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SOME COMMON TRIPLED FIXED POINT THEOREMS IN TWO QUASI-PARTIAL b-METRIC SPACES

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Abstract. In this paper some common tripled fixed-point theorems are proved for mappings defined on a set equipped with two quasi-partial b-metric spaces and some examples are provided to support the results.

Keywords: common tripled fixed point; tripled coincidence point; quasi-partial metric space; w-compatible mappings.

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1. Introduction

Matthews [16] in 1994 introduced the notion of partial metric space which is a generalization of usual metric space obtained by replacing the d(x,x)= 0 by $d(x,x) \leq d(x,y)$ for all x, y in the definition of metric. He extended the Banach contraction principle from metric spaces to partial metric spaces. Bakhtin [6] introduced the concept of b-metric spaces which was further extended by Czerwick [8]. Later in the year 2013, Shukla [19] generalized both the concept of b-metric and partial metric spaces by introducing the partial b-metric spaces. Many authors ([3,4,5,13,18]) worked on this notion of partial metric spaces and obtained fixed point results for mappings satisfying different contractive conditions.

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2. Preliminaries

In 2012, Karapinar *et al.* [14] introduced the concept of quasi-partial metric spaces. The definition of partial metric space is given as follows:

Definition 2.1. (Matthews, [16]) A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(P_1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,

(P_2) $p(x,x) \leq p(x,y)$,

(P_3) $p(x,y) = p(y,x)$,

(P_4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X, p) such that X is a non-empty set and p is a partial metric on X. For a partial metric p on X, the function $d_p : X \times X \to \mathbb{R}^+$ defined by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y) \text{ is a metric on } X.$$

Definition 2.2. (Karapinar *et al.* [14]) A *quasi-partial metric* on non-empty set X is a function $q : X \times X \to \mathbb{R}^+$ which satisfies:

(QPM_1) If $q(x,x) = q(x,y) = q(y,y)$, then $x = y$,

(QPM_2) $q(x,x) \leq q(x,y)$,

(QPM_3) $q(x,x) \leq q(y,x)$, and

(QPM_4) $q(x,y) + q(z,z) \leq q(x,z) + q(z,y)$

for all $x, y, z \in X$.

A *quasi-partial metric space* is a pair (X, q) such that X is a non-empty set and q is a quasi-partial metric on X.

Let q be a quasi-partial metric on the set X. Then

$$d_q(x,y) = q(x,y) + q(y,x) - q(x,x) - q(y,y) \text{ is a metric on } X.$$

Lemma 2.1. (Karapinar *et. al* [14]) *Let (X, q) be a quasi-partial metric space. Let (X, p_q) be the corresponding partial metric space, and let (X, d_{p_q}) be the corresponding metric space. Then the following statements are equivalent:*

(1) (X,q) *is complete,*

(2) (X,p_q) *is complete,*

(3) (X,d_{p_q}) *is complete.*

Moreover,

$$\begin{aligned}\lim_{n\to\infty} d_{p_q}(x,x_n)=0 \quad &\Leftrightarrow \quad p_q(x,x)=\lim_{n\to\infty} p_q(x,x_n)=\lim_{n,m\to\infty} p_q(x_n,x_m)\\ &\Leftrightarrow \quad q(x,x)=\lim_{n\to\infty} q(x,x_n)=\lim_{n,m\to\infty} q(x_n,x_m)\\ &\qquad\qquad =\lim_{n\to\infty} q(x_n,x)=\lim_{n,m\to\infty} q(x_m,x_n)\,.\end{aligned}$$

Definition 2.3. (Shukla [19]) A *partial b-metric* on a non-empty set X is a mapping p_b : $X\times X\to\mathbb{R}^+$ such that for some real number $s\geq 1$ and for all $x,y,z\in X$:

(P_{b_1}) $x=y$ if and only if $p_b(x,x)=p_b(x,y)=p_b(y,y)$,

(P_{b_2}) $p_b(x,x)\leq p_b(x,y)$,

(P_{b_3}) $p_b(x,y)=p_b(y,x)$,

(P_{b_4}) $p_b(x,y)\leq s[p_b(x,z)+p_b(z,y)]-p_b(z,z)$.

A *partial b-metric space* is a pair (X,p_b) such that X is a non-empty set and p_b is a partial b-metric on X. The number s is called the coefficient of (X,p_b).

For simplicity, We denote $X\times X\times ....X$ by X^k where $k\in\mathbb{N}$ and X is a non-empty set.

Definition 2.4. (Bhaskar and Lakshmikantham [7]) Let X be a non-empty set. An element $(x,y)\in X^2$ is a *coupled fixed point* of the mapping

$$F:X^2\to X \text{ if } F(x,y)=x \text{ and } F(y,x)=y\,.$$

Definition 2.5. (Lakshmikantham and Ćirić [15]) An element $(x,y)\in X^2$ is called

(1) a *coupled coincidence point* of the mappings $F:X^2\to X$ and $g:X\to X$ if $F(x,y)=gx$ and $F(y,x)=gy$; in this case (gx,gy) is called *coupled point of coincidence* of mappings F and g;

(2) a *common coupled fixed point* of mappings $F:X^2\to X$ and $g:X\to X$ if $F(x,y)=gx=x$ and $F(y,x)=gy=y$.

Definition 2.6. (Samet and Vetro [18]) An element $(x,y,z) \in X^3$ is a *tripled fixed point* of the mapping

$$F : X^3 \to X \text{ if } F(x,y,z) = x,\ F(y,z,x) = y \text{ and } F(z,x,y) = z.$$

Definition 2.7. (Aydi *et al.* [15]) An element $(x,y,z) \in X^3$ is called

(1) a *tripled coincidence point* of the mappings $F : X^3 \to X$ and $g : X \to X$ if $F(x,y,z) = gx$, $F(y,z,x) = gy$ and $F(z,x,y) = gz$; in this case (gx,gy,gz) is called *tripled point of coincidence* of mappings F and g;

(2) a *common tripled fixed point* of mappings $F : X^3 \to X$ and $g : X \to X$ if $F(x,y,z) = gx = x$, $F(y,z,x) = gy = y$ and $F(z,x,y) = gz = z$.

Definition 2.8. (Aydi *et al.* [1]) Let X be a non-empty set. The mappings $F : X^3 \to X$ and $g : X \to X$ are *w-compatible* if $gF(x,y,z) = F(gx,gy,gz)$ whenever $gx = F(x,y,z)$, $gy = F(y,z,x)$ and $gy = F(z,x,y)$.

Theorem 2.1. [9] *Let q_1 and q_2 be two quasi partial metrics on X such that $q_2(x,y) \le q_1(x,y)$, for all $x,y \in X$, and let $F : X^3 \to X$, $g : X \to X$ be two mappings. Suppose that there exists k_1, k_2, k_3, k_4, and k_5 in $[0,1)$ with*

$$k_1 + k_2 + k_3 + 2k_4 + k_5 < 1$$

such that the condition

$$\begin{aligned} &q_1(F(x,y,z),F(u,v,w)) + q_1(F(y,z,x),F(v,w,u)) + q_1(F(z,x,y),F(w,u,v)) \\ &\quad \le k_1[q_2(gx,gu) + q_2(gy,gv)] + q_2(gz,gw) \\ &\qquad + k_2[q_2(gx,F(x,y,z)) + q_2(gy,F(y,z,x)) + q_2(gz,F(z,x,y))] \\ &\qquad + k_3[q_2(gu,F(u,v,w)) + q_2(gv,F(v,w,u)) + q_2(gw,F(w,u,v))] \\ &\qquad + k_4[q_2(gx,F(u,v,w)) + q_2(gy,F(v,w,u)) + q_2(gz,F(w,u,v))] \\ &\qquad + k_5[q_2(gu,F(x,y,z)) + q_2(gv,F(y,z,x)) + q_2(gw,F(z,x,y))] \end{aligned}$$

holds for all $x,y,z,u,v,w \in X$. Also, suppose we have the following hypotheses:

(1) $F(X^3) \subseteq g(X)$.

(2) $g(X)$ is complete subspace of X with respect to the quasi-partial metric q_1.

Then the mapping F and g have a tripled coincidence point (x,y,z) satisfying $gx = F(x,y,z) = F(y,z,x) = gy = F(z,x,y) = gz$. Moreover, if F and g are w-compatible, then F and g have a unique common tripled fixed point of the form (u,u,u).

Recently, Gupta and Gautam [11] has introduced quasi-partial b-metric spaces which is the generalization of the concept of quasi-partial-metric spaces.

Definition 2.9. (Gupta and Gautam [11]) A quasi-partial b-metric on a non-empty set X is a mapping $qp_b : X \times X \to \mathbb{R}^+$ such that for some real number $s \geq 1$ and for all $x,y,z \in X$:

(QP_{b_1}) $qp_b(x,x) = qp_b(x,y) = qp_b(y,y) \Rightarrow x = y$,

(QP_{b_2}) $qp_b(x,x) \leq qp_b(x,y)$,

(QP_{b_3}) $qp_b(x,x) \leq qp_b(y,x)$,

(QP_{b_4}) $qp_b(x,y) \leq s[qp_b(x,z) + qp_b(z,y)] - qp_b(z,z)$.

A *quasi-partial b-metric space* is a pair (X, qp_b) such that X is a non-empty set and qp_b is a quasi-partial b-metric on X.

Let qp_b be a quasi-partial b-metric on the set X.Then

$$d_{qp_b}(x,y) = qp_b(x,y) + qp_b(y,x) - qp_b(x,x) - qp_b(y,y)$$

is a b-metric on X.

Lemma 2.2. (Gupta and Gautam [11]) *Every quasi-partial metric space is a quasi-partial b-metric space. But the converse need not be true.*

Lemma 2.3. (Gupta and Gautam [11]) *Let (X, qp_b) be a quasi-partial b-metric space. Then the following hold:*

(1) If $qp_b(x,y) = 0$ then $x = y$,

(2) If $x \neq y$, then $qp_b(x,y) > 0$ and $qp_b(y,x) > 0$.

Definition 2.10. (Gupta and Gautam [11]) Let (X, qp_b) be a quasi-partial b-metric space. Then:

(1) A sequence $\{x_n\} \subset X$ *converges* to $x \in X$ if and only if

$$qp_b(x,x) = \lim_{n\to\infty} qp_b(x,x_n) = \lim_{n\to\infty} qp_b(x_n,x)$$

(2) A sequence $\{x_n\} \subset X$ is called a *Cauchy sequence* if and only if

$$\lim_{n,m\to\infty} qp_b(x_n,x_m) \quad \text{and} \lim_{n,m\to\infty} qp_b(x_m,x_n) \quad \text{exist (and are finite).}$$

(3) The quasi partial b-metric space (X,qp_b) is said to be *complete* if every Cauchy sequence $\{x_n\} \subset X$ converges with respect to τ_{qp_b} to a point $x \in X$ such that

$$qp_b(x,x) = \lim_{n,m\to\infty} qp_b(x_m,x_n) = \lim_{n,m\to\infty} qp_b(x_n,x_m).$$

(4) A mapping $f : X \to X$ is said to be *continuous* at $x_0 \in X$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B(x_0,\delta)) \subset B(f(x_0),\varepsilon)$.

Recently, Aydi and Abbas [2] obtained some tripled coincidence and fixed point theorems in partial metric space. Also, Shatanawi and Pitea [17] derived some common coupled fixed point theorems for a pair of mappings in quasi-partial metric space. Gu and Wang [9,10] obtained some results on coupled and tripled fixed-point theorems in two quasi-partial metric spaces. Very recently, Gupta and Gautam [12] discussed some coupled fixed point results on quasi-partial b-metric spaces. The aim of this paper is to explore some common tripled fixed-point theorems for mappings defined on a set equipped with two quasi-partial b-metric spaces.

3. Main results

In this section we prove our main theorem which gives conditions for existence and uniqueness of a tripled fixed point on quasi-partial b-metric spaces.

Theorem 3.1. *Let qp_{b_1} and qp_{b_2} be two quasi-partial b-metrics on X such that $qp_{b_2}(x,y) \leq qp_{b_1}(x,y)$, for all $x,y \in X$. Let $F : X^3 \to X$, $g : X \to X$ be two mappings. Suppose that there exist k_1,k_2,k_3,k_4, and k_5 in $[0,1)$ with*

$$k_1+k_2+k_3+2sk_4+k_5 < \frac{1}{s} \tag{3.1}$$

such that the condition

$$\begin{aligned}
&qp_{b_1}(F(x,y,z),F(u,v,w))+qp_{b_1}(F(y,z,x),F(v,w,u))+qp_{b_1}(F(z,x,y),F(w,u,v))\\
&\quad\leq k_1[qp_{b_2}(gx,gu)+qp_{b_2}(gy,gv)]+qp_{b_2}(gz,gw)\\
&\qquad+k_2[qp_{b_2}(gx,F(x,y,z))+qp_{b_2}(gy,F(y,z,x))+qp_{b_2}(gz,F(z,x,y))]\\
&\qquad+k_3[qp_{b_2}(gu,F(u,v,w))+qp_{b_2}(gv,F(v,w,u))+qp_{b_2}(gw,F(w,v,u))]\\
&\qquad+k_4[qp_{b_2}(gx,F(u,v,w))+qp_{b_2}(gy,F(v,w,u))+qp_{b_2}(gz,F(w,u,v))]\\
&\qquad+k_5[qp_{b_2}(gu,F(x,y,z))+qp_{b_2}(gv,F(y,z,x))+qp_{b_2}(gw,F(z,x,y))]
\end{aligned} \tag{3.2}$$

holds for all $x,y,z,u,v,w \in X$. Also, suppose we have the following hypotheses:

(1) $F(X^3) \subset g(X)$

(2) $g(X)$ is a complete subspace of X with respect to the quasi-partial b-metric qp_{b_1}.

Then the mappings F and g have a tripled coincidence point (x,y,z) satisfying $gx = F(x,y,z) = F(y,z,x) = gy = F(z,x,y) = gz$. Moreover, if F and g are w-compatible, then F and g have a unique common tripled fixed point of the form (u,u,u).

Proof. Let $x_0,y_0,z_0 \in X$. Since $F(X^3) \subset g(X)$, we can choose $x_1,y_1,z_1 \in X$ such that $gx_1 = F(x_0,y_0,z_0)$, $gy_1 = F(y_0,z_0,x_0)$ and $gz_1 = F(z_0,x_0,y_0)$. Similarly, we can choose $x_2,y_2,z_2 \in X$ such that $gx_2 = F(x_1,y_1,z_1)$, $gy_2 = F(y_1,z_1,x_1)$ and $gz_2 = F(z_1,x_1,y_1)$.

Continuing in this manner we can construct three sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X such that

$$gx_{n+1} = F(x_n,y_n,z_n), gy_{n+1} = F(y_n,z_n,x_n) \text{ and } gz_{n+1} = F(z_n,x_n,y_n)\ \forall\ n \geq 0. \tag{3.3}$$

Consider

$$\begin{aligned}
&qp_{b_1}(gx_n,gx_{n+1})+qp_{b_1}(gy_n,gy_{n+1})+qp_{b_1}(gz_n,gz_{n+1})\\
&\quad= qp_{b_1}(F(x_{n-1},y_{n-1},z_{n-1}),F(x_n,y_n,z_n))\\
&\qquad+qp_{b_1}(F(y_{n-1},z_{n-1},x_{n-1}),F(y_n,z_n,x_n))\\
&\qquad+qp_{b_1}(F(z_{n-1},x_{n-1},y_{n-1}),F(z_n,x_n,y_n)).
\end{aligned}$$

It follows from (3.2), (QP_{b_4}) and (QP_{b_2}) that,

$$\begin{aligned} &qp_{b_1}(gx_n,gx_{n+1})+qp_{b_1}(gy_n,gy_{n+1})+qp_{b_1}(gz_n,gz_{n+1})\\ &\quad\leq (k_1+k_2)[qp_{b_2}(gx_{n-1},gx_n)+qp_{b_2}(gy_{n-1},gy_n)++qp_{b_2}(gz_{n-1},gz_n)]\\ &\qquad+k_3[qp_{b_2}(gx_n,gx_{n+1})+qp_{b_2}(gy_n,gy_{n+1})+qp_{b_2}(gz_n,gz_{n+1})]\\ &\qquad+k_4[qp_{b_2}(gx_{n-1},gx_{n+1})+qp_{b_2}(gy_{n-1},gy_{n+1})+qp_{b_2}(gz_{n-1},gz_{n+1})]\\ &\qquad+k_5[qp_{b_2}(gx_n,gx_n)+qp_{b_2}(gy_n,gy_n)+qp_{b_2}(gz_n,gz_n)] \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\leq (k_1+k_2)[qp_{b_2}(gx_{n-1},gx_n)+qp_{b_2}(gy_{n-1},gy_n)+qp_{b_2}(gz_{n-1},gz_n)]\\ &\quad+k_3[qp_{b_2}(gx_n,gx_{n+1})+qp_{b_2}(gy_n,gy_{n+1})qp_{b_2}(gz_n,gz_{n+1})]\\ &\quad+k_4[s\{qp_{b_2}(gx_{n-1},gx_n)+qp_{b_2}(gx_n,gx_{n+1})\}-qp_{b_2}(gx_n,gx_n)]\\ &\quad+sk_4[\{qp_{b_2}(gy_{n-1},gy_n)+qp_{b_2}(gy_n,gy_{n+1})\}-qp_{b_2}(gy_n,gy_n)]\\ &\quad+sk_4[\{qp_{b_2}(gz_{n-1},gz_n)+qp_{b_2}(gz_n,gz_{n+1})\}-qp_{b_2}(gz_n,gz_n)]\\ &\quad+k_5[qp_{b_2}(gx_n,gx_{n+1})+qp_{b_2}(gy_n,gy_{n+1})+qp_{b_2}(gz_n,gz_{n+1})]\\ &\leq (k_1+k_2+sk_4)[qp_{b_1}(gx_{n-1},gx_n)+qp_{b_1}(gy_{n-1},gy_n)+qp_{b_1}(gz_{n-1},gz_n)]\\ &\quad+(k_3+sk_4+k_5)[qp_{b_1}(gx_n,gx_{n+1})+qp_{b_1}(gy_n,gy_{n+1})+qp_{b_1}(gz_n,gz_{n+1})], \end{aligned} \tag{3.5}$$

which implies that

$$\begin{aligned} &qp_{b_1}(gx_n,gx_{n+1})+qp_{b_1}(gy_n,gy_{n+1})+qp_{b_1}(gz_n,gz_{n+1})\\ &\quad\leq \frac{k_1+k_2+sk_4}{1-k_3-sk_4-k_5}[qp_{b_1}(gx_{n-1},gx_n)+qp_{b_1}(gy_{n-1},gy_n)+qp_{b_1}(gz_{n-1},gz_n)]. \end{aligned}$$

Put $k=\dfrac{k_1+k_2+sk_4}{1-k_3-sk_4-k_5}$. Clearly, $0\leq k<\dfrac{1}{s}<1$. By repetition of inequality (3.4) n times we get

$$\begin{aligned} &qp_{b_1}(gx_n,gx_{n+1})+qp_{b_1}(gy_n,gy_{n+1})+qp_{b_1}(gz_n,gz_{n+1})\\ &\quad\leq k^n[qp_{b_1}(gx_0,gx_1)+qp_{b_1}(gy_0,gy_1)+qp_{b_1}(gz_0,gz_1)]. \end{aligned}$$

Next, we shall prove that $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$are Cauchy sequences in $g(X)$. For each $n,m \in \mathbb{N}$, $m > n$, from (QP_{b_4}) and (3.5), we have

$$\begin{aligned}
&qp_{b_1}(gx_n,gx_m)+qp_{b_1}(gy_n,gy_m)+qp_{b_1}(gz_n,gz_m)\\
&\quad\leq \sum_{i=n}^{m-1} s^{m-i}\cdot k^i[qp_{b_1}(gx_0,gx_1)+qp_{b_1}(gy_0,gy_1)+qp_{b_1}(gz_0,gz_1)]\\
&\quad= \sum_{i=n}^{m-1}\left(\frac{k}{s}\right)^i s^m[qp_{b_1}(gx_0,gx_1)+qp_{b_1}(gy_0,gy_1)+qp_{b_1}(gz_0,gz_1)]\\
&\quad\leq \sum_{i=n}^{\infty}\left(\frac{k}{s}\right)^i s^m[qp_{b_1}(gx_0,gx_1)+qp_{b_1}(gy_0,gy_1)+qp_{b_1}(gz_0,gz_1)]\\
&\quad= \frac{\left(\frac{k}{s}\right)^n}{\left(1-\frac{k}{s}\right)}\cdot s^m[qp_{b_1}(gx_0,gx_1)+qp_{b_1}(gy_0,gy_1)+qp_{b_1}(gz_0,gz_1)].
\end{aligned}\tag{3.6}$$

Taking limit as $n\to\infty$ in (3.6) and keeping m fixed, we get

$$\lim_{n\to\infty}[qp_{b_1}(gx_n,gx_m)+qp_{b_1}(gy_n,gy_m)+qp_{b_1}(gz_n,gz_m)]\leq 0.$$

But

$$\lim_{n\to\infty}[qp_{b_1}(gx_n,gx_m)+qp_{b_1}(gy_n,gy_m)+qp_{b_1}(gz_n,gz_m)]\geq 0.$$

This gives

$$\lim_{n\to\infty}[qp_{b_1}(gx_n,gx_m)]=\lim_{n\to\infty}[qp_{b_1}(gy_n,gy_m)]=\lim_{n\to\infty}[qp_{b_1}(gz_n,gz_m)]=0.$$

Now taking limit as $m\to+\infty$, one has

$$\lim_{n,m\to\infty}qp_{b_1}(gx_n,gx_m)=\lim_{n,m\to\infty}qp_{b_1}(gy_n,gy_m)=\lim_{n,m\to\infty}qp_{b_1}(gz_n,gz_m)=0.\tag{3.7}$$

Similarly, we can show that

$$\lim_{n,m\to\infty}qp_{b_1}(gx_m,gx_n)=0\quad\text{and}\quad\lim_{n,m\to\infty}qp_{b_1}(gz_m,gz_n)=0.\tag{3.8}$$

So, $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are Cauchy sequences in $(g(X),qp_{b_1})$. Since $(g(X),qp_{b_1})$ is complete, there exist $gx,gy,gz\in g(X)$ such that $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ converges to gx, gy and gz

with respect to $\tau_{qp_{b_1}}$, that is,

$$qp_{b_1}(gx,gx) = \lim_{n\to\infty} qp_{b_1}(gx,gx_n) = \lim_{n\to\infty} qp_{b_1}(gx_n,gx) = \lim_{n,m\to\infty} qp_{b_1}(gx_m,gx_n) = \lim_{n,m\to\infty} qp_{b_1}(gx_n,gx_m), \tag{3.9}$$

$$qp_{b_1}(gy,gy) = \lim_{n\to\infty} qp_{b_1}(gy,gy_n) = \lim_{n\to\infty} qp_{b_1}(gy_n,gy) = \lim_{n,m\to\infty} qp_{b_1}(gy_m,gy_n) = \lim_{n,m\to\infty} qp_{b_1}(gy_n,gy_m) and \tag{3.10}$$

$$qp_{b_1}(gz,gz) = \lim_{n\to\infty} qp_{b_1}(gz,gz_n) = \lim_{n\to\infty} qp_{b_1}(gz_n,gz) = \lim_{n,m\to\infty} qp_{b_1}(gz_m,gz_n) = \lim_{n,m\to\infty} qp_{b_1}(gz_n,gz_m). \tag{3.11}$$

Combining (3.7)-(3.11), we obtain

$$qp_{b_1}(gx,gx) = \lim_{n\to\infty} qp_{b_1}(gx,gx_n) = \lim_{n\to\infty} qp_{b_1}(gx_n,gx) = \lim_{n,m\to\infty} qp_{b_1}(gx_m,gx_n) = \lim_{n,m\to\infty} qp_{b_1}(gx_n,gx_m) = 0, \tag{3.12}$$

$$qp_{b_1}(gy,gy) = \lim_{n\to\infty} qp_{b_1}(gy,gy_n) = \lim_{n\to\infty} qp_{b_1}(gy_n,gy) = \lim_{n,m\to\infty} qp_{b_1}(gy_m,gy_n) = \lim_{n,m\to\infty} qp_{b_1}(gy_n,gy_m) = 0 \tag{3.13}$$

and

$$qp_{b_1}(gz,gz) = \lim_{n\to\infty} qp_{b_1}(gz,gz_n) = \lim_{n\to\infty} qp_{b_1}(gz_n,gz) = \lim_{n,m\to\infty} qp_{b_1}(gz_m,gz_n) = \lim_{n,m\to\infty} qp_{b_1}(gz_n,gz_m) = 0. \tag{3.14}$$

By QP_{b_4}, we have

$$\begin{aligned} qp_{b_1}(gx_{n+1},F(x,y,z)) &\leq s\{qp_{b_1}(gx_{n+1},gx) + qp_{b_1}(gx,F(x,y,z))\} - qp_{b_1}(gx,gx) \\ &\leq s\{qp_{b_1}(gx_{n+1},gx) + qp_{b_1}(gx,F(x,y,z))\} \\ &\leq s\ [qp_{b_1}(gx_{n+1},gx) + s\{qp_{b_1}(gx,gx_{n+1}) \\ &\qquad + qp_{b_1}(gx_{n+1},F(x,y,z))\} - qp_{b_1}(gx_{n+1},gx_{n+1})] \\ &\leq s[qp_{b_1}(gx_{n+1},gx)] + s^2[qp_{b_1}(gx,gx_{n+1})] \\ &\qquad + s^2[qp_{b_1}(gx_{n+1},F(x,y,z))]. \end{aligned}$$

Taking limit as $n \to \infty$ in the above inequalities and using (3.14), we have

$$\begin{aligned} \frac{1}{s} qp_{b_1}(gx, F(x,y,z)) &\leq \lim_{n\to\infty} qp_{b_1}(gx_{n+1}, F(x,y,z)) \\ &\leq s qp_{b_1}(gx, F(x,y,z)). \end{aligned} \tag{3.15}$$

Similarly using (3.15), one has

$$\begin{aligned} \frac{1}{s} qp_{b_1}(gy, F(y,z,x)) &\leq \lim_{n\to\infty} qp_{b_1}(gy_{n+1}, F(y,z,x)) \\ &\leq s qp_{b_1}(gy, F(y,z,x)). \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} \frac{1}{s} qp_{b_1}(gz, F(z,x,y)) &\leq \lim_{n\to\infty} qp_{b_1}(gz_{n+1}, F(z,x,y)) \\ &\leq s qp_{b_1}(gz, F(z,y,x)). \end{aligned} \tag{3.17}$$

Now, we prove that $F(x,y,z) = gx$, $F(y,z,x) = gy$ and $F(z,x,y) = gz$. In fact, it follows from (3.1) and (3.2) that

$$\begin{aligned} & qp_{b_1}(gx_{n+1}, F(x,y,z)) + qp_{b_1}(gy_{n+1}, F(y,z,x)) + qp_{b_1}(gz_{n+1}, F(z,x,y)) \\ &= qp_{b_1}(F(x_n,y_n,z_n), F(x,y,z)) + qp_{b_1}(F(y_n,z_n,x_n), F(y,z,x) \\ &\quad + qp_{b_1}(F(z_n,x_n,y_n), F(z,y,x)) \\ &\leq k_1[qp_{b_2}(gx_n, gx) + qp_{b_2}(gy_n, gy) + qp_{b_2}(gz_n, gz)] \\ &\quad + k_2[qp_{b_2}(gx_n, F(x_n,y_n,z_n)) + qp_{b_2}(gy_n, F(y_n,z_n,x_n)) + qp_{b_2}(gz_n, F(z_n,x_n,y_n))] \\ &\quad + k_3[qp_{b_2}(gx, F(x,y,z)) + qp_{b_2}(gy, F(y,z,x)) + qp_{b_2}(gz, F(z,x,y))] \\ &\quad + k_4[qp_{b_2}(gx_n, F(x,y,z)) + qp_{b_2}(gy_n, F(y,z,x)) + qp_{b_2}(gz_n, F(z,x,y))] \\ &\quad + k_5[qp_{b_2}(gx, F(x_n,y_n,z_n)) + qp_{b_2}(gy, F(y_n,z_n x_n)) + qp_{b_2}(gz, F(z_n,x_n,y_n))] \\ &\leq k_1[qp_{b_1}(gx_n, gx) + qp_{b_1}(gy_n, gy) + qp_{b_1}(gz_n, gz)] \\ &\quad + k_2[qp_{b_1}(gx_n, gx_{n+1}) + qp_{b_2}(gy_n, gy_{n+1}) + qp_{b_2}(gz_n, gz_{n+1})] \\ &\quad + k_3[qp_{b_1}(gx, F(x,y,z)) + qp_{b_1}(gy, F(y,z,x)) + qp_{b_1}(gz, F(z,x,y))] \\ &\quad + k_4[qp_{b_1}(gx_n, F(x,y,z)) + qp_{b_1}(gy_n, F(y,z,x)) + qp_{b_1}(gz_n, F(z,x,y))] \\ &\quad + k_5[qp_{b_1}(gx, gx_{n+1}) + qp_{b_1}(gy, gy_{n+1}) + qp_{b_1}(gz, gz_{n+1})]. \end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality, using (3.12)-(3.17), we get

$$\begin{aligned}
&\lim_{n\to\infty}[qp_{b_1}(gx_{n+1},F(x,y,z))+qp_{b_1}(gy_{n+1},F(y,z,x))+qp_{b_1}(gz_{n+1},F(z,x,y))]\\
&\quad\le \lim_{n\to\infty}\{[k_1(qp_{b_1}(gx_n,gx)+qp_{b_1}(gy_n,gy)+qp_{b_1}(gz_n,gz)]\\
&\qquad + k_2[qp_{b_1}(gx_n,gx_{n+1})+qp_{b_1}(gy_n,gy_{n+1})+qp_{b_1}(gz_n,gz_{n+1})]\\
&\qquad + k_3[qp_{b_1}(gx,F(x,y,z))+qp_{b_1}(gy,F(y,z,x))+qp_{b_1}(gz,F(z,x,y))]\\
&\qquad + k_4[qp_{b_1}(gx_n,F(x,y,z))+qp_{b_1}(gy_n,F(y,z,x))+qp_{b_1}(gz_n,F(z,x,y))]\\
&\qquad + k_5[qp_{b_1}(gx,gx_{n+1})+qp_{b_1}(gy,gy_{n+1})+qp_{b_1}(gz,gz_{n+1})]\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\lim_{n\to\infty}[qp_{b_1}(gx_{n+1},F(x,y,z))+qp_{b_1}(gy_{n+1},F(y,z,x))+qp_{b_1}(gz_{n+1},F(z,x,y))]\\
&\quad\le k_1[qp_{b_1}(gx,gx)+qp_{b_1}(gy,gy)+qp_{b_1}(gz,gz)]+k_2[qp_{b_1}(gx,gx)+qp_{b_1}(gy,gy)+qp_{b_1}(gz,gz)]\\
&\qquad + k_3[qp_{b_1}(gx,F(x,y,z))+qp_{b_1}(gy,F(y,z,x))+qp_{b_1}(gz,F(z,x,y))]\\
&\qquad + \lim_{n\to\infty} k_4[qp_{b_1}(gx_n,F(x,y,z))+qp_{b_1}(gy_n,F(y,z,x))+qp_{b_1}(gz_n,F(z,x,y))]\\
&\qquad + k_5[qp_{b_1}(gx,gx)+qp_{b_1}(gy,gy)+qp_{b_1}(gz,gz)]
\end{aligned}$$

$$\begin{aligned}
&= k_3[qp_{b_1}(gx,F(x,y,z))+qp_{b_1}(gy,F(y,z,x))+qp_{b_1}(gz,F(z,x,y))]\\
&\quad + \lim_{n\to\infty} k_4[qp_{b_1}(gx_n,F(x,y,z))+qp_{b_1}(gy_n,F(y,z,x))+qp_{b_1}(gz_n,F(z,x,y))].
\end{aligned} \tag{3.19}$$

By using (3.12)-(3.17), we get

$$\begin{aligned}
&\lim_{n\to\infty}[qp_{b_1}(gx_{n+1},F(x,y,z))+qp_{b_1}(gy_{n+1},F(y,z,x))+qp_{b_1}(gz_{n+1},F(z,x,y))]\\
&\quad\le k_3[qp_{b_1}(gx,F(x,y,z))+qp_{b_1}(gy,F(y,z,x))+qp_{b_1}(gz,F(z,x,y))]\\
&\qquad + k_4\cdot s[qp_{b_1}(gx,F(x,y,z))+qp_{b_1}(gy,F(y,z,x))+qp_{b_1}(gz,F(z,x,y))]\\
&\quad = (k_3+sk_4)[qp_{b_1}(gx,F(x,y,z))+qp_{b_1}(gy,F(y,z,x))+qp_{b_1}(gz,F(z,x,y))].
\end{aligned}$$

And also

$$\begin{aligned}
&\frac{1}{s}[qp_{b_1}(gx,F(x,y,z))+qp_{b_1}(gy,F(y,z,x))+qp_{b_1}(gz,F(z,x,y))]\\
&\quad\le (k_3+sk_4)[qp_{b_1}(gx,F(x,y,z))+qp_{b_1}(gy,F(y,z,x))+qp_{b_1}(gy,F(z,x,y))]\\
\Rightarrow\quad &\left[\frac{1}{s}-k_3-sk_4\right][qp_{b_1}(gx,F(x,y,z))+qp_{b_1}(gy,F(y,z,x))+qp_{b_1}(gz,F(z,x,y))]\le 0.
\end{aligned}\tag{3.18}$$

Since $k_3+sk_4<\frac{1}{s}$. Thus it follows from (3.18) that

$$qp_{b_1}(gx,F(x,y,z))=qp_{b_1}(gy,F(y,z,x))=qp_{b_1}(gz,F(z,x,y))=0.$$

By Lemma 2.3, we get $F(x,y,z)=gx$, $F(y,z,x)=gy$ and $F(z,x,y)=gz$. Hence, (gx,gy,gz) is a tripled point of coincidence of mappings F and g.

Next, we will show that the tripled point of coincidence is unique. Suppose that $(x',y',z')\in X^3$ with $F(x',y',z')=gx'$, $F(y',z',x')=gy'$ and $F(z',x',y')=gz'$.

Using (3.2), (3.14)-(3.16), and (QP_{b_3}), we obtain

$$\begin{aligned}
&qp_{b_1}(gx,gx')+qp_{b_1}(gy,gy')+qp_{b_1}(gz,gz')\\
&\quad=qp_{b_1}(F(x,y,z),F(x',y',z'))+qp_{b_1}(F(y,z,x),F(y',z',x')\\
&\qquad+qp_{b_1}(F(z,x,y),F(z',x',y'))\\
&\quad\le k_1[qp_{b_2}(gx,gx')+qp_{b_2}(gy,gy')+qp_{b_2}(gz,gz')]\\
&\qquad+k_2[qp_{b_2}(gx,F(x,y,z))+qp_{b_2}(gy,F(y,z,x))+qp_{b_2}(gz,F(z,x,y))]\\
&\qquad+k_3[qp_{b_2}(gx',F(x',y',z'))+qp_{b_2}(gy',F(y',z',x'))+qp_{b_2}(gz',F(z',x',y'))]\\
&\qquad+k_4[qp_{b_2}(gx,F(x',y',z'))+qp_{b_2}(gy,F(y',z',x'))+qp_{b_2}(gz,F(z',y',x'))]\\
&\qquad+k_5[qp_{b_2}(gx',F(x,y,z))+qp_{b_2}(gy',F(y,z,x))+qp_{b_2}(gz',F(z,y,x))]\\
&\quad=k_1[qp_{b_2}(gx,gx')+qp_{b_2}(gy,gy')+qp_{b_2}(gz,gz')]\\
&\qquad+k_2[qp_{b_2}(gx,gx)+qp_{b_2}(gy,gy)+qp_{b_2}(gz,gz)]\\
&\qquad+k_3[qp_{b_2}(gx',gx')+qp_{b_2}(gy',gy')+qp_{b_2}(gz',gz')]
\end{aligned}$$

$$
\begin{aligned}
&+k_4[qp_{b_2}(gx,gx')+qp_{b_2}(gy,gy')+qp_{b_2}(gz,gz')]\\
&+k_5[qp_{b_2}(gx',gx)+qp_{b_2}(gy',gy)+qp_{b_2}(gz',gz)]\\
\leq{}& (k_1+k_4)[qp_{b_1}(gx,gx')+qp_{b_1}(gy,gy')+qp_{b_1}(gz,gz')]\\
&+k_2[qp_{b_1}(gx,gx)+qp_{b_1}(gy,gy)+qp_{b_1}(gz,gz)]\\
&+k_3[qp_{b_1}(gx',gx')+qp_{b_1}(gy',gy')+qp_{b_1}(gz',gz')]\\
&+k_5[qp_{b_1}(gx',gx)+qp_{b_1}(gy',gy)+qp_{b_1}(gz',gz)]\\
\leq{}& (k_1+k_3+k_4)[qp_{b_1}(gx,gx')+qp_{b_1}(gy,gy')+qp_{b_1}(gz,gz')]\\
&+k_5[qp_{b_1}(gx',gx)+qp_{b_1}(gy',gy)+qp_{b_1}(gz',gz)].
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&qp_{b_1}(gx,gx')+qp_{b_1}(gy,gy')+qp_{b_1}(gz,gz')\\
&\quad\leq \frac{k_5}{1-k_1-k_3-k_4}[qp_{b_1}(gx',gx)+qp_{b_1}(gy',gy)+qp_{b_1}(gz',gz)].
\end{aligned}
\tag{3.19}
$$

Similarly, we have

$$
\begin{aligned}
&qp_{b_1}(gx',gx)+qp_{b_1}(gy',gy)+qp_{b_1}(gz',gz)\\
&\quad\leq \frac{k_5}{1-k_1-k_3-k_4}[qp_{b_1}(gx,gx')+qp_{b_1}(gy,gy')+qp_{b_1}(gz,gz')].
\end{aligned}
\tag{3.20}
$$

Substituting (3.20) into (3.19), we obtain

$$
\begin{aligned}
&qp_{b_1}(gx,gx')+qp_{b_1}(gy,gy')+qp_{b_1}(gz,gz')\\
&\quad\leq \left(\frac{k_5}{1-k_1-k_3-k_4}\right)^2[qp_{b_1}(gx,gx')+qp_{b_1}(gy,gy')+qp_{b_1}(gz,gz')].
\end{aligned}
\tag{3.21}
$$

Since $\frac{k_5}{1-k_1-k_3-k_4}<1$, from (2.21), we must have

$$qp_{b_1}(gx,gx')=qp_{b_1}(gy,gy')=qp_{b_1}(gz,gz')=0.$$

By Lemma 2.3, we get $gx=gx'$, $gy=gy'$ and $gz=gz'$. This gives the uniqueness of the tripled point of coincidence of F and g, that is, (gx,gy,gz).

Next, we will show that $gx = gy = gz$. In fact, from (3.2), (3.14)-(3.16), we have

$$
\begin{aligned}
&qp_{b_1}(gx,gy)+qp_{b_1}(gy,gz)+qp_{b_1}(gz,gx)\\
&\quad = qp_{b_1}(F(x,y,z),F(y,z,x))+qp_{b_1}(F(y,z,x),F(z,x,y))+qp_{b_1}(F(z,x,y),F(x,y,z))\\
&\quad \leq k_1[qp_{b_2}(gx,gy)+qp_{b_2}(gy,gz)]+qp_{b_2}(gz,gx)]\\
&\qquad +k_2[qp_{b_2}(gx,F(x,y,z))+qp_{b_2}(gy,F(y,z,x))+qp_{b_2}(gz,F(z,x,y))]\\
&\qquad +k_3[qp_{b_2}(gy,F(y,z,x))+qp_{b_2}(gz,F(z,x,y))+qp_{b_2}(gx,F(x,y,z))]\\
&\qquad +k_4[qp_{b_2}(gx,F(y,z,x))+qp_{b_2}(gy,F(z,x,y))+qp_{b_2}(gz,F(x,y,z))]\\
&\qquad +k_5[qp_{b_2}(gy,F(x,y,z))+qp_{b_2}(gz,F(y,z,x))+qp_{b_2}(gx,F(z,x,y))]
\end{aligned}
$$

$$
\begin{aligned}
&= k_1[qp_{b_2}(gx,gy)+qp_{b_2}(gy,gz)]+qp_{b_2}(gz,gx)]\\
&\quad +k_2[qp_{b_2}(gx,gx)+qp_{b_2}(gy,gy)+qp_{b_2}(gz,gz)]\\
&\quad +k_3[qp_{b_2}(gy,gy)+qp_{b_2}(gz,gz)+qp_{b_2}(gx,gx)]\\
&\quad +k_4[qp_{b_2}(gx,gy)+qp_{b_2}(gy,gz)+qp_{b_2}(gz,gx)]\\
&\quad +k_5[qp_{b_2}(gy,gx)+qp_{b_2}(gz,gy)+qp_{b_2}(gx,gz)]\\
&\leq k_1[qp_{b_1}(gx,gy)+qp_{b_1}(gy,gz)+qp_{b_1}(gz,gx)] \qquad (3.22)\\
&\quad +k_2[qp_{b_1}(gx,gx)+qp_{b_1}(gy,gy)+qp_{b_1}(gz,gz)]\\
&\quad +k_3[qp_{b_1}(gy,gy)+qp_{b_1}(gz,gz)+qp_{b_1}(gx,gx)]\\
&\quad +k_4[qp_{b_1}(gx,gy)+qp_{b_1}(gy,gz)+qp_{b_1}(gz,gx)]\\
&\quad +k_5[qp_{b_1}(gy,gx)+qp_{b_1}(gz,gy)+qp_{b_1}(gx,gz)]\\
&= (k_1+k_4+k_5)[qp_{b_1}(gx,gy)+qp_{b_1}(gy,gz)+qp_{b_1}(gz,gx)].
\end{aligned}
$$

Since $k_1+k_4+k_5<1$ from (3.22) we have

$$qp_{b_1}(gx,gy)=qp_{b_1}(gy,gz)=qp_{b_1}(gz,gx)=0.$$

By Lemma 2.3, we get $gx = gy = gz$.

Finally, assume that g and F are w-compatible. Let $u = gx$, then we have $u = gx = F(x,y,z) = gy = F(y,z,x) = gz = F(z,x,y)$, so that

$$gu = ggx = g(F(x,y,z)) = F(gx,gy,gz) = F(u,u,u). \tag{3.23}$$

Consequently, (u,u,u) is a tripled coincidence point of F and g, and therefore (gu,gu,gu) is a tripled point of coincidence of F and g, and by its uniqueness, we get $gu = gx$. Thus, we obtain $F(u,u,u) = gu = u$. Therefore, (u,u,u) is the unique common tripled fixed point of F and g. This completes the proof.

Corollary 3.1. *Let qp_b be a quasi-partial b-metrics on X, $F : X^3 \to X$, $g : X \to X$ be two mappings. Suppose that there exist k_1, k_2, k_3, k_4, and k_5 in $[0,1)$ with*

$$k_1 + k_2 + k_3 + 2sk_4 + k_5 < \frac{1}{s} \tag{3.1.1}$$

such that the condition

$$\begin{aligned}
&qp_b(F(x,y,z),F(u,v,w)) + qp_b(F(y,z,x),F(v,w,u)) + qp_b(F(z,x,y),F(w,u,v)) \\
&\quad \le k_1[qp_b(gx,gu) + qp_b(gy,gv)] + qp_{b_2}(gz,gw) \\
&\qquad + k_2[qp_b(gx,F(x,y,z)) + qp_b(gy,F(y,z,x)) + qp_b(gz,F(z,x,y))] \\
&\qquad + k_3[qp_b(gu,F(u,v,w)) + qp_b(gv,F(v,w,u)) + qp_b(gw,F(w,v,u))] \\
&\qquad + k_4[qp_b(gx,F(u,v,w)) + qp_b(gy,F(v,w,u)) + qp_b(gz,F(w,u,v))] \\
&\qquad + k_5[qp_b(gu,F(x,y,z)) + qp_b(gv,F(y,z,x)) + qp_b(gw,F(z,x,y))]
\end{aligned} \tag{3.1.2}$$

holds for all $x,y,z,u,v,w \in X$. Also, suppose we have the following hypotheses:

(1) $F(X^3) \subset g(X)$

(2) $g(X)$ is a complete subspace of X with respect to the quasi-partial b-metric qp_b.

Then the mappings F and g have a tripled coincidence point (x,y,z) satisfying $gx = F(x,y,z) = F(y,z,x) = gy = F(z,x,y) = gz$.

Moreover, if F and g are w-compatible, then F and g have a unique common tripled fixed point of the form (u,u,u).

Corollary 3.2. *Let qp_{b_1} and qp_{b_2} be two quasi-partial b-metrics on X and $qp_{b_2}(x,y) \le qp_{b_1}(x,y)$, for all $x,y \in X$. Let $F : X^3 \to X$, $g : X \to X$ be two mappings. Suppose that there exist $a_i \in [0,1)$*

$(i = 1,2,3,\dots,15)$ *with*

$$a_1+a_2+a_3+a_4+a_5+a_6+a_7+a_8+a_9+2s(a_{10}+a_{11}+a_{12})+a_{13}+a_{14}+a_{15}<\frac{1}{s} \tag{3.2.1}$$

such that the condition

$$\begin{aligned} &qp_{b_1}(F(x,y,z),F(u,v,w))+qp_{b_1}(F(y,z,x),F(v,w,u))+qp_{b_1}(F(z,x,y),F(w,u,v))\\ &\quad\le a_1qp_{b_2}(gx,gu)+a_2qp_{b_2}(gy,gv)+a_3qp_{b_2}(gz,gw)\\ &\qquad+a_4qp_{b_2}(gx,F(x,y,z))+a_5qp_{b_2}(gy,F(y,z,x))+a_6qp_{b_2}(gz,F(z,x,y))\\ &\qquad+a_7qp_{b_2}(gu,F(u,v,w))+a_8qp_{b_2}(gv,F(v,w,u))++a_9qp_{b_2}(gv,F(v,w,u))\\ &\qquad+a_{10}qp_{b_2}(gx,F(u,v,w))+a_{11}qp_{b_2}(gy,F(v,w,u))+a_{12}qp_{b_2}(gz,F(w,u,v))\\ &\qquad+a_{13}qp_{b_2}(gu,F(x,y))+a_{14}qp_{b_2}(gv,F(y,z,x))+a_{15}qp_{b_2}(gw,F(z,y,x)) \end{aligned} \tag{3.2.2}$$

holds for all $x,y,z,u,v,w \in X$. *Also suppose we have the following hypotheses:*

(1) $F(X^3) \subseteq g(X)$

(2) $g(X)$ *is a complete subspace of* X *with respect to the quasi-partial b-metric* qp_{b_1}.

Then the mappings F *and* g *have a tripled coincidence point* (x,y,z) *satisfying* $gx = F(x,y,z) = F(y,z,x) = gy = F(z,x,y) = gz$.

Moreover, if F *and* g *are w-compatible, then* F *and* g *have a unique common tripled fixed point of the form* (u,u,u).

Proof. Given $x,y,z,u,v,w \in X$, it follows from (3.2.2) that

$$\begin{aligned} &qp_{b_1}(F(x,y,z),F(u,v,w))\\ &\quad\le a_1qp_{b_2}(gx,gu)+a_2qp_{b_2}(gy,gv)+a_3qp_{b_2}(gz,gw)\\ &\qquad+a_4qp_{b_2}(gx,F(x,y,z))+a_5qp_{b_2}(gy,F(y,z,x))+a_6qp_{b_2}(gz,F(z,x,y))\\ &\qquad+a_7qp_{b_2}(gu,F(u,v,w))+a_8qp_{b_2}(gv,F(v,w,u))++a_9qp_{b_2}(gv,F(v,w,u))\\ &\qquad+a_{10}qp_{b_2}(gx,F(u,v,w))+a_{11}qp_{b_2}(gy,F(v,w,u))+a_{12}qp_{b_2}(gz,F(w,u,v))\\ &\qquad+a_{13}qp_{b_2}(gu,F(x,y))+a_{14}qp_{b_2}(gv,F(y,z,x))+a_{15}qp_{b_2}(gw,F(z,y,x)) \end{aligned} \tag{3.2.3}$$

holds for all $x,y,z,u,v,w \in X$. Also suppose we have the following hypotheses: and

$$\begin{aligned} qp_{b_1}&(F(y,z,x),F(v,w,u)) \\ &\leq a_1qp_{b_2}(gy,gv)+a_2qp_{b_2}(gz,gw)+a_3qp_{b_2}(gx,gu) \\ &\quad +a_4qp_{b_2}(gy,F(y,z,x))+a_5qp_{b_2}(gz,F(z,x,y))+a_6qp_{b_2}(gx,F(x,y,z)) \\ &\quad +a_7qp_{b_2}(gv,F(v,w,u))++a_8qp_{b_2}(gw,F(v,w,u))+a_9qp_{b_2}(gu,F(u,v,w)) \\ &\quad +a_{10}qp_{b_2}(gy,F(v,w,u))+a_{11}qp_{b_2}(gz,F(w,u,v))+a_{12}qp_{b_2}(gx,F(u,v,w)) \\ &\quad +a_{13}qp_{b_2}(gv,F(y,z,x))+a_{14}qp_{b_2}(gw,F(z,y,x))+a_{15}qp_{b_2}(gu,F(x,y)) \end{aligned} \tag{3.2.4}$$

holds for all $x,y,z,u,v,w \in X$. Also suppose we have the following hypotheses:

$$\begin{aligned} qp_{b_1}&(F(z,x,y),F(w,u,v)) \\ &\leq +a_1qp_{b_2}(gz,gw)+a_2qp_{b_2}(gx,gu)a_3qp_{b_2}(gy,gv) \\ &\quad +a_6qp_{b_2}(gy,F(y,z,x))+a_4qp_{b_2}(gz,F(z,x,y))+a_5qp_{b_2}(gx,F(x,y,z)) \\ &\quad +a_7qp_{b_2}(gw,F(w,u,v))+a_8qp_{b_2}(gu,F(u,v,w))+a_9qp_{b_2}(gv,F(v,w,u)) \\ &\quad +a_{10}qp_{b_2}(gz,F(w,u,v))+a_{11}qp_{b_2}(gx,F(u,v,w))+a_{12}qp_{b_2}(gy,F(v,w,u)) \\ &\quad +a_{13}qp_{b_2}(gw,F(z,y,x))+a_{14}qp_{b_2}(gu,F(x,y))+a_{15}qp_{b_2}(gv,F(y,z,x)) \end{aligned} \tag{3.2.5}$$

holds for all $x,y,z,u,v,w \in X$. Adding inequalities (3.2.3) and (3.2.4) to inequality (3.2.5), we get

$$\begin{aligned} qp_{b_1}&(F(x,y,z),F(u,v,w)))+qp_{b_1}(F(y,z,x),F(v,w,u))+qp_{b_1}(F(z,x,y),F(w,u,v)) \\ &\leq (a_1+a_2+a_3)[qp_{b_2}(gx,gu)+qp_{b_2}(gy,gv)+qp_{b_2}(gz,gw)] \\ &\quad +(a_4+a_5+a_6)[qp_{b_2}(gx,F(x,y,z))+qp_{b_2}(gy,F(y,z,x))+qp_{b_2}(gz,F(z,x,y))] \\ &\quad +(a_7+a_8+a_9)[qp_{b_2}(gu,F(u,v,w))+qp_{b_2}(gv,F(v,w,u))+qp_{b_2}(gw,F(w,u,v))] \\ &\quad +(a_{10}+a_{11}+a_{12})[qp_{b_2}(gx,F(u,v,w))+qp_{b_2}(gy,F(v,w,u))+qp_{b_2}(gz,F(w,u,v))] \\ &\quad +(a_{13}+a_{14}+a_{15})[qp_{b_2}(gu,F(x,y,z))+qp_{b_2}(gv,F(y,z,x))+qp_{b_2}(gw,F(z,x,y))]. \end{aligned}$$

Therefore, letting $a_1+a_2+a_3=k_1$, $a_4+a_5+a_6=k_2$, $a_7+a_8+a_9=k_3$, $a_{10}+a_{11}+a_{12}=k_4$, $a_{13}+a_{14}+a_{15}=k_5$, the result follows from Theorem 3.1.

Corollary 3.3. *Let qp_{b_1} and qp_{b_2} be two quasi-partial b-metrics such that $qp_{b_2}(x,y) \leq qp_{b_1}(x,y)$, for all $x,y \in X$. Let $F : X^3 \to X$, $g : X \to X$ be two mappings. Suppose that there exists $k \in [0,1)$ such that the condition*

$$\begin{aligned} &qp_{b_1}(F(x,y,z),F(u,v,w)) + qp_{b_1}(F(y,z,x),F(v,w,u)) + qp_{b_1}(F(z,x,y),F(w,u,v)) \\ &\qquad \leq k[qp_{b_2}(gx,gu) + qp_{b_2}(gy,gv) + qp_{b_2}(gz,gw)] \end{aligned}$$

holds for all $x,y,z,u,v,w \in X$. Also, suppose we have the following hypotheses:

(1) $F(X^3) \subseteq g(X)$

(2) $g(X)$ is a complete subspace of X with respect to the quasi-partial b-metric qp_{b_1}.

Then the mappings F and g have a tripled coincidence point (x,y,z) satisfying $gx = F(x,y,z) = F(y,z,x) = gy = F(z,x,y) = gz$.

Moreover, if F and g are w-compatible, then F and g have a unique common tripled fixed point of the form (u,u,u).

Proof. By putting $k_1 = k$ and $k_2 = k_3 = k_4 = k_5 = 0$ in Theorem 3.1 we get the result.

Corollary 3.4. *Let qp_{b_1} and qp_{b_2} be two quasi-partial b-metrics on X such that $qp_{b_2}(x,y) \leq qp_{b_1}(x,y)$, for all $x,y \in X$. Let $F : X^3 \to X$, $g : X \to X$ be two mappings. Suppose that there exists $k \in \left[0, \frac{1}{2s}\right)$ such that the condition*

$$\begin{aligned} &qp_{b_1}(F(x,y,z),F(u,v,w)) + qp_{b_1}(F(y,z,x),F(v,w,u)) + qp_{b_1}(F(z,x,y),F(w,u,v)) \\ &\qquad \leq k[qp_{b_2}(gx,F(u,v,w)) + qp_{b_2}(gy,F(v,w,u)) + qp_{b_2}(gz,F(w,u,v))] \end{aligned} \tag{3.2}$$

holds for all $x,y,z,u,v,w \in X$. Also, suppose we have the following hypotheses:

(1) $F(X^3) \subseteq g(X)$

(2) $g(X)$ is a complete subspace of X with respect to the quasi-partial b-metric qp_{b_1}.

Then the mappings F and g have a tripled coincidence point (x,y) satisfying $gx = F(x,y,z) = F(y,z,x) = gy = F(z,x,y) = gz$.

Moreover, if F and g are w-compatible, then F and g have a unique common tripled fixed point of the form (u,u,u).

Proof. By putting $k_4 = k$ and $k_1 = k_2 = k_3 = k_5 = 0$ in Theorem 3.1 we get the desired result.

Example 3.1. *Let $X=[0,1]$ and two quasi-partial b-metrics qp_{b_1} and qp_{b_2} on X be given as*

$$qp_{b_1}(x,y)=|x-y|+x \quad and \quad qp_{b_2}(x,y)=\frac{1}{2}(|x-y|+x)$$

for all $x,y\in X$. Also, define $F:X^3\to X$ and $g:X\to X$ as $F(x,y)=\dfrac{x+y+z}{36}$ and $g(x)=\dfrac{x}{2}$ for all $x,y\in X$. Then

(1) (X,qp_{b_1}) is a complete quasi-partial b-metric space.
(2) $F(X^3)\subseteq g(X)$
(3) F and g is w-compatible.
(4) For any $x,y,z,u,v,w\in X$, we have

$$\begin{aligned}&qp_{b_1}(F(x,y,z),F(u,v,w))+qp_{b_1}(F(y,z,x),F(v,w,u))+qp_{b_1}(F(z,x,y),F(w,u,v))\\&\quad\leq\frac{1}{3}(qp_{b_2}(gx,gu)+qp_{b_2}(gy,gv)+qp_{b_2}(gz,gz)).\end{aligned}$$

Proof. The proof of (i), (ii) and (iii) are clear. Next, we prove (iv). For $x,y,z,u,v,w\in X$, we have

$$\begin{aligned}&qp_{b_1}(F(x,y,z),F(u,v,w))+qp_{b_1}(F(y,z,x),F(v,w,u))+qp_{b_1}(F(z,x,y),F(w,u,v))\\
&=qp_{b_1}\left(\frac{x+y+z}{36},\frac{u+v}{36}\right)+qp_{b_1}\left(\frac{y+z+x}{36},\frac{v+w+u}{36}\right)+qp_{b_1}\left(\frac{z+x+y}{36},\frac{w+u+v}{36}\right)\\
&=\left|\frac{x+y+z}{36}-\frac{u+v+w}{36}\right|+\left|\frac{y+z+x}{36}-\frac{v+w+u}{36}\right|\\
&+\left|\frac{z+x+y}{36}-\frac{w+u+v}{36}\right|+\frac{3(x+y+z)}{36}\\
&=\frac{1}{12}\left[|(x+y+z)-(u+v+w)|+(x+y+z)\right]\\
&=\frac{1}{12}\left[|(x-u)+(y-v)+(z-w)|+(x+y+z)\right]\\
&\leq\frac{1}{12}\left[|x-u|+|y-v|+|z-w|+(x+y+z)\right]\\
&=\frac{1}{3}\left[\frac{1}{4}|x-u|+\frac{1}{4}|y-v|+\frac{1}{4}|z-w|+\frac{x}{4}+\frac{y}{4}+\frac{z}{4}\right]\\
&=\frac{1}{3}(qp_{b_2}(\frac{x}{2},\frac{u}{2})+qp_{b_2}(\frac{y}{2},\frac{v}{2}))+qp_{b_2}(\frac{z}{2},\frac{w}{2})\\
&=\frac{1}{3}(qp_{b_2}(gx,gu)+qp_{b_2}(gy,gv)+qp_{b_2}(gz,gw)).\end{aligned}$$

Thus, F and g satisfy all the hypotheses of Corollary 2.4. So, F and g have a unique common tripled fixed point. Here $(0,0,0)$ is the unique common tripled fixed point of F and g.

Example 3.2. *Let $X=[0,1]$ and two quasi-partial b-metrics qp_{b_1} and qp_{b_2} on X be given as*

$$qp_{b_1}(x,y)=qp_{b_2}(x,y)=|x-y|+x$$

for all $x,y\in X$. Also, define $F:X^3\to X$ and $g:X\to X$ as $F(x,y)=\dfrac{x+y+z}{3^n m}$ and $g(x)=\dfrac{x}{m}$ for all $x,y\in X$ and $n,m\in\mathbb{N}$. Then

(1) (X,qp_{b_1}) is a complete quasi-partial b-metric space.

(2) $F(X^3)\subseteq g(X)$

(3) F and g is w-compatible.

(4) For any $x,y,z,u,v,w\in X$, we have

$$\begin{aligned}&qp_{b_1}(F(x,y,z),F(u,v,w))+qp_{b_1}(F(y,z,x),F(v,w,u))+qp_{b_1}(F(z,x,y),F(w,u,v))\\&\quad\leq\frac{1}{3^{n-1}}(qp_{b_2}(gx,gu)+qp_{b_2}(gy,gv)+qp_{b_2}(gz,gz)).\end{aligned}$$

Conflict of Interests

The author declares that there is no conflict of interests.

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