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CALDERON'S REPRODUCING FORMULA FOR DUNKL CONVOLUTION

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Abstract. Calderon-type reproducing formula for Dunkl convolution is established using the theory of Dunkl transform.

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1. Introduction

Calderon formula [8] involving convolution related to the Fourier transform is useful in obtaining reconstruction formula for wavelet transform besides many other applications in decomposition of certain function spaces. It is expressed as follows:

$$f(x) = \int_0^\infty (\phi_t * \phi_t * f)(x) \frac{dt}{t}, \tag{1.1}$$

where $\phi : \mathbb{R}^n \to C \ and \ \phi_t(x) = t^{-n}\phi(x/t),\ t > 0.$ For conditions of validity of identity (1.1), we may refer to [8].

On the real line, the Dunkl operator are differential-difference operator introduced by Dunkl [1] and are denoted by Λ_α, where α is real parameter $> -1/2$. These operator associated with the reflection group $\mathbb{Z}_2 \, on \, \mathbb{R}$. The Dunkl kernel E_α is used to define the Dunkl transform which was introduced by Dunkl in [2]. Rosler in [3] show that the Dunkl kernels verify a product formula. This allows to define the Dunkl translation. As a result, we have the Dunkl convolution.

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Dunkl Operator has a unique solution $E_\alpha(\lambda x)$, called Dunkl kernel and given by

$$E_\alpha(\lambda x) = j_\alpha(i\lambda x) + \frac{\lambda x}{2(\alpha+1)} j_{\alpha+1}(i\lambda x), \quad x \in R, \tag{1.2}$$

where j_α is the normalized Bessel function of the first kind and order α.

Let $\alpha > -1/2$ be a fixed number and μ_α be the weighted Lebesgue measure on *R*, given by

$$d\mu_\alpha(x) := \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^{-1} |x|^{2\alpha+1} dx. \tag{1.3}$$

We define $\mathrm{L}_{p,\alpha}(0,\infty)$, $1 \le p \le \infty$, as the spaces of those real measurable function f on $(0,\infty)$ for which

$$\|f\|_{p,\alpha} = \left(\int_R |f(x)|^p d\mu_\alpha(x)\right)^{1/p} < \infty \quad if \quad p \in [1,\infty) \tag{1.4}$$

and $\|f\|_\infty = ess \sup_{x \in R} |f(x)|$ if $p = \infty$.

The Dunkl kernel gives rise to an integral transform, called Dunkl transform on *R*, which was introduced and studied in [7].

The Dunkl transform F_α of a function $f \in L_{1,\alpha}(R)$, is given by

$$F_\alpha f(\lambda) = \hat{f}(\lambda) = \int_R E_\alpha(-i\lambda x) f(x) d\mu_\alpha(x) \quad ; \lambda \in R \tag{1.5}$$

An inversion formula for this transform is given by

$$F_\alpha^{-1}\left(\hat{f}(\lambda)\right) = \left(\hat{f}(\lambda)\right)^\vee = f(x) = \int_R E_\alpha(i\lambda x) \hat{f}(\lambda) d\mu_\alpha(\lambda) \tag{1.6}$$

An Parseval formula for this transform is given by

$$\int_{-\infty}^{\infty} f(x) g(x) dx = \int_{-\infty}^{\infty} \hat{f}(\lambda) \hat{g}(\lambda) \tag{1.7}$$

To define Dunkl convolution $*_\alpha$, we define

$$W_\alpha(x, y, z) = \int_0^\infty E_\alpha(\lambda x) E_\alpha(\lambda y) E_\alpha(\lambda z) d\mu_\alpha(\lambda) \tag{1.8}$$

$$= \left(1 - \sigma_{x,y,z} + \sigma_{z,x,y} + \sigma_{z,y,x}\right) \Delta_\alpha(x, y, z)$$

where $\sigma_{x,y,z} = \begin{cases} \dfrac{x^2+y^2+z^2}{2xy}, if & x, y \in R \setminus 0 \\ 0 & otherwise \end{cases}$

and Δ_α is the Bessel kernel. Clearly $W_\alpha(x, y, z)$ is symmetric in x, y, z. Apply inversion formula (1.6) to (1.8), we get

$$\int_0^\infty E_\alpha(\lambda z) W_\alpha(x, y, z)\, d\mu_\alpha(z) = E_\alpha(\lambda x) E_\alpha(\lambda y)\,. \tag{1.9}$$

Now setting $\lambda = 0$, we obtain

$$\int_0^\infty W_\alpha(x, y, z)\, d\mu_\alpha(z) = 1\,. \tag{1.10}$$

Let $p, q, r \in [1,\infty)$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. Then Dunkl convolution of $f \in L_{p,\alpha}(R)$ and $g \in L_{q,\alpha}(R)$ is defined by [7]

$$(f *_\alpha g)(x) = \int_R \int_R f(z) g(y) W_\alpha(x, y, z)\, d\mu_\alpha(y)\, d\mu_\alpha(z) \tag{1.11}$$

Let $p, q, r \in [1, \infty[$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, $f \in L_{p,\alpha}(R)$ and $g \in L_{q,\alpha}(R)$. Then convolution $f *_\alpha g(x)$ satisfies the following norm inequality

(i) $$\| f *_\alpha g \|_{r,\alpha} \le 4 \| f \|_{p,\alpha} \| g \|_{q,\alpha} \tag{1.12}$$

Moreover for all $f \in L_{1,\alpha}(R)$ and $g \in L_{2,\alpha}(R)$, we have

(ii) $$(f *_\alpha g)(\lambda) = f(\lambda) g(\lambda) \tag{1.13}$$

2. Calderon's formula

In this section, we obtain Calderon's reproducing identity using the properties of Dunkl transform and Dunkl convolutions.

Theorem 2.1 Let ϕ *and* $\psi \in L_{1,\mu_\alpha}[0,\infty)$ be such that following admissibility condition holds:

$$\int_0^\infty \hat{\phi}(\xi) \hat{\psi}(\xi) \frac{d\mu_\alpha(\xi)}{\xi} = 1 \tag{2.1}$$

for all $\xi \in [0,\infty)$. Then the following Calderon's reproducing identity holds:

$$f(x) = \int_0^\infty (f *_\alpha \phi_a *_\alpha \psi_a)(x) \frac{d\mu_\alpha(a)}{a}, \quad \forall\, f \in L^1(R)\,. \tag{2.2}$$

Proof: Taking Dunkl transform of the right hand side of (2.2), we get

$$F_\alpha\left[\int_0^\infty \left(f *_\alpha \phi_a *_\alpha \psi_a\right)(x)\frac{d\mu_\alpha(a)}{a}\right]$$

$$= \int_0^\infty \hat{f}(\xi)\,\phi_a(\xi)\hat{\psi}_a(\xi)\frac{d\mu_\alpha(a)}{a}$$

$$= \hat{f}(\xi)\int_0^\infty \phi_a(\xi)\hat{\psi}_a(\xi)\frac{d\mu_\alpha(a)}{a} \tag{2.3}$$

$$= \hat{f}(\xi)\int_0^\infty \left(\phi(a\xi)\hat{\psi}(a\xi)\right)\frac{d\mu_\alpha(a)}{a}$$

$$= \hat{f}(\xi)$$

Now, by putting $a\xi = \omega$

$$\int_0^\infty \hat{\phi}(a\xi)\hat{\psi}(a\xi)\frac{d\mu_\alpha(a)}{a} = \int_0^\infty \hat{\phi}(\omega)\hat{\psi}(\omega)\frac{d\mu_\alpha(\omega)}{\omega} \tag{2.4}$$

$$= 1.$$

Hence the result follows.

Theorem 2.2 Suppose $\phi \in L_{1,\mu_\alpha}[0,\infty)$ is real valued and satisfies

$$\int_0^\infty \left[\hat{\phi}(a\xi)\right]^2 \frac{d\mu_\alpha(a)}{a} = 1. \tag{2.5}$$

For $f \in L_{1,\mu_\alpha}[0,\infty) \cap L_{2,\mu_\alpha}[0,\infty)$, suppose that

$$f_{\varepsilon,\delta}(x) = \int_\varepsilon^\delta \left(f *_\alpha \phi_a *_\alpha \phi_a\right)(x)\frac{d\mu_\alpha(a)}{a} \tag{2.6}$$

Then $\left\| f - f_{\varepsilon,\delta}\right\|_{2,\mu_\alpha} \to 0 \ as\ \varepsilon \to 0 \ \&\ \delta \to 0$.

Proof: Taking Dunkl transform of both sides of (2.6) and using Fubini's theorem, we get

$$\hat{f}_{\varepsilon,\delta}(\xi) = \hat{f}(\xi)\int_\varepsilon^\delta \left[\hat{\phi}(a\xi)\right]^2 \frac{d\mu_\alpha(a)}{a} \tag{2.7}$$

By [4], we have

$$\left\|\phi_a *_\alpha \phi_a *_\alpha f\right\|_{2,\mu_\alpha} \le \left\|\phi_a *_\alpha \phi_a\right\|_{1,\mu_\alpha} \left\|f\right\|_{2,\mu_\alpha}$$

$$\le \left\|\phi_a\right\|_{1,\mu_\alpha}^2 \left\|f\right\|_{2,\mu_\alpha}. \tag{2.8}$$

Now using above inequality and Minkowski's inequality [6, page 41], we get

$$\|f\|_{2,\mu_\alpha}^2=\int_0^\infty d\mu_\alpha(x)\left|\int_\varepsilon^\delta(\phi_a *_\alpha \phi_a *_\alpha f)(x)\frac{d\mu_\alpha(a)}{a}\right|^2$$

$$\leq\int_\varepsilon^\delta\int_0^\infty\left|(\phi_a *_\alpha \phi_a *_\alpha f)(x)\right|^2 d\mu_\alpha(x)\frac{d\mu_\alpha(a)}{a}$$

$$\leq\int_\varepsilon^\delta\left\|(\phi_a *_\alpha \phi_a *_\alpha f)(x)\right\|_{2,\mu_\alpha}\frac{d\mu_\alpha(a)}{a} \tag{2.9}$$

$$\leq\|\phi_a\|_{1,\mu_\alpha}^2\|f\|_{2,\mu_\alpha}\int_\varepsilon^\delta\frac{dt}{t}$$

$$=\|\phi_a\|_{1,\mu_\alpha}^2\|f\|_{2,\mu_\alpha}\log\left(\frac{\delta}{\varepsilon}\right).$$

Hence by Parseval formula, we get

$$\lim_{\substack{\varepsilon\to0\\\delta\to\infty}}\left\|f-f_{\varepsilon,\delta}\right\|_{2,\mu_\alpha}^2=\lim_{\substack{\varepsilon\to0\\\delta\to\infty}}\left\|\hat f-\hat f_{\varepsilon,\delta}\right\|_{2,\mu_\alpha}^2$$

$$\lim_{\substack{\varepsilon\to0\\\delta\to\infty}}\int_0^\infty\left|\hat f(\xi)\left(1-\int_\varepsilon^\delta\left[\hat\phi(a\xi)\right]^2\frac{d\mu_\alpha(a)}{a}\right)\right|^2 d\mu_\alpha(x) \tag{2.10}$$

$$=0.$$

Since $\left|\hat f(\xi)\left(1-\int_\varepsilon^\delta\left[\hat\phi(a\xi)\right]^2\frac{d\mu_\alpha(a)}{a}\right)\right|\leq\hat f(\xi)$, therefore by the dominated convergence theorem, the result follows.

The reproducing identity (2.2) holds in the point wise sense under different set of nice conditions.

Theorem 2.3 Suppose $f,\hat f\in L_{1,\mu_\alpha}[0,\infty)$. *Let* $\phi\in L_{1,\mu_\alpha}[0,\infty)$ be real valued and satisfies

$$\int_0^\infty\left[\hat\phi(a\xi)\right]^2\frac{d\mu_\alpha(a)}{a}=1,\quad \xi\in R-\{0\}. \tag{2.11}$$

Then

$$\lim_{\substack{\varepsilon\to0\\\delta\to\infty}}\int_\varepsilon^\delta(f *_\alpha \phi_a *_\alpha \phi_a)(x)\frac{d\mu_\alpha(a)}{a}=f(x)\qquad. \tag{2.12}$$

Proof: Let

$$f_{\varepsilon,\delta}(x)=\int_\varepsilon^\delta(f *_\alpha \phi_a *_\alpha \phi_a)(x)\frac{d\mu_\alpha(a)}{a}. \tag{2.13}$$

By [4, page 311], we have

$$\|\phi_a *_\alpha \phi_a *_\alpha f\|_{1,\mu_\alpha} \le \|\phi_a *_\alpha \phi_a\|_{1,\mu_\alpha} \|f\|_{1,\mu_\alpha}$$

$$\le \|\phi_a\|_{1,\mu_\alpha}^2 \|f\|_{1,\mu_\alpha} \tag{2.14}$$

Now

$$\|f_{\varepsilon,\delta}\|_{1,\mu_\alpha} = \int_0^\infty d\mu_\alpha(x) \left| \int_\varepsilon^\delta (\phi_a *_\alpha \phi_a *_\alpha f)(x) \frac{d\mu_\alpha(a)}{a} \right|$$

$$\le \int_\varepsilon^\delta \int_0^\infty |(\phi_a *_\alpha \phi_a *_\alpha f)(x)| d\mu_\alpha(x) \frac{d\mu_\alpha(a)}{a}$$

$$\le \int_\varepsilon^\delta \|(\phi_a *_\alpha \phi_a *_\alpha f)(x)\|_{1,\mu_\alpha} \frac{d\mu_\alpha(a)}{a} \tag{2.15}$$

$$\le \|\phi_a\|_{1,\mu_\alpha}^2 \|f\|_{1,\mu_\alpha} \int_\varepsilon^\delta \frac{dt}{t}$$

$$= \|\phi_a\|_{1,\mu_\alpha}^2 \|f\|_{1,\mu_\alpha} \log\left(\frac{\delta}{\varepsilon}\right).$$

Therefore, $f_{\varepsilon,\delta} \in L^1(0,\infty)$. Also using Fubini's, we get theorem and taking Dunkl transform of (2.13), we get

$$\hat{f}_{\varepsilon,\delta}(\xi) = \int_0^\infty E_\alpha(x\xi) \left(\int_\varepsilon^\delta (\phi_a *_\alpha \phi_a *_\alpha f)(x) \frac{d\mu_\alpha(a)}{a} \right) d\mu_\alpha(a)$$

$$= \int_\varepsilon^\delta \int_0^\infty E_\alpha(x\xi)(\phi_a *_\alpha \phi_a *_\alpha f)(x)\, d\mu_\alpha(x) \frac{d\mu_\alpha(a)}{a} \tag{2.16}$$

$$= \int_\varepsilon^\delta \hat{\phi}_a(\xi)\hat{\phi}_a(\xi)\, \hat{f}(\xi) \frac{d\mu_\alpha(a)}{a}$$

$$= \hat{f}(\xi) \int_\varepsilon^\delta [\hat{\phi}(a\xi)]^2 \frac{d\mu_\alpha(a)}{a} \quad .$$

Therefore, by (2.11), $\left|\hat{f}_{\varepsilon,\delta}(\xi)\right| \le \left|\hat{f}(\xi)\right|$.

It follows that $\hat{f}_{\varepsilon,\delta} \in L_{1,\mu_\alpha}[0,\infty)$ .By inversion, we have

$$f(x) - f_{\varepsilon,\delta}(x) = \int_0^\infty E_\alpha(x\xi)[\hat{f}(\xi) - \hat{f}_{\varepsilon,\delta}(\xi)]\, d\mu_\alpha(\xi),\ x \in [0,\infty) \tag{2.17}$$

Putting

$$h_{\varepsilon,\delta}(\xi : x) = E_\alpha(x\xi)\left[\hat{f}(\xi) - \hat{f}_{\varepsilon,\delta}(\xi)\right]$$

$$= \hat{f}(\xi)\, E_\alpha(x\xi)\left[1 - \int_\varepsilon^\delta [\hat{\phi}(a\xi)]^2 \frac{d\mu_\alpha(a)}{a}\right] \qquad (2.18)$$

we get

$$f(x) - f_{\varepsilon,\delta}(x) = \int_0^\infty E_\alpha(x\xi)\left[\hat{f}(\xi) - \hat{f}_{\varepsilon,\delta}(\xi)\right] d\mu_\alpha(\xi) \qquad (2.19)$$

$$= \int_0^\infty h_{\varepsilon,\delta}(\xi : x)\, d\mu_\alpha(\xi).$$

Now using (2.11) in (2.18), we get

$$\lim_{\substack{\varepsilon\to 0\\ \delta\to\infty}} h_{\varepsilon,\delta}(\xi : x) = 0,\ \ \xi \in R - \{0\}. \qquad (2.20)$$

Since $\left|h_{\varepsilon,\delta}(\xi : x)\right| \le \left|\hat{f}(\xi)\right|$, the Lebsegue dominated convergence theorem yields

$$\lim_{\substack{\varepsilon\to 0\\ \delta\to\infty}} \left[f(x) - f_{\varepsilon,\delta}(x)\right] = 0,\ \ \forall x. \qquad (2.21)$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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