leading parameter (denoted by ε) to the exact solutions at hand. In this method, it is assumed that an intricate dependency between the solution and ε exists and an expression is derived for the desired solution (in terms of a power series in ε, for example, using Taylor expansion). The final product of this process is the exact solution of initial problem with further terms which describe the deviation in the solution. Finally, an error analysis is necessary to evaluate the performance of this method in the case of study. This method is applicable to solve algebraic and differential equations and for approximating integrals and gradient and Monte Carlo estimations (see Nayfeh(1981)).

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The perturbation theory is also applicable for statistical problems. For example, it can be used for design of control chart (see Fu and Hu (1997)), analysis of discrete-event dynamic systems (refer to Gong and Ho (1987)) and studying generalized semi-Markov process (see Fu and Hu (1992)). Fouque et al. (2003) applied this method for option pricing models under stochastic volatility assumptions.

The first order autoregressive model $AR(1)$ is often used for prediction in finance. This model is non-stationary if it has a unit root. Therefore, it is assumed that the slop parameter remains between -1 and 1. However, this parameter may be very close to 1. This phenomena is referred to near unit root problem. For large sample size, Bobkoski (1983) showed this process converges to Ornstein-Uhlenbeck (OU) process. He also derived the asymptotic properties of least square estimate of slop. For more general results, see Cox and Llatas (1991). But this is definitely not the end of the story. Results may vary under slop perturbation. In the current paper, we study the perturbation analysis in near unit root $AR(1)$ model. This paper is organized as follows. In section 2, we propose Bobkoski's limiting distributions under perturbation assumption. Error analysis is also studied in this section. Simulation results are given in section 3.

2. Perturbation analysis.

Let $Z_1, Z_2, ...$ be a sequence of independent and identically distributed zero mean random variables with σ^2. Suppose that $Y_{k+1}^n = \phi_n Y_k^n + Z_{k+1}$, $k = 0, 1, ...$, where $\phi_n = 1 - \beta/n$. One can note that ϕ_n is very close to 1 as $n \to \infty$. Bobkoski (1983) proved that $X_n(t) = n^{(-1/2)} Y_{[nt]}^n$ converges weakly to a OU process with parameters β and σ, i.e.,

$$dX = -\beta X dt + \sigma dW,$$

where W is standard Brownian motion on (0,1). In this note, we let

$$\phi_n^\varepsilon = 1 - \varepsilon\beta/n,$$

where ε is a small number and replace Y_{k+1}^n, $X_n(t)$ with $Y_{k+1}^{n,\varepsilon}$, $X_n^\varepsilon(t)$, respectively. Notice that the slop can be close to 1 because of two factors, first as $n \to \infty$ and second as $\varepsilon \to 0$. Following Bobkoski (1983), as follows, we show that the convergence process is

OU process with parameters $\varepsilon\beta$ and σ. One can note that

$$X_n^\varepsilon(t) = (\phi_n^\varepsilon)^{[nt]} X_n^\varepsilon(0) + \int_0^t (\phi_n^\varepsilon)^{[nt]-1-[ns]} dW_n(s),$$

where $W_n(s) = n^{(-1/2)} \sum_{i=1}^{[nt]} Z_i$. If $X_n^\varepsilon(0) \to X_0^\varepsilon$, then $X_n^\varepsilon \Longrightarrow X^\varepsilon$, where

$$X_t^\varepsilon = e^{-\beta\varepsilon t} X_0^\varepsilon + \int_0^t e^{-\beta\varepsilon(t-s)} dW(s),$$

or equivalently, $dX_t^\varepsilon = -\varepsilon\beta X_t^\varepsilon dt + \sigma dW_t$. When, $\varepsilon = 0$ then X_t^ε is σW_t. If $\varepsilon \to 0$, then

$$e^{-\beta\varepsilon t} = (1 - \beta\varepsilon t + \frac{(\beta\varepsilon t)^2}{2}) + O(\varepsilon^2).$$

Therefore, we can approximate X_t^ε in the form of $X_t^\varepsilon = \widehat{X}_t^\varepsilon + O(\varepsilon^2)$, where

$$\begin{aligned}\widehat{X}_t^\varepsilon &= (1 - \beta\varepsilon t + \frac{(\beta\varepsilon t)^2}{2}) X_0^\varepsilon \\ &\quad + \int_0^t (1 - \beta\varepsilon(t-s) + \frac{(\beta\varepsilon(t-s))^2}{2}) dW(s).\end{aligned}$$

Here, we consider the error analysis for evaluate the performance of our method. A direct computation shows that mean function is zero, i.e., $E(\widehat{X}_t^\varepsilon) = 0$ and the variance function of perturbed process is

$$var(\widehat{X}_t^\varepsilon) = t(1 + \beta\varepsilon t)\sigma^2,$$

and for $s < t$, the covariance function is

$$cov(\widehat{X}_s^\varepsilon, \widehat{X}_t^\varepsilon) = s(1 + \beta\varepsilon t)\sigma^2.$$

One can see that these function are perturbation approximation of mean, variance and covariance functions of a OU process with parameters $\varepsilon\beta$ and σ (i.e., X_t^ε) as $\varepsilon \to 0$. This shows that our method approximates these functions well. Since the main properties of each OU is derived from this functions, we find that our method works well.

3. Simulations. In this section, we study the behavior of slop estimation in perturbed OU process. We consider two cases, the fixed volatility model and second the stochastic volatility case.

Fixed volatility model. Let $\widehat{\phi}_n^\varepsilon$ be the least square estimate of ϕ_n^ε. Then

$$\sum_{k=0}^{n-1} Y_k^{n,\varepsilon} \{(\phi_n^\varepsilon - \widehat{\phi}_n^\varepsilon) Y_k^{n,\varepsilon} + Z_{k+1}\} = 0.$$

This equations says that

$$n(\phi_n^\varepsilon - \widehat{\phi}_n^\varepsilon)\int_0^{\frac{n-1}{n}} X_n^\varepsilon(s)ds = \int_0^{\frac{n-1}{n}} X_n^\varepsilon(s^-)dW_n(s).$$

As $n \to \infty$, then $n(\widehat{\phi}_n^\varepsilon - \phi_n^\varepsilon)$ converges to in distribution to U^ε where

$$U^\varepsilon = \frac{\sigma\int_0^1 X_s^\varepsilon dW(s)}{\int_0^1 X_s^{2\varepsilon}ds}.$$

One should note that $\widehat{\beta} - \beta = \varepsilon^{-1}U^\varepsilon$. For small ε process X_t^ε is approximated by $\widehat{X}_t^\varepsilon$ in above formula and $\widehat{U}^\varepsilon$ is obtained. It is expected $\widehat{U}^\varepsilon$ is close to U^ε. Here, using a Monte Carlo simulation study with $M = 1000$ repetitions, we derive the maximum (max) and median (med) of errors $|\widehat{U}^\varepsilon - U^\varepsilon|$ for various values of ε. We fixed $\beta = 0.5$ and $\sigma = 1$. The results are given in the following table.

Table 1: Max and Med of errors

ε	0.1	0.05	0.025	0.001	0.0001
max	0.1	0.08	0.08	0.05	0.01
med	0.05	0.045	0.033	0.022	0.006

Stochastic volatility case. The OU process plays an important role in financial engineering. The volatility of this process isn't fixed in practice and a stochastic volatility model is fitted much better to financial time series. Therefore, we consider the following models

$$\begin{cases} dX_t^\varepsilon = \varepsilon\beta X_t^\varepsilon dt + \sigma_t dW_t \\ d\sigma_t^2 = \mu\sigma_t^2 dt + \xi\sigma_t^2 dB_t, \end{cases}$$

where W_t and B_t are two independent Wiener processes. Following Kutoyants (2004), the least square estimate $\widehat{\zeta}^\varepsilon = \varepsilon\widehat{\beta}^\varepsilon$ is the minimizer of

$$\int_0^1 \sigma_t^{-2}(dX_t^\varepsilon - \varepsilon\beta X_t^\varepsilon dt)^2.$$

It is seen that

$$\widehat{\zeta}^\varepsilon = \frac{\int_0^1 \sigma_s^{-2}X_s^\varepsilon dW(s)}{\int_0^1 \sigma_s^{-2}X_s^{2\varepsilon}ds}.$$

It is easy to see that $\widehat{\zeta}^\varepsilon - \zeta^\varepsilon = U_{sv}^\varepsilon$, where

$$U_{sv}^\varepsilon = \frac{\int_0^1 \sigma_s^{-1}X_s^\varepsilon dW(s)}{\int_0^1 \sigma_s^{-2}X_s^{2\varepsilon}ds}.$$

Again, in practice X_s^ε is approximated by perturbed process $\widehat{X}_t^\varepsilon$. The following table gives the value of maximum and median errors as we did for table 2. Here, we let $\mu = 0.4$, $\xi = 0.75$ and $\beta = 0.5$.

Table 2: Max and Med of errors; SV model

ε	0.1	0.05	0.025	0.001	0.0001
max	0.12	0.076	0.083	0.075	0.01
med	0.08	0.032	0.013	0.068	0.003

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