Available online at http://scik.org
Eng. Math. Lett. 2015, 2015:5
ISSN: 2049-9337

COMMON FIXED POINT THEOREM IN FUZZY SYMMETRIC SPACES

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Abstract. The main aim of this paper is to prove a unique common fixed point theorem for six self mappings in fuzzy symmetric spaces for occasionally weakly compatible mappings.

Keywords: Fuzzy symmetric space; Occasionally weakly compatible mappings; Weakly compatible mappings; Coincidence point; Fixed point.

2010 AMS Subject Classification: 54H25, 47H10.

1. Introduction

Wilson introduced the concept of semi-metric spaces in 1931, many fixed point theorems have been proved in this space. Jungck and Rhoads [1] initiated the the concept of weakly compatible mappings which are weaker than compatible mappings. Recently Jungck and Rhoads [9] introduced the concept of occasionally weakly compatible mappings which are more general among compatible mappings. The purpose of this paper is to obtain a common fixed point theorem for six self maps in fuzzy symmetric space.

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Received November 11, 2014

2. Preliminaries

Definition 2.1. A binary operation $* : [0,1] * [0,1] \to [0,1]$ is called a continuous t-norm if $*$ satisfies the following conditions:

(i) $*$ is commutative and associative
(ii) $*$ is continuous
(iii) $a * 1 = a$
(iv) $a * b \leq c * d$, where $a \leq c, b \leq d$ and $a, b, c, d \in [0,1]$.

Definition 2.2. The 3-tupple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary non empty set, $*$ is continuous t-norm and M is a fuzzy set in $X^2 \times (0, \infty)$ which satisfies the following conditions:

(i) $M(x,y,t) > 0$,
(ii) $M(x,y,t) = 1$ if and only if $x = y$,
(iii) $M(x,y,t) = M(y,x,t)$,
(iv) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$,
(v) $M(x,y,.) : (0,\infty) \to [0,1]$ is continuous for all $x,y,z \in X$ and $t,s > 0$.

Definition 2.3. The pair (X, M) is called fuzzy symmetric space if X is an arbitrary non empty set and M is fuzzy set in $X^2 \times (0,\infty)$ satisfying the following conditions:

(i) $M(x,y,t) > 0$,
(ii) $M(x,y,t) = 1$ if and only if $x = y$,
(iii) $M(x,y,t) = M(y,x,t)$,
(iv) $M(x,y,.) : (0,\infty) \to [0,1]$ is continuous for all $x,y,z \in X$ and $t,s > 0$.

If (X, M) is a symmetric space, then M is called fuzzy symmetric space for X.

Every fuzzy metric space is a fuzzy symmetric space but not conversely.

Example 2.4. *Consider* $X = [0,2)$ *and* $M(x,y,t) = \frac{t}{t+e^{|x-y|}-1}$ *Let* $x = 1, y = 1/2, z = 0, t = 1, s = 0$ *then (iv) of definition is not satisfied and hence* (X, M) *is fuzzy semi-metric space but a not fuzzy metric space.*

Definition 2.5. Let A and B be two self mappings of a fuzzy symmetric space (X,M). Then A and B are said to be compatible if $\lim_{n\to\infty}(ABx_n, BAx_n, t) = 1$ whenever a sequence $\{xn\}$ in X such that $\lim_{n\to\infty}(Ax_n, x, t) = \lim_{n\to\infty}(Bx_n, x, t) = 1$ for some $t \in X$.

Definition 2.6. Let X be a set A and B be self mappings of X. A point x in X is called a Coincidence point of A and B if and only if $Ax = Bx$. We denote $w = Ax = Bx$ a point of coincidence of A and B.

Definition 2.7. Let A and B be two self mappings of a fuzzy symmetric space (X,M) then, A and B are said to be weakly compatible if they commute at their coincidence point.

Definition 2.8. Let A and B be two self maps of a fuzzy symmetric space (X,M) then A and B are to be occasionally weekly compatible if there is a point $x \in X$ which is coincidence point of A and B at which A and B commute.

Lemma 2.9. Let X be a set. Let A and B be occasionally weakly compatible self maps of X. If A and B have a unique point of coincidence $w = Ax = Bx$ then w is the unique common fixed point of A and B.

Lemma 2.10. *If for all $x, yX, t > 0$ and for a number $k \in (0,1)$, then $M(x,y,kt) \geq M(x,y,t)$ then $x = y$.*

Proof. Suppose that there exists $k \in (0,1)$ such that $M(x,y,kt) \geq M(x,y,t)$,for all x,y in X and $t > 0$. Then $M(x,y,t) \geq M(s,y,t/k)$ and after n-th iteration $M(x,y,t) \geq M(x,y,t/k_n)$ for some positive integer taking limit as n $\to \infty$ we have M(x,y,t)$\geq$ 1. Hence x=y.

3. Main results

Theorem 3.1. *Let (X,M) be a fuzzy symmetric space. A, B, S, T, P and Q be self maps of X such that*

- **I:** . *(AP,S) ,(BQ, T) are occasionally weakly compatible*
- **II:** . *M(APx, BQy, qt)$\geq$ Min $\{$M(Sx, Ty,t), M(APx, Sx,t) , M(BQy,Ty,t), M(APx, Ty,t)$\}$ for all x,y$\in$ X ,q$\in$ (0.1).*

Then AP, BQ, S, and T have unique common fixed point. Further if (A, P) and (B, Q) are commuting pair of mapping s then A, B, S, T, P and Q have a unique common fixed point.

Proof. Since (AP,S) and (BQ,T) are occasionally weekly compatible then there exits $x,y \in X$ Such that $APx = x$ and $BQy = Ty$. We claim $APx = BQy$.
$M(APx,BQy,qt) \geq \min\{M(Sx,Ty,t),M(APx,Sx,t),M(BQy,Ty,t),M(APx,Ty,t)\}$.
$M(APx,BQy,qt) \geq \min\{M(APx,BQy,t),M(APx,APx,t),M(BQy,BQy,t),M(APx,BQy,t)\}$.
$M(APx,BQy,qt) \geq \min\{(APx,BQy,t)1,1,M(APx,BQy,t)\}$.
$M(APx,BQy,qt) \geq M(APx,BQy,t).APx = BQy.APx = BQy = Sx = Ty$ (3.1.1). If there is another point of coincidence t such that $Apt = St$ then using (II) we get $APt = BQy = St = Ty$ (3.1.2). Also from (3.1.1) and (3.1.2) $APx = APz \Rightarrow t = x$. Hence $w = APx = Sx$ for $w \in X$ is the unique point of coincidence of AP and S. By the lemma, w is a unique common fixed point of AP and S Hence $APw = Sw = w$. Similarly there is a unique common fixed point of BQ and T. Hence $BQu = Tu = u$ Suppose $u \neq w$ which contradicts the inequality (3.I-I). Hence w is unique common fixed point AP,BQ,S and T. We show that w is only common fixed point of A,B,S,T,P and Q. If the pairs (A,Q) (B,Q) are commuting pairs then for this we have $A(APw) = A(PAw) = AP(Aw) = Aw$. $x = Aw, y = w$ in (3.II). $M(AP(Aw)BQw,qt) \geq \min\{M(S(Aw),Tw,t,),M(AP(Aw),S(Aw),t),M(BQw,Tw,t),M(AP(Aw),Tw,t)\}$. $M(Aw,w,qt) \geq \min\{M(Aw,w,t)M(Aw,Aw,t),M(w,w,t),M(Aw,w,t)\} \Rightarrow (Aw,w,qt) \geq M(Aw,w,t)$. Therefore $Aw = wAPw = w \Rightarrow Pw = w$. Put $x = w, y = Qw$ in (3.II). $M(APw,BQ(Qw),qt) \geq$
$\min\{M(Sw,T(Qw),t),M(APw,Sw,t),M(BQ(Qw),T(Qw)t),M(AQw,T(Qw),t)\}$. $M(w,Qw,qt) \geq \min\{M(w,Qw,t),M(w,w,t)M(Qw,Qw,t)M(w,Qw,t)\} whichgives Qw = w$. $BQw = w$ implies $Bw = w$. Therefore, we have $Sw = Tw = Pw = Aw = Qw = Bw = w$. Hence A,B,S,T,Q and P have a unique common fixed point.

Example 3.2. *consider* $X = [0,2)$ *with the fuzzy semi metric space* (X,M) *defined by* $M(x,y,t) = \frac{t}{t+e^{|x-y|}-1}$ *for* x,y *all in* X *.Define self mappings* A,B,S,T,P *and* Q *as*

$$A(x) = Q(x) = \left\{ x \quad \textit{if } x \in [0,2); \right.$$

$$B(x) = BQ(x) = \begin{cases} 3/4 & \textit{if } x \in [0,1); \\ 1 & \textit{if } x \in [1,2). \end{cases}$$

$$T(x) = \begin{cases} 3x/2 & \textit{if } x \in [0,1); \\ 1 & \textit{if } x \in [1,2). \end{cases}$$

$$P(x) = AP(x) = \begin{cases} x/2 & \textit{if } x \in [0,1); \\ 1 & \textit{if } x=1; \\ 0.95 & \textit{if } x \in (1,2). \end{cases}$$

$$S(x) = \begin{cases} 1/4 & \textit{if } x \in [0,1); \\ 1/x & \textit{if } x=1; \\ 1/x^2 & \textit{if } x \in (1,2). \end{cases}$$

It is easy to verify that the pairs (AP, S) and (BQ, T) are occasionally weakly compatible mappings and 1 is common fixed point.
The above example reveals that occasionally weakly compatible mappings are not weakly compatible. Since it has two coincidence points 1/2 and 1 (AP,S) and (BQ,T) are not commuting at x=1/2. We observed that the self mappings (A, P) and (B, Q) are commuting and the mappings A, B, S, T, P and Q have unique common fixed point.

Corollary 3.3. *Let (X, M) be a fuzzy symmetric space. A, B, S, T, P and D be self maps of X such that*

- **I:** . *(AP,S) and (BQ,T) are occasionally weekly compatible*
- **II:** . $[M(APx, BQy, kt)]^2 * [M(APx, BQy, kt) * M(Sx, Ty,t)]$
  $\geq k_1\, [M(BQy,Sx,1.25kt) * M(APx,Ty,1.25kt)] + k_2 [M(APx, Sx,2.5kt) * M(BQy,Ty,2.5kt)] M(Sx, Ty,t)$.

for all x,y in X And $k_1,\ k_2 \geq 0,\ k_1 + k_2 \geq 1$. *Then AP, BQ, S, and T have unique common fixed point. If (A, P) and (B, Q) are commuting pair of mapping s then A, B, S, T, P and Q have a unique common fixed point theorem.*

Remark 3.4. Our result is partially generalizes Bijendra Singh, Arihant Jain and Aijaz Ahmed Masoodi [11] and Srinivas, Reddy and Umamaheswarrao [12].

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] G.Jungck, B.E. Rhoades, Fixed point for set valued function without continuity, Indian J. Pure Appl. Math. 293 (1998) 227-238.

[2] S.H. Cho, On common fixed point theorems in fuzzy metric spaces, J. Appl. Math. Comput. 20 (2006), 523-533.

[3] K. Jha, M. Imdad, U. Rajopadhyaya, Fixed point theorem for occasionally weakly compatible mappings in semi-metric space, Annals Pure Appl. Math. 5 (2014), 153-157.

[4] T.K. Samanta, B. Dinda, S. Mohinta, S. Roy, J. Ghosh, On coincidence and fixed point theorems in Fuzzy symmetric spaces, J. Hyperstructures 1 (2012), 74-91.

[5] K. Menger, Untersuchungenuberallgemeine, Math Annalen, 100 (1928), 75-165

[6] W.A. Wilson, On semi-metric space, Amer. J. Math. 53 (1931), 361-373.

[7] B.D. Pant, S. Chauhan, Common fixed theorems for occasionally weakly compatible mappings in Menger space, Surveys Math. Appl. 6 (2006) 1-7.

[8] B. Singh, S. Jain, Weakly compatibility and fixed point theorem in fuzzy metric spaces, Ganita. 56 (2005), 167-176.

[9] G. Jungck, B.E. Rhoades, Fixed point theorems for occasionally weakly compatible mappings, Fixed Point Theory 7 (2006) 280-296.

[10] B. Singh, M.S. Chauhan, Common fixed points of compatible maps in fuzzy metric Spaces, Fuzzy sets and Systems, 155 (2000), 471-475

[11] B. Singh, A. Jain and Aijaz Ahmed Masoodi, Semi -compatibilitry and fixed point theorem in fuzzy metric spaces, Intern. Math. Forum 5 (2010), 3041-3051.

[12] V. Srinivas, B.V.B. Reddy, R. Umamaheswarrao, A common fixed point Theorem on Fuzzy metric Spaces, Kathmandu University Journal of Science, Eng. Tech. 8 (2012), 77-82.