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UNIFIED INTEGRALS ASSOCIATED WITH GENERALIZED BESSEL FUNCTIONS AND STRUVE FUNCTIONS

SAIFUL R. MONDAL

Department of Mathematics and Statistics, King Faisal University, Ahsaa 31982, Saudi Arabia

Abstract. Integral formulas involving generalized Bessel and Struve functions of the first kind are expressed in term of generalized Wright function and generalized hypergeometric series. Many special cases, including cosine and sine function are also discussed.

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1. Introduction

The generalized Bessel function of the first kind $\mathscr{W}_{p,b,c}$ defined for complex $z \in \mathbb{C}$ and $b, c, p \in \mathbb{C}$ $(\mathrm{Re}(p) > -1)$ by

(1) $$\mathscr{W}_{p,b,c}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k c^k}{\Gamma(p+\frac{b+1}{2}+k)k!}\left(\frac{z}{2}\right)^{2k+p},$$

with $p+(b+1)/2 \neq 0, -1, -2, -3, \cdots$. More details related to the function $\mathscr{W}_{p,b,c}$ and its particular cases can be found in [1, 7] and references therein. It is worth mentioning that, $\mathscr{W}_{p,1,1} = J_p$ is the Bessel function of order p and $\mathscr{W}_{p,1,-1} = I_p$ is the modified Bessel function

*Corresponding author

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of order p. Also, $\mathscr{W}_{p,2,1} = 2j_p/\sqrt{\pi}$ is the spherical Bessel function of order p and $\mathscr{W}_{p,2,-1} = 2i_p/\sqrt{\pi}$ is the modified spherical Bessel function of order p.

The generalized Struve function of the first kind $\mathscr{H}_{p,b,c}(z)$ defined for complex $z \in \mathbb{C}$ and $b,c,p \in \mathbb{C}$ ($\mathrm{Re}(p) > -1$) by

$$\mathscr{H}_{p,b,c}(z) := \sum_{k=0}^{\infty} \frac{(-1)^k c^k}{\Gamma(p+1+\frac{b}{2}+k)\Gamma(k+3/2)} \left(\frac{z}{2}\right)^{2k+p+1}. \tag{2}$$

Details related to the function $\mathscr{H}_{p,b,c}$ and its particular cases can be seen in [3] and the references therein. Like Bessel function $\mathscr{H}_{p,1,1}$ represent Struve function of order p and $\mathscr{H}_{p,1,-1}$ is nothing but modified Struve function of order p and the study of $\mathscr{H}_{p,b,c}$ will cover all possible known cases.

Following integral formula [6, 2.47, p.22] is required in sequel:

$$\int_0^{\infty} x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} dx = 2\lambda a^{-\lambda} \left(\frac{a}{2}\right)^{\mu} \frac{\Gamma(2\mu)\Gamma(\lambda-\mu)}{\Gamma(1+\lambda+\mu)}. \tag{3}$$

The article emphasis on investigating compositions of integral (3) with generalized Bessel function(GBF) (1) and generalized Struve function (GSF) (2). Such compositions are expressed in terms of the generalized Wright hypergeometric (GWHF) function ${}_p\psi_q(z)$ is given by the series

$${}_p\psi_q(z) = {}_p\psi_q\left[\begin{array}{c} (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{array}\middle| z\right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i k)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j k)} \frac{z^k}{k!}, \tag{4}$$

where $a_i, b_j \in \mathbb{C}$, and real $\alpha_i, \beta_j \in \mathbb{R}$ ($i = 1,2,\ldots,p; j = 1,2,\ldots,q$). Asymptotic behavior of this function for large values of argument of $z \in \mathbb{C}$ were studied in [5] and under the condition

$$\sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i > -1 \tag{5}$$

in [18, 19]. Properties of this GWHF were investigated in [8], (see also [10, 9]. In particular, it was proved [8] that ${}_p\psi_q(z)$, $z \in \mathbb{C}$ is an entire function under the condition (5). Compositions of GBF and GSF with the integral (3) can also be represented in terms of the generalized hypergeometric function(GHF) ${}_pF_q(z)$ defined for complex $a_i, b_j \in \mathbb{C}$, $b_j \neq 0,1,\ldots$ ($i = 1,2,\ldots,p; j = 1,2,\ldots,q$) [11]. Certain interesting results on the fractional derivatives of generalized hypergeometric functions for several variables can be found in [16].

For purpose in sequel, we state and proof a generalization of the well-known Legendre duplication type formula for $(a,\alpha)=\Gamma(a+\alpha k)$. Legendre duplication formula [4] is defined as follwos:

$$\Gamma(2x)=\frac{2^{2x-1}}{\sqrt{\pi}}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) \quad \text{or} \quad (x)_{2k}=2^{2k}\left(\frac{x}{2}\right)_k\left(\frac{x+1}{2}\right)_k,\ (k\in\mathbb{N}_0). \tag{6}$$

Now using the identity $\Gamma(a+k)=\Gamma(a)(a)_k$, following result can be obtained by method of induction.

Lemma 1.1. *For any $n,k\in\mathbb{N}_0$, the following identity holds:*

$$(a,2^n)=\Gamma(a)2^{n2^nk}\prod_{i=1}^{2^n}\left(\frac{a+i-1}{2^n}\right)_k. \tag{7}$$

Proof. We prove the identity (7) by using method of induction. For $n=0$,

$$(a,1)=\Gamma(a)2^0\prod_{i=1}^{1}\left(\frac{a+i-1}{2^0}\right)_k=\Gamma(a)(a)_k,$$

is trivially true by definition of (a,α). Now assume that the identity (7) holds for $n=m\in\mathbb{N}$, i.e.

$$(a,2^m)=\Gamma(a)2^{m2^mk}\prod_{i=1}^{2^m}\left(\frac{a+i-1}{2^m}\right)_k. \tag{8}$$

Then

$$\begin{aligned}
(a,2^{m+1}) &= \Gamma(a)(a)_{2^{m+1}k}\\
&= \Gamma(a)2^{2^{m+1}k}\left(\frac{a}{2}\right)_{2^mk}\left(\frac{a+1}{2}\right)_{2^mk}\\
&= \frac{\Gamma(a)}{\Gamma(\frac{a}{2})\Gamma(\frac{a+1}{2})}2^{2^{m+1}k}\left(\frac{a}{2},2^m\right)\left(\frac{a+1}{2},2^m\right)\\
&= \Gamma(a)2^{2^{m+1}k}\times 2^{m2^mk}\prod_{i=1}^{2^m}\left(\frac{a+2i-2}{2^{m+1}}\right)_k\times 2^{m2^mk}\prod_{i=1}^{2^m}\left(\frac{a+2i-1}{2^{m+1}}\right)_k\\
&= \Gamma(a)2^{(m+1)2^{m+1}k}\prod_{i=1}^{2^{m+1}}\left(\frac{a+i-1}{2^{m+1}}\right)_k.
\end{aligned}$$

This complete the proof of the result.

2. Integrals involving GBF

This section consist of integral of generalized Bessel function(GBF) which are represented in terms of GWHF or GHF.

Theorem 2.1. *Let $b \in \mathbb{R}$ and λ, μ, c, $p \in \mathbb{C}$ with $\operatorname{Re}(p) > -(b+1)/2$, $\operatorname{Re}(\lambda+p) > \operatorname{Re}(\mu) > 0$ and $x > 0$. Then following integral identity holds:*

$$\begin{aligned}
(9) \qquad & \int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}\mathscr{W}_{p,b,c}\left(\tfrac{y}{a^2}(x+a-\sqrt{x^2+2ax})\right)dx \\
&= 2^{1-\mu-p}a^{\mu-\lambda-p}y^p\Gamma(2\mu) \\
&\quad \times {}_2\Psi_3\left[\begin{array}{l} (\lambda-\mu+p,2),(1+\lambda+p,2); \\ (\lambda+p,2),(1+\lambda+p+\mu,2),(p+\frac{b+1}{2},1) \end{array} \bigg| -\frac{cy^2}{4a^2}\right].
\end{aligned}$$

Proof. Consider the series representation of $\mathscr{W}_{p,b,c}(y(x+a-\sqrt{x^2+2ax})/a^2)$ and then by a computation it can be shown that the involved series is uniformly convergent. Thus it is allow to interchange the order of integration and summation, and that leads left side of (9) to the expression

$$\begin{aligned}
(10) \qquad & \int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}\mathscr{W}_{p,b,c}\left(\frac{y}{a^2}(x+a-\sqrt{x^2+2ax})\right)dx \\
&= \sum_{k=0}^\infty \frac{(-c)^k}{\Gamma(\kappa+k)k!}\left(\frac{y}{2}\right)^{2k+p}\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda-p-2k}dx.
\end{aligned}$$

The given condition $\operatorname{Re}(\lambda+p) > \operatorname{Re}(\mu) > 0$, yields

$$\operatorname{Re}(\lambda+p+2k) \geq \operatorname{Re}(\lambda+p) > \operatorname{Re}(\mu) > 0, \quad k \in \mathbb{N}\cup 0.$$

Thus using (3), the integration in the right-hand side of (10) reduce to

$$\begin{aligned}
& \int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda-p-2k}dx \\
&= 2(\lambda+p+2k)a^{-(\lambda+p+2k)}\left(\frac{a}{2}\right)^\mu\frac{\Gamma(2\mu)\Gamma(\lambda+p+2k-\mu)}{\Gamma(1+\lambda+p+2k+\mu)}.
\end{aligned}$$

This along with (10) yields

$$\begin{aligned}
&\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}\mathscr{W}_{p,b,c}\left(\tfrac{y}{x+a+\sqrt{x^2+2ax}}\right)dx\\
&=2a^{-\lambda}\left(\tfrac{a}{2}\right)^{\mu}\Gamma(2\mu)\sum_{k=0}^{\infty}\tfrac{(-c)^k\Gamma(1+\lambda+p+2k)\Gamma(\lambda+p+2k-\mu)}{\Gamma(\kappa+k)\Gamma(\lambda+p+2k)\Gamma(1+\lambda+p+2k+\mu)k!}\left(\tfrac{y}{2a}\right)^{2k+p}\\
&=2^{1-p-\mu}a^{\mu-\lambda-p}\Gamma(2\mu)y^p\\
&\qquad\times{}_2\psi_3\left[\begin{array}{l}(\lambda-\mu+p,2),(1+\lambda+p,2);\\(\lambda+p,2),(1+\lambda+p+\mu,2),(p+\frac{b+1}{2},1)\end{array}\middle|-\tfrac{cy^2}{4a^2}\right].
\end{aligned}$$

This complete the proof.

By adopting similar technique as in the proof of Theorem 2.1, a computation will give the following result.

Theorem 2.2. *Let $b\in\mathbb{R}$ and $\lambda,\ \mu,\ p\in\mathbb{C}$ with $\operatorname{Re}(p)>-(b+1)/2,\ \operatorname{Re}(\lambda)>\operatorname{Re}(\mu)\geq(b+1)/2$ and $x>0$. Then the following identity holds:*

$$\begin{aligned}
(11)\qquad&\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda}\mathscr{W}_{p,b,c}\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)dx\\
&=2^{1-\mu}a^{\mu-\lambda}\left(\frac{y}{4}\right)^{p}\Gamma(\lambda-\mu)\\
&\qquad\times{}_2\psi_3\left[\begin{array}{c}(2\mu+2p,4),(1+\lambda+p,2);\\(\lambda+p,2),(1+\lambda+2p+\mu,4),(p+\frac{b+1}{2},1)\end{array}\middle|-\frac{cy^2}{16}\right].
\end{aligned}$$

Next we establish some integral formula for $\mathscr{W}_{p,b,c}$ expressed in terms of generalized hypergeometric functions ${}_pF_q$. Considering $n=0,1$ in the identity (7), Theorem 2.1 can be rewrite as follows:

Corollary 2.1. *Let the conditions of Theorem 2.1 be satisfied and $\mu,\lambda+\mu,\lambda-\mu+p\in\mathbb{C}\backslash\mathbb{Z}_0^-$. Then the following integral formula holds:*

$$\begin{aligned}
&\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}\mathscr{W}_{p,b,c}\left(\tfrac{y}{x+a+\sqrt{x^2+2ax}}\right)dx\\
&=\tfrac{2^{1-\mu}a^{\mu-\lambda}(\lambda+p)}{\Gamma\left(p+\frac{b+1}{2}\right)(\lambda+\mu+p)}\left(\tfrac{y}{2a}\right)^{p}\mathscr{B}(2\mu,\lambda-\mu+p)\\
&\qquad\times{}_3F_4\left[\begin{array}{l}\frac{\lambda-\mu+p}{2},\frac{\lambda-\mu+p+1}{2},\frac{2+\lambda+p}{2};\\\frac{\lambda+p}{2},\frac{1+\lambda+p+\mu}{2},\frac{2+\lambda+p+\mu}{2},p+\frac{b+1}{2}\end{array}\middle|-\tfrac{cy^2}{4a^2}\right].
\end{aligned}$$

Here $\mathscr{B}(a,b)=\Gamma(a)\Gamma(b)/\Gamma(a+b)$ *is the well-known Beta functions.*

The integration in Theorem 2.2 can also be expressed in terms of GHF using the identity (7) for $n=0,1,2$.

Corollary 2.2. *Let the conditions of Theorem* 2.2 *be satisfied and* $\mu+p,\lambda+\mu+2p,\lambda+p\in\mathbb{C}\backslash\mathbb{Z}_0^-$. *Then the following integral formula holds:*

$$\begin{aligned}&\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda}\mathscr{W}_{p,b,c}\left(\tfrac{xy}{x+a+\sqrt{x^2+2ax}}\right)dx\\ &=\frac{2^{1-\mu}a^{\mu-\lambda}(\lambda+p)}{\Gamma\left(p+\frac{b+1}{2}\right)(\lambda+\mu+p)}\left(\tfrac{y}{4}\right)^p\mathscr{B}(\lambda-\mu,2\mu+2p)\\ &\qquad\times{}_5F_6\left[\begin{array}{c}\frac{\mu+p}{2},\frac{2\mu+2p+1}{4},\frac{\mu+p+1}{2},\frac{2\mu+2p+3}{4},\frac{2+\lambda+p}{2}\\ \frac{\lambda+p}{2},\frac{1+\lambda+\mu+2p}{4},\frac{2+\lambda+\mu+2p}{4},\frac{3+\lambda+\mu+2p}{4},\frac{4+\lambda+\mu+2p}{4},p+\frac{b+1}{2}\end{array}\middle|-\frac{cy^2}{16}\right].\end{aligned}$$

Remark 2.1. For $b=c=1$, the result obtain in this section has also been obtained in [3]. For other special values of b,c, the results will be discussed in Section 4.

3. Integrals involving GSF

The unified integral, as in section 2, but with generalized Struve function are considered in this section. The results are expressed in terms of GWHF and GHF.

Theorem 3.1. *Let* λ, μ, $p\in\mathbb{C}$ *with* $\operatorname{Re}(p)>-(2+b)/2$, $\operatorname{Re}(\lambda+p)>\operatorname{Re}(\mu)>0$ *and* $x>0$. *Then following integral identity holds:*

$$\begin{aligned}&\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}\mathscr{S}_{p,b,c}\left(\tfrac{y}{a^2}(x+a-\sqrt{x^2+2ax})\right)dx\\ &=2^{1-\mu}a^{\mu-\lambda}\left(\tfrac{y}{2a}\right)^{p+1}\Gamma(2\mu)\\ &\qquad\times{}_2\psi_3\left[\begin{array}{l}(1,1),(\lambda-\mu+p+1,2),(2+\lambda+p,2);\\ (\frac{3}{2},1),(1+\lambda+p,2),(2+\lambda+p+\mu,2),(p+1+\frac{b}{2},1)\end{array}\middle|-\frac{cy^2}{4a^2}\right].\end{aligned}$$

Since $(3/2,1)=\Gamma(3/2+k)$ *and* $(1,1)=\Gamma(1+k)=k!$, *the above representation can be obtain by proceeding as the proof of Theorem 2.1.*

Theorem 3.2. *Let λ, μ, $p \in \mathbb{C}$ with $\operatorname{Re}(p) > -(2+b)/2$, $\operatorname{Re}(\lambda) > \operatorname{Re}(\mu) \geq 1$ and $x > 0$. Then following identity holds:*

$$\begin{aligned}
&\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} \mathscr{S}_{p,b,c}\left(\tfrac{xy}{a^2}(x+a-\sqrt{x^2+2ax})\right)dx \\
&= 2^{1-\mu}a^{\mu-\lambda}\left(\tfrac{y}{4}\right)^p \Gamma(\lambda-\mu) \\
&\quad \times {}_2\psi_3\left[\begin{array}{l} (2\mu+2p,4),(1+\lambda+p,2); \\ (\lambda+p,2),(1+\lambda+2p+\mu,4),(p+\frac{b+1}{2},1) \end{array} \Bigg| -\tfrac{cy^2}{16}\right].
\end{aligned}$$

Using the identity (7), the integral involving $\mathscr{S}_{p,b,c}$ similar to Theorem 3.1 and Theorem 3.2 can be expressed in terms of generalized hypergeometric functions ${}_pF_q$, and the representations are given in following results. We omit the details proof due to similar computation as earlier results.

Corollary 3.1. *Let the conditions of Theorem 3.1 be satisfied and $\mu, \lambda+\mu, \lambda-\mu+p \in \mathbb{C}\backslash\mathbb{Z}_0^-$. Then the following integral formula holds:*

$$\begin{aligned}
&\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} \mathscr{S}_{p,b,c}\left(\tfrac{y}{x+a+\sqrt{x^2+2ax}}\right)dx \\
&= \frac{2^{1-\mu}a^{\mu-\lambda}(\lambda+p)}{\Gamma\left(p+\frac{b+1}{2}\right)(\lambda+\mu+p)}\left(\tfrac{y}{2a}\right)^p \mathscr{B}(2\mu,\lambda-\mu+p) \\
&\quad \times {}_3F_4\left[\begin{array}{l} \frac{\lambda-\mu+p}{2},\frac{\lambda-\mu+p+1}{2},\frac{2+\lambda+p}{2}; \\ \frac{\lambda+p}{2},\frac{1+\lambda+p+\mu}{2},\frac{2+\lambda+p+\mu}{2},p+\frac{b+1}{2} \end{array} \Bigg| -\tfrac{cy^2}{4a^2}\right].
\end{aligned}$$

Here $\mathscr{B}(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is well-known Beta functions.

Corollary 3.2. *Let the conditions of Theorem 3.2 be satisfied and $\mu+p, \lambda+\mu+2p, \lambda+p \in \mathbb{C}\backslash\mathbb{Z}_0^-$. Then the following integral formula holds:*

$$\begin{aligned}
&\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} \mathscr{S}_{p,b,c}\left(\tfrac{xy}{x+a+\sqrt{x^2+2ax}}\right)dx \\
&= \frac{2^{1-\mu}a^{\mu-\lambda}(\lambda+p)}{\Gamma\left(p+\frac{b+1}{2}\right)(\lambda+\mu+p)}\left(\tfrac{y}{4}\right)^p \mathscr{B}(\lambda-\mu,2\mu+2p) \\
&\quad \times {}_5F_6\left[\begin{array}{l} \frac{\mu+p}{2},\frac{2\mu+2p+1}{4},\frac{\mu+p+1}{2},\frac{2\mu+2p+3}{4},\frac{2+\lambda+p}{2} \\ \frac{\lambda+p}{2},\frac{1+\lambda+\mu+2p}{4},\frac{2+\lambda+\mu+2p}{4},\frac{3+\lambda+\mu+2p}{4},\frac{4+\lambda+\mu+2p}{4},p+\frac{b+1}{2} \end{array} \Bigg| -\tfrac{cy^2}{16}\right].
\end{aligned}$$

4. Integrals involving trigonometric functions

For all $b \in \mathbb{R}$, if $p = -b/2$, then the generalized Bessel function $\mathscr{W}_{p,b,c}(z)$ have the form

$$\mathscr{W}_{-\frac{b}{2},b,c^2}(z) = \left(\frac{2}{z}\right)^{\frac{b}{2}} \frac{\cos cz}{\sqrt{\pi}} \quad \text{and} \quad \mathscr{W}_{-\frac{b}{2},b,-c^2}(z) = \left(\frac{2}{z}\right)^{\frac{b}{2}} \frac{\cosh cz}{\sqrt{\pi}}. \tag{12}$$

Hence following results are consequence of Theorem 2.1 and Theorem 2.2 respectively.

Corollary 4.1. *Let* $\lambda, \mu, c \in \mathbb{C}$ *such that* $\mathrm{Re}(\lambda) > \mathrm{Re}(\mu) > 0$. *For* $x > 0$, *following identity holds:*

$$\begin{aligned} &\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} \cos\left(\tfrac{cy}{x+a+\sqrt{x^2+2ax}}\right) dx \\ &= \sqrt{\pi}\, 2^{1-\mu} a^{\mu-\lambda} \Gamma(2\mu) \times {}_2\psi_3 \left[\begin{array}{c} (\lambda-\mu,2),(1+\lambda,2); \\ (\lambda,2),(1+\lambda+\mu,2),(\frac{1}{2},1) \end{array} \middle| -\frac{c^2y^2}{4a^2} \right]. \end{aligned} \tag{13}$$

and

$$\begin{aligned} &\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} \cosh\left(\tfrac{cy}{x+a+\sqrt{x^2+2ax}}\right) dx \\ &= \sqrt{\pi}\, 2^{1-\mu} a^{\mu-\lambda} \Gamma(2\mu) \times {}_2\psi_3 \left[\begin{array}{c} (\lambda-\mu,2),(1+\lambda,2); \\ (\lambda,2),(1+\lambda+\mu,2),(\frac{1}{2},1) \end{array} \middle| \frac{c^2y^2}{4a^2} \right]. \end{aligned} \tag{14}$$

Proof. On setting $p = -b/2$ and replacing c by c^2 in to (9) and using (12), we have

$$\begin{aligned} &\tfrac{1}{\sqrt{\pi}} \int_0^\infty x^{\mu-1} \left(x+a+\sqrt{x^2+2ax}\right)^{\frac{b}{2}-\lambda} \left(\tfrac{2}{y}\right)^{\frac{b}{2}} \cos\left(\tfrac{cy}{x+a+\sqrt{x^2+2ax}}\right) dx \\ &= 2^{1-\mu} a^{\mu-\lambda} \left(\tfrac{y}{2a}\right)^{-\frac{b}{2}} \Gamma(2\mu) \times {}_2\psi_3 \left[\begin{array}{c} (\lambda-\mu-\frac{b}{2},2),(1+\lambda-\frac{b}{2},2); \\ (\lambda-\frac{b}{2},2),(1+\lambda-\frac{b}{2}+\mu,2),(\frac{1}{2},1) \end{array} \middle| -\frac{c^2y^2}{4a^2} \right]. \end{aligned}$$

After some computation and finally replacing $(\lambda - b/2)$ by λ will gives the identity (13).

Similarly the identity (14) can be obtain from (9), with the aid of identity (12), taking $p = -b/2$ and replacing c by $-c^2$. □

Adopting similar method as in Corollary 4.1 following result can be obtained from Theorem 2.2, we omit the details.

Corollary 4.2. *Let $b \in \mathbb{R}$ and $\lambda, \mu, c \in \mathbb{C}$ such that* $\operatorname{Re}(\lambda) > \operatorname{Re}(\mu) > 0$. *For $x > 0$, following identity holds:*

$$\int_0^\infty x^{\mu-\frac{b}{2}-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda+\frac{b}{2}} \cos\left(\tfrac{cxy}{x+a+\sqrt{x^2+2ax}}\right) dx$$
$$= \sqrt{\pi}\, 2^{1-\mu+\frac{b}{2}} a^{\mu-\lambda} \Gamma(\lambda-\mu) \times {}_2\psi_3 \left[\begin{array}{l} (2\mu-b,2),(1+\lambda-\frac{b}{2},2); \\ (\lambda-\frac{b}{2},2),(1+\lambda-b+\mu,2),(\frac{1}{2},1) \end{array} \middle| -\frac{c^2y^2}{16} \right].$$

and

$$\int_0^\infty x^{\mu-\frac{b}{2}-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda+\frac{b}{2}} \cosh\left(\tfrac{cxy}{x+a+\sqrt{x^2+2ax}}\right) dx$$
$$= \sqrt{\pi}\, 2^{1-\mu+\frac{b}{2}} a^{\mu-\lambda} \Gamma(\lambda-\mu) \times {}_2\psi_3 \left[\begin{array}{l} (2\mu-b,2),(1+\lambda-\frac{b}{2},2); \\ (\lambda-\frac{b}{2},2),(1+\lambda-b+\mu,2),(\frac{1}{2},1) \end{array} \middle| \frac{c^2y^2}{16} \right].$$

For all $b \in \mathbb{R}$, if $p = 1 - b/2$, then the generalized Bessel function $\mathscr{W}_{p,b,c}(z)$ is related with sine and hyperbolic sine function as follows:

$$\mathscr{W}_{1-\frac{b}{2},b,c^2}(z) = \left(\tfrac{2}{z}\right)^{\frac{b}{2}} \tfrac{\sin cz}{c\sqrt{\pi}} \quad \textit{and} \quad \mathscr{W}_{1-\frac{b}{2},b,-c^2}(z) = \left(\tfrac{2}{z}\right)^{\frac{b}{2}} \tfrac{\sinh cz}{c\sqrt{\pi}}. \tag{15}$$

Next we represent integrals involving sine and hyperbolic sine functions in term of generalized (Wright) hypergeometric functions.

Corollary 4.3. *Let $\lambda, \mu, c \in \mathbb{C}$ such that* $\operatorname{Re}(\lambda) > \operatorname{Re}(\mu) > 0$. *For $x > 0$, following identity holds:*

$$\begin{aligned}&\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} \sin\left(\tfrac{cy}{a^2}(x+a-\sqrt{x^2+2ax})\right) dx \\ &= c\sqrt{\pi}\, 2^{1-\mu} a^{\mu-\lambda} \Gamma(2\mu) \times {}_2\psi_3 \left[\begin{array}{l} (\lambda-\mu,2),(1+\lambda,2); \\ (\lambda,2),(1+\lambda+\mu,2),(\frac{1}{2},1) \end{array} \middle| -\frac{c^2y^2}{4a^2} \right].\end{aligned} \tag{16}$$

and

$$\begin{aligned}&\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} \sinh\left(\tfrac{cy}{x+a+\sqrt{x^2+2ax}}\right) dx \\ &= c\sqrt{\pi}\, 2^{1-\mu} a^{\mu-\lambda} \Gamma(2\mu) \times {}_2\psi_3 \left[\begin{array}{l} (\lambda-\mu,2),(1+\lambda,2); \\ (\lambda,2),(1+\lambda+\mu,2),(\frac{1}{2},1) \end{array} \middle| \frac{c^2y^2}{4a^2} \right].\end{aligned} \tag{17}$$

Remark 4.1. Using (7), integrals in Corollary 4.1 – Corollary 4.3 can also be represent in terms of generalized hypergeometric functions.

Conflict of Interests

The authors declare that there is no conflict of interests.

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