SOLITON SOLUTION FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS BY THE ($\frac{G'}{G}$)-EXPANSION METHOD AND ITS APPLICATIONS

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Abstract：We established a traveling wave solution by the ($\frac{G'}{G}$) -expansion method for nonlinear partial differential e：quations (PDEs).The proposed method gives more general exact solutions for four different types of nonlinear partial differential equations such as the modified regularized long wave equation, the improved Korteweg de Vries equation, the two dimension Korteweg de Vries (2D KdV) equations and Coupled equal width wave equations (CEWE).

Keywords: ($\frac{G'}{G}$) -expansion method; modified regularized long wave equation; improved Korteweg de Vries equation.

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1. Introduction

The nonlinear partial differential equations (NPDEs) are widely used to describe many important phenomena and dynamic processes in physics, chemistry, biology, fluid dynamics, plasma, optical fibers and other areas of engineering. Many efforts have been made to study NPDEs. One of the most exciting advances of nonlinear science and theoretical physics has been a development of methods that look for exact solutions for nonlinear evolution equations. The availability of symbolic computations such as Mathematica or Maple, has popularized direct seeking for exact solutions of nonlinear equations. Therefore, exact solution methods of nonlinear evolution equations have become more and more important resulting in methods like the tanh method [1–3], extended tanh function method [4, 5], the modified extended tanh

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function method [6], the generalized hyperbolic function [7] . Most of exact solutions have been obtained by these methods, including the solitary wave solutions, shock wave solutions, periodic wave solutions, and the like. In this letter, we propose the extended $(\frac{G'(\xi)}{G(\xi)})$-expansion method to find the exact solutions of the modified regularized long wave (MRLW)equation, the improved Korteweg de Vries (IKdV) equation , the two dimension Korteweg de Vries (2D KdV) equation and Coupled equal width wave equations (CEWE)[12] . In order to illustrate the effectiveness and convenience of the method, we consider the modified regularized long wave equation, the improved Korteweg de Vries (IKdV) equation and the two dimension Korteweg de Vries (2D KdV) equation.Our main goal in this study is to present the improved $(\frac{G'(\xi)}{G(\xi)})$-expansion method [13-18] for constructing the travelling wave solutions. In section 2, we describe the $(\frac{G'(\xi)}{G(\xi)})$-expansion method. In section 3, we apply the method to four physically important nonlinear evolution equations.

2. Outline of the $(\frac{G'(\xi)}{G(\xi)})$-expansion method

The $(\frac{G'(\xi)}{G(\xi)})$-expansion method will be introduced as presented by A.Hendi [8] and by [13–18].The method is applied to find out an exact solution of a nonlinear ordinary differential equation. Consider the nonlinear partial differential equation in the form

$$N(u, u_t, u_x, u_{tx}, u_{xx}, \dots) = 0 \tag{1}$$

Where $u(\text{x}, \text{t})$ is the solution of nonlinear partial differential equation Eq. (1). We use the transformation, $\xi = (x - c\,t)$, to transform $u(\text{x}, \text{t})$ to $u\,(\xi)$ give :

$$\frac{\partial}{\partial\,t} = -\text{c}\,\frac{\text{d}}{\text{d}\xi}, \frac{\partial}{\partial\,\text{x}} = \frac{\text{d}}{\text{d}\xi}, \frac{\partial^2}{\partial\,\text{x}^2} = \frac{\text{d}^2}{\text{d}\xi^2}, \frac{\partial^3}{\partial\,\text{x}^3} = \frac{\text{d}^3}{\text{d}\xi^3}, \tag{2}$$

 and so on, then Eq. (1) becomes an ordinary differential equation

$$N(u, -c\,u', \, u'\,, c^2\,u'', -c\,u'', \dots \dots) = 0, \tag{3}$$

The solution of Eq.(3) can be expressed by a polynomial in $\frac{G'(\,\xi)}{G(\,\xi)}$

$$u(\xi)\; = \sum_{i=-N}^{N}\,a_i\left(\frac{G'(\,\xi)}{G(\,\xi)}\right)^i, \tag{4}$$

Where $G\; = \; G(\xi)$ satisfies,

$$G''(\xi) + \lambda\,G'(\xi) + \mu\,G(\xi) = 0, \tag{5}$$

Where $G'(\xi) = \frac{d\,G(\xi)}{d\xi}$, $G''(\xi) = \frac{d^2\,G(\xi)}{d\xi}$, a_i, λ and μ are constants to be determined later, $a_i \neq 0$,

the unwritten part in (4) is also a polynomial in $\frac{G'(\xi)}{G(\xi)}$),but the degree of which is generally equal

to or less than $m - 1$,the positive integer m can be determined by balancing the highest order

derivative terms with nonlinear term appearing in Eq.(3). The solutions of Eq.(5) for ($\frac{G'}{G}$) can be

written in the form of hyperbolic, trigonometric and rational functions as given below[8].

$$
\frac{G'(\xi)}{G(\xi)} = \begin{cases}
\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1 \sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)+C_2 \cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{C_1 \cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)+C_2 \sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}\right) - \frac{\lambda}{2}, & \text{when } \lambda^2 - 4\mu > 0, \\[2em]
\frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-C_1 \sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)+C_2 \cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)}{C_1 \cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)+C_2 \sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)}\right) - \frac{\lambda}{2}, & \text{when } \lambda^2 - 4\mu > 0, \\[2em]
\frac{C2}{C1+C2\,\xi} - \frac{\lambda}{2}, & \text{when } \lambda^2 - 4\mu = 0,
\end{cases}
\tag{6}
$$

Where C_1 and C_2 are integration constants. Inserting Eq.(4) into (3) and using Eq.(5),collecting

all terms with the same order $\frac{G'(\xi)}{G(\xi)}$ together, the left hand side of Eq.(3) is converted into another

polynomial in ($\frac{G'(\xi)}{G(\xi)}$).Equating each coefficients of this polynomial to zero, yields a set of

algebraic equations for a_i, λ, and μ .with the knowledge of the coefficients a_i and general

solution of Eq.(5) we have more travelling wave solutions of the nonlinear evolution Eq.(1).

3. Applications

In order to illustrate the effectiveness of the proposed method four examples in mathematical are

chosen as follows

3.1 Test problem

3.1.1 The modified regularized long wave (MRLW) equation

Consider the modified regularized long wave equation in the form [9-11]

$$u_t + u_x + \epsilon u^2 u_x - u_{xxt} = 0, \tag{7}$$

By using the wave variable $\xi = x - ct$ and $u(x,t) = u(\xi))$ then Eq. (7) becomes,

$$-c\,u' + u' + \epsilon\,u^2 u' + c\,u''' = 0, \tag{8}$$

Integrating the above equation with respect to ξ, we get

$$(1 - c)u + \frac{\epsilon}{3}u^3 + c\,u'' = 0, \tag{9}$$

Balancing u^3 with u'' gives $m = 1$. Thus we suppose solutions of Eq. (7) can be expressed by

$$u(\xi) = a_0 + a_1\left(\frac{G'(\xi)}{G(\xi)}\right), \tag{10}$$

By solving this system with the aid of Mathematica we obtain

$$a_0 = \pm\frac{i\sqrt{3}\lambda}{\sqrt{\epsilon}\sqrt{2+(\lambda^2-4\mu)}}, a_1 = \pm\frac{2i\sqrt{3}}{\sqrt{\epsilon}\sqrt{(2+\lambda^2-4\mu)}} \quad \text{and} \quad c = \frac{2}{2+(\lambda^2-4\mu)}, \tag{11}$$
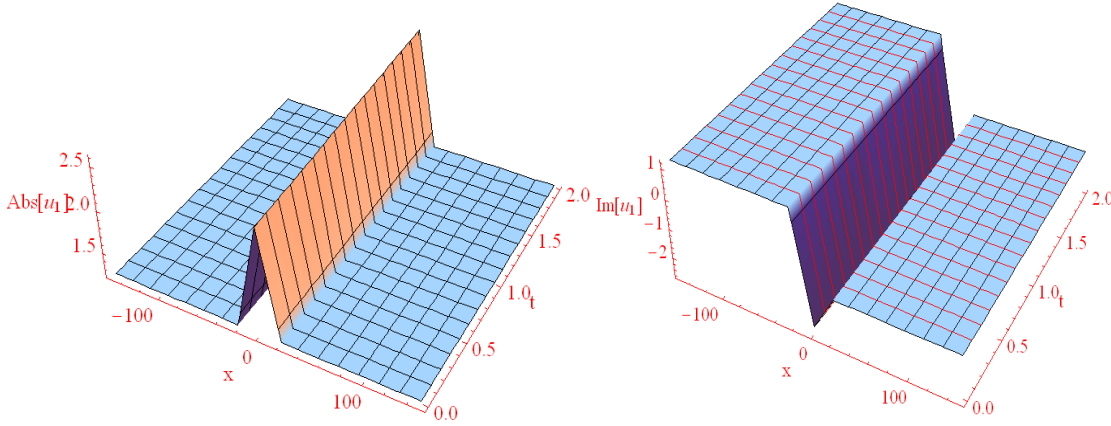
Substituting Eq.(11) into Eq.(10) yields

$$u(\xi) = \pm\frac{i\sqrt{3}\,\lambda}{\sqrt{\epsilon}\sqrt{2+(\lambda^2-4\mu)}} \pm \frac{2i\sqrt{3}}{\sqrt{\epsilon}\sqrt{(2+\lambda^2-4\mu)}}\left(\frac{G'(\xi)}{G(\xi)}\right), \tag{12}$$

Where $\quad \xi = x - \frac{2t}{2+(\lambda^2-4\mu)}$, With the knowledge of the solution of Eq.(5), we have three types

of travelling wave solutions of Eq.(7) as

The first type: when $\quad \lambda^2 - 4\mu > 0,$

$$u_1(\xi) = \left(\pm\frac{i\sqrt{3}\sqrt{\lambda^2-4\mu}}{\sqrt{\epsilon}\sqrt{2+(\lambda^2-4\mu)}}\right)\left(\frac{C_1\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)+C_2\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{C_1\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)+C_2\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}\right), \tag{13}$$

The behavior of the solutions absolute $u_1(x,t)$ and $iu_1(x,t)$ for MRLW equation in Figures(1,2)

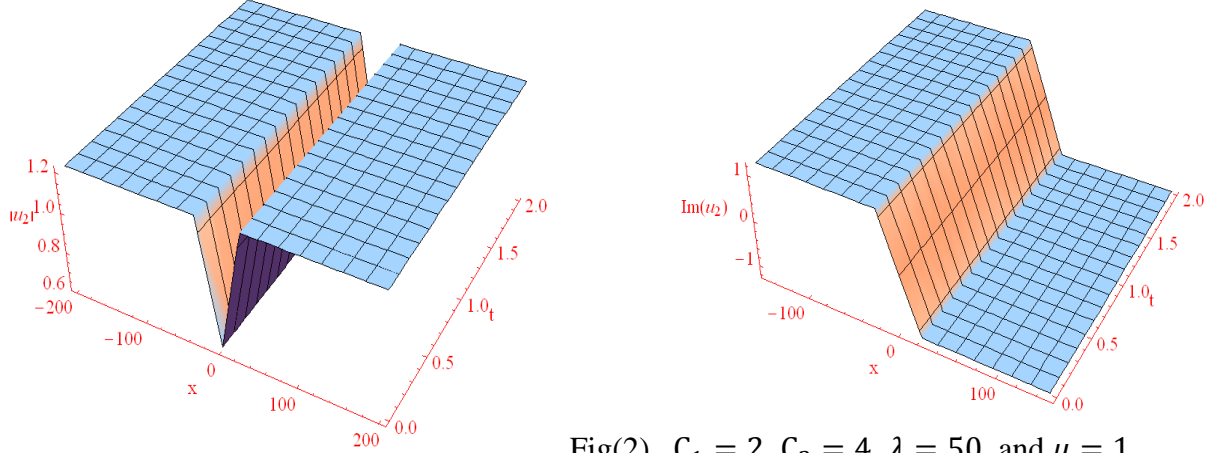Fig(1), $C_1 = 1, C_2 = 2, \lambda = 50,$ and $\mu = 1$

The second type: when $\quad \lambda^2 - 4\mu < 0,$

$$u_2(\xi) = \left(\pm\frac{i\sqrt{3}\sqrt{4\mu-\lambda^2}}{\sqrt{\epsilon(2+\lambda^2-4\mu)}}\right)\left(\frac{C_1\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)-C_2\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)}{C_1\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)+C_2\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)}\right) \tag{14}$$

The behavior of the solutions absolute $u_2(x,t)$ and $iu_2(x,t)$ are shown in Figure(2)

Fig(2), $C_1 = 2, C_2 = 4, \lambda = 50$, and $\mu = 1$

The Third type: when $\lambda^2 - 4\mu = 0$

$$u_3(\xi) = \frac{2i\sqrt{3}}{\sqrt{2\epsilon}}\left(\frac{C_2}{C_1 + C_2\xi}\right), \tag{15}$$

3.1.2. The improved Korteweg de Vries (IKdV) equation

We Consider the IKdV equation in the form [11]

$$u_t + \epsilon u u_x + \beta u_{xxx} - \delta u_{xxt} = 0, \tag{16}$$

We make the transformation

$$u(x,t) = u(\xi), \ \xi = x - ct, \tag{17}$$

Eq. (16) becomes

$$-cu' + \epsilon u u' + \beta u''' + \delta c u''' = 0, \tag{18}$$

Integrating the above equation with respect to ξ, we get

$$-cu + \frac{\epsilon}{2}u^2 + (\beta + \delta c)u'' = 0, \tag{19}$$

Balancing u^2 with u'' gives $m = 2$. thus we suppose solutions of Eq. (18) can be expressed by

$$u(\xi) = a_0 + a_1\left(\frac{G'(\xi)}{G(\xi)}\right) + a_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2, \tag{20}$$

Where a_0, a_1, a_2 are constants, Substituting Eq.(20) into Eq.(19),collecting the coefficients of

$\left(\frac{G'(\xi)}{G(\xi)}\right)$ we obtain a set of algebraic equations for a_0, a_1, a_2 and c ,and solving this system we

obtain the two sets of solutions as

Case (1)

$$a_0 = -\frac{2\beta(\lambda^2+2\mu)}{\epsilon(1+\delta(\lambda^2-4\mu))}, a_1 = -\frac{12\beta\lambda}{\epsilon(1+\delta(\lambda^2-4\mu))}, \ a_2 = -\frac{12\beta}{\epsilon(1+\delta(\lambda^2-4\mu))}, \text{ and } c = \frac{-\beta(\lambda^2-4\mu)}{1+\delta(\lambda^2-4\mu)}, \quad (21)$$

Case (2)

$$a_0 = \frac{12\beta\mu}{\epsilon(-1+\delta(\lambda^2-4\mu))} \ , a_1 = \frac{12\beta\lambda}{\epsilon(-1+\delta(\lambda^2-4\mu))}, a_2 = \frac{12\beta}{\epsilon(-1+\delta(\lambda^2-4\mu))}, \text{ and } c = \frac{-\beta(\lambda^2-4\mu)}{-1+\delta(\lambda^2-4\mu)}, \quad (22)$$

By using Eq.(21) , Eq.(20)can written as

$$u(\xi) = -\frac{2\beta(\lambda^2+2\mu)}{\epsilon(1+\delta(\lambda^2-4\mu))} - \frac{12\beta\lambda}{\epsilon(1+\delta(\lambda^2-4\mu))}\left(\frac{G'(\xi)}{G(\xi)}\right) - \frac{12\beta}{\epsilon(1+\delta(\lambda^2-4\mu))}\left(\frac{G'(\xi)}{G(\xi)}\right)^2, \quad (23)$$

or by using Eq.(22),Eq.(20)can written as

$$u(\xi) = \frac{12\beta\mu}{\epsilon(-1+\delta(\lambda^2-4\mu))} + \frac{12\beta\lambda}{\epsilon(-1+\delta(\lambda^2-4\mu))}\left(\frac{G'(\xi)}{G(\xi)}\right) + \frac{12\beta}{\epsilon(-1+\delta(\lambda^2-4\mu))}\left(\frac{G'(\xi)}{G(\xi)}\right)^2 \quad (24)$$

We have three types of travelling wave solutions of the IKdV equation as

The first type: when $\lambda^2 - 4\mu > 0$, $u_1(\xi) =$

$$\left(\frac{3\beta(4\mu-\lambda^2)}{\epsilon(1+\delta(\lambda^2-4\mu))}\right)\left(\left(\frac{C_1\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)+C_2\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{C_1\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)+C_2\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}\right)^2 - \frac{1}{3}\right), \quad (25)$$

Where $\xi = x + \frac{\beta(\lambda^2-4\mu)}{1+\delta(\lambda^2-4\mu)}t$, or

$$u_2(\xi) = \left(\frac{3\beta(4\mu-\lambda^2)}{\epsilon(-1+\delta(\lambda^2-4\mu))}\right)\left(1 - \left(\frac{C_1\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)+C_2\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{C_1\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)+C_2\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}\right)^2\right), \quad (26)$$

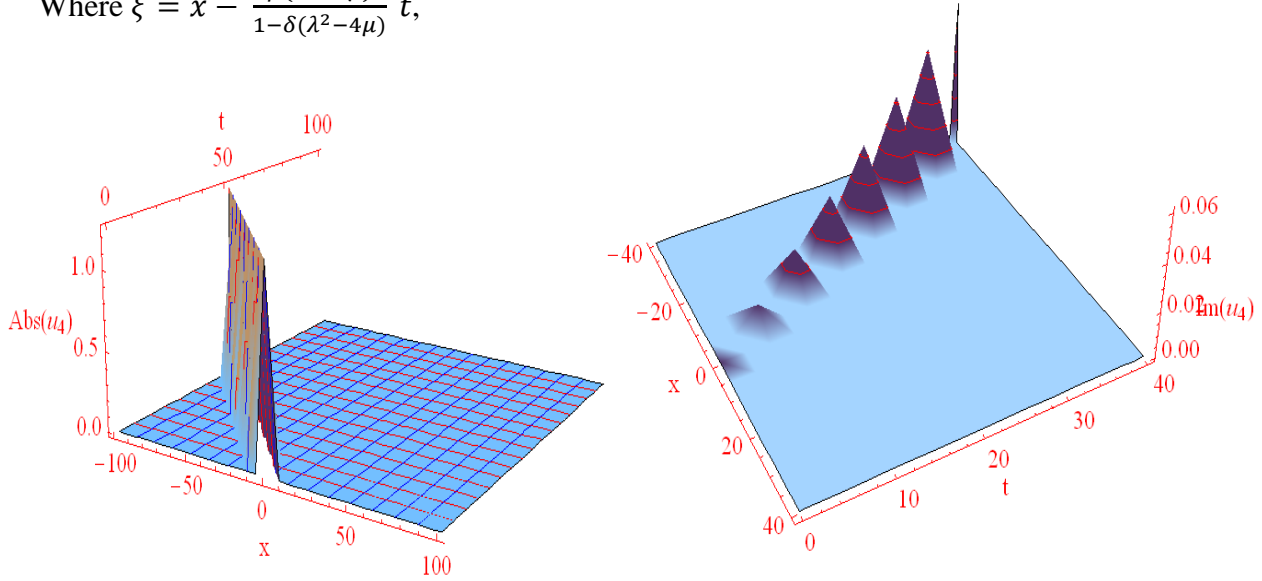Where $\xi = x + \frac{\beta(\lambda^2-4\mu)}{1-\delta(\lambda^2-4\mu)}t$,

The second type: when $\lambda^2 - 4\mu < 0$,

$$u_3(\xi) = \left(\frac{-3\beta(4\mu-\lambda^2)}{\epsilon(1+\delta(\lambda^2-4\mu))}\right)\left(\left(\frac{-C_1\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)+C_2\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)}{C_1\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)+C_2\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)}\right)^2 + \frac{1}{3}\right), \quad (27)$$

Where $\xi = x + \frac{\beta(\lambda^2-4\mu)}{1+\delta(\lambda^2-4\mu)}t$, or

$$u_4(\xi) = \left(\frac{3\beta(4\mu-\lambda^2)}{\epsilon(-1+\delta(\lambda^2-4\mu))}\right)\left(\left(\frac{-C_1\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)+C_2\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)}{C_1\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)+C_2\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)}\right)^2 + 1\right), \quad (28)$$

Where $\xi = x - \frac{\beta(\lambda^2 - 4\mu)}{1 - \delta(\lambda^2 - 4\mu)} t$,

Fig(3) for IKdV equation where $C_1 = 2, C_2 = 4, \lambda = 150$, and $\mu = 1, \beta = 2, \beta = 1$;

The Third type: when $\lambda^2 - 4\mu = 0$

$$u_5(\xi) = \frac{-12\beta}{\epsilon}\left(\frac{C2}{C1 + C2\xi}\right)^2, \quad \text{Where } \xi = x - \frac{\beta(\lambda^2 - 4\mu)}{1 - \delta(\lambda^2 - 4\mu)} t, \tag{29}$$

3.13. The two dimension Korteweg de Vries (2D KdV) equation [11]

Consider the two dimensions Korteweg de Vries in the form

$$(u_t - \epsilon u \, u_x + u_{xxx})_x + 3u_{yy} = 0, \tag{30}$$

Put $u(x, t) = u(\xi)$, $\xi = x + \beta y - c t$, Eq. (30) become

$$(3\beta^2 - c)u'' - \epsilon(u \, u')' + u'''' = 0, \tag{31}$$

Integrating the above equation with respect to ξ , we get

$$(3\beta^2 - c)u - \frac{\epsilon}{2}u^2 + u'' = 0, \tag{32}$$

Balancing u^2 with u'' gives $m = 2$. thus we suppose solutions of Eq. (30) can be expressed by

$$u(\xi) = a_0 + a_1\left(\frac{G'(\xi)}{G(\xi)}\right) + a_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2, \tag{33}$$

Where a_0, a_1, a_2 are constants, Substituting Eq.(33) into Eq.(32),collecting the coefficients of

$\left(\frac{G'(\xi)}{G(\xi)}\right)$ we obtain a set of algebraic equations for a_0, a_1, a_2 and c ,and solving this system we

obtain the two sets of solutions as

Case (1)

$$a_0 = \frac{12\mu}{\epsilon}, a_1 = \frac{12\lambda}{\epsilon}, a_2 = \frac{12}{\epsilon} \text{ and } c = 3\beta^2 + \lambda^2 - 4\mu, \tag{34}$$

Case (2)

$$a_0 = \frac{2(\lambda^2+2\mu)}{\epsilon}, \quad a_1 = \frac{12\lambda}{\epsilon}, \quad a_2 = \frac{12}{\epsilon}, \quad c = 3\beta^2 - \lambda^2 + 4\mu, \tag{35}$$

By using Eq.(34)and Eq(35),Eq.(33) can written as

$$u_1(\xi) = \frac{12\mu}{\epsilon} + \frac{12\lambda}{\epsilon}\left(\frac{G'(\xi)}{G(\xi)}\right) + \frac{12}{\epsilon}(\frac{G'(\xi)}{G(\xi)})^2, \text{ or} \tag{36}$$

$$u_2(\xi) = \frac{2(\lambda^2+2\mu)}{\epsilon} + \frac{12\lambda}{\epsilon}\left(\frac{G'(\xi)}{G(\xi)}\right) + \frac{12}{\epsilon}(\frac{G'(\xi)}{G(\xi)})^2, \tag{37}$$

With the knowledge of the solution of Eq.(5) and Eqs.(36-37),we have three types of travelling wave solutions of the Eq.(30) as

The first type: when $\lambda^2 - 4\mu > 0$,

$$u_1(\xi) = -\frac{3(C_1^2-C_2^2)(\lambda^2-4\mu)}{\epsilon(C_1\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)+C_2\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi))^2}, \tag{38}$$

Where $\xi = x + \beta y - (3\beta^2 + \lambda^2 - 4\mu)t$, or

$$u_2(\xi) = \frac{(\lambda^2-4\mu)}{\epsilon}(2 - \frac{3(C_1^2-C_2^2)}{(C_1\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)+C_2\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi))^2}), \tag{39}$$

Where $\xi = x + \beta y - (3\beta^2 - \lambda^2 + 4\mu)t$

The second type: when $\lambda^2 - 4\mu < 0$

$$u_3(\xi) = \frac{(\lambda^2-4\mu)}{\epsilon}(2 - \frac{3(C_1^2+C_2^2)}{(C_1\cos(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi)+C_2\sin(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi))^2}), \tag{40}$$

Where $\xi = x + \beta y - (3\beta^2 + \lambda^2 - 4\mu)t$, or

$$u_4(\xi) = -\frac{3(C_1^2+C_2^2)(\lambda^2-4\mu)}{\epsilon(C_1\cos(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi)+C_2\sin(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi))^2}, \tag{41}$$
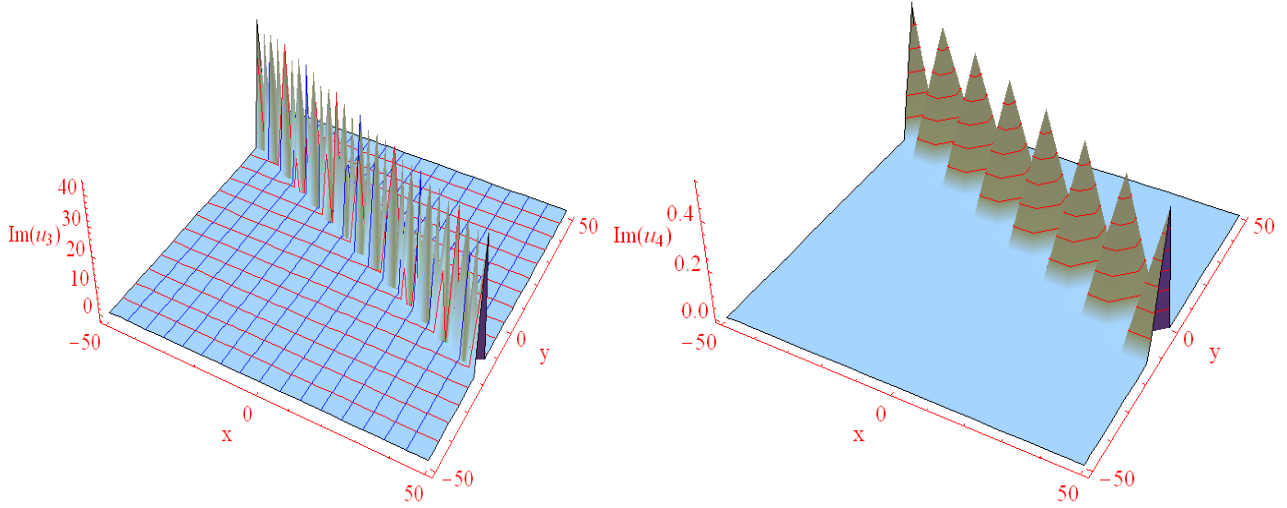
Where $\xi = x + \beta y - (3\beta^2 - \lambda^2 + 4\mu)t$

The Third type: when $\lambda^2 - 4\mu = 0$,

$$u_5(\xi) = \frac{12C_2^2}{\epsilon(C_2+C_2\xi)^2}, \tag{42}$$

Where $\xi = x + \beta y - 3\beta^2 t$,

The behavior of the solutions $u_3(x,t)$ and $iu_3(x,t)$ for 2DKdV equation are shown in Figure(4)

Fig(4) $C_1 = 2, C_2 = 4, \lambda = 3,$ and $\mu = 1,$

3.1.4 . Coupled equal width wave equation (CEWE)[12]

In this case we consider the coupled equal width wave equations, in the normalized form

$$u_t + u\,u_x - u_{xxt} + v\,v_x = 0, \qquad v_t + v\,v_x - v_{xxt} = 0, \qquad (43)$$

We put $u(x,t) = u(\xi), v(x,t) = v(\xi),$ and $\xi = x - c\,t$ Eq. (43) become

$$-c\,u' + u\,u' + c\,u''' + v\,v' = 0, \quad \text{and} \quad -c\,v' + v\,v' + c\,v''' = 0, \qquad (44)$$

Integrating the above equation we get

$$c\,u'' + \tfrac{1}{2}(u^2 + v^2) - c\,u = 0, \quad \text{and} \quad c\,v'' + \tfrac{1}{2}v^2 - c\,v = 0, \qquad (45)$$

Balancing u^2 with u'' gives $m = 2$. thus we suppose solutions of Eq. (43) can be expressed by

$$u(\xi) = a_0 + a_1\left(\frac{G'(\xi)}{G(\xi)}\right) + a_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2,$$

$$v(\xi) = b_0 + b_1\left(\frac{G'(\xi)}{G(\xi)}\right) + b_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2, \qquad (46)$$

Substituting Eq.(46) into Eq.(45) and solving this system we obtain the two sets of solutions as

Case (1)

$$a_0 = \tfrac{3}{2}\left(1 + i\sqrt{3}\right)c(1 - \lambda^2), a_1 = -6c\lambda\left(1 + i\sqrt{3}\right),, \quad a_2 = -6c(1 + i\sqrt{3}),$$

$$b_0 = 3c(1 - \lambda^2); b_1 = -12c\lambda; b_2 = -12c; and\ \mu = \tfrac{1}{4}(-1 + \lambda^2), \qquad (47)$$

Case (2)

$$a_0 = -\tfrac{1}{2}(1 + i\sqrt{3})c(1 + 3\lambda^2), a_1 = -6c\lambda\left(1 + i\sqrt{3}\right), a_2 = -6c(1 + i\sqrt{3}),$$

$$b_0 = -c(1 + 3\lambda^2); b_1 = -12c\lambda; b_2 = -12c; and\ \mu = \tfrac{1}{4}(1 + \lambda^2), \qquad (48)$$

By using Eq.(47),Eq.(46)can written as

$$u(x,t) = \frac{3}{2}\big(1 + i\sqrt{3}\big)c(1 - \lambda^2) - 6c\lambda\big(1 + i\sqrt{3}\big)\big(\frac{G'(\xi)}{G(\xi)}\big) - 6c(1 + i\sqrt{3})\big(\frac{G'(\xi)}{G(\xi)}\big)^2,$$

$$v(x,t) = 3c(1 - \lambda^2) - 12c\lambda\big(\frac{G'(\xi)}{G(\xi)}\big) - 12c\big(\frac{G'(\xi)}{G(\xi)}\big)^2, \tag{49}$$

or by using Eq.(48),Eq.(46)can written as

$$u(x,t) = -\frac{1}{2}\big(1 + i\sqrt{3}\big)c(1 + 3\lambda^2) - 6c\lambda\big(1 + i\sqrt{3}\big)\left(\frac{G'(\xi)}{G(\xi)}\right) - 6c\big(1 + i\sqrt{3}\big)\left(\frac{G'(\xi)}{G(\xi)}\right)^2,$$

$$v(x,t) = -c(1 + 3\lambda^2) - 12c\lambda\big(\frac{G'(\xi)}{G(\xi)}\big) - 12c\big(\frac{G'(\xi)}{G(\xi)}\big)^2, \tag{50}$$

We have two types of travelling wave solutions of the CEWE equation as

The first type: when $\lambda^2 - 4\mu > 0$

$$u_1(x,t) = \frac{6c(1+i\sqrt{3})(C_1^2-C_2^2)e^{ct+x}}{[(C_1-C_2)e^{ct}+(C_1+C_2)e^x]^2},$$

$$v_1(x,t) = \frac{3c(C_1^2-C_2^2)}{[C_1\cosh(\frac{1}{2}(x-ct))+C_2\sinh(\frac{1}{2}(x-ct))]^2}, \tag{51}$$

or

$$u_2(x,t) = \frac{(1-i\sqrt{3})c[(C_1^2+C_2^2)\cos(x-ct)-2(C_1^2-C_2^2-iC_1C_2\sin(x-ct))]}{2(C_1\cos(\frac{1}{2}(x-ct))-iC_2\sin(\frac{1}{2}(x-ct))]^2}$$

$$v_2(x,t) = (2c + \frac{3c(C_1^2-C_2^2)}{[iC_1\cos(\frac{1}{2}(x-ct))-C_2\sin(\frac{1}{2}(x-ct))]^2}), \tag{52}$$

The second type: when $\lambda^2 - 4\mu < 0$,

$$u_3(x,t) = \frac{3(1-i\sqrt{3})c(C_1^2+C_2^2)}{2[C_1\cosh(\frac{1}{2}(x-ct))-iC_2\sinh(\frac{1}{2}(x-ct))]^2},$$

$$v_3(x,t) = \frac{3c(C_1^2+C_2^2)}{[C_1\cosh\big(\frac{1}{2}(x-ct)\big)-iC_2\sinh(\frac{1}{2}(x-ct))]^2}, \quad \text{or} \tag{53}$$

$$u_4(x,t) = \frac{(1-i\sqrt{3})c((C_1^2-C_2^2)\cos[ct-x]-2(C1^2+C2^2+C1C2\text{Sin}[ct-x]))}{2[C_1\cos(\frac{1}{2}(x-ct))-C_2\sin(\frac{1}{2}(x-ct))]^2},$$

$$v_4(x,t) = c(2 - \frac{3(C_1^2+C_2^2)}{(C_1\cos(\frac{1}{2}(x-ct))+C_2\sin(\frac{1}{2}(x-ct)))^2}), \tag{54}$$

The behavior of the solutions $u_3(x,t)$ and $v_3(x,t)$ for CEWE equation are shown in Fig.(5)
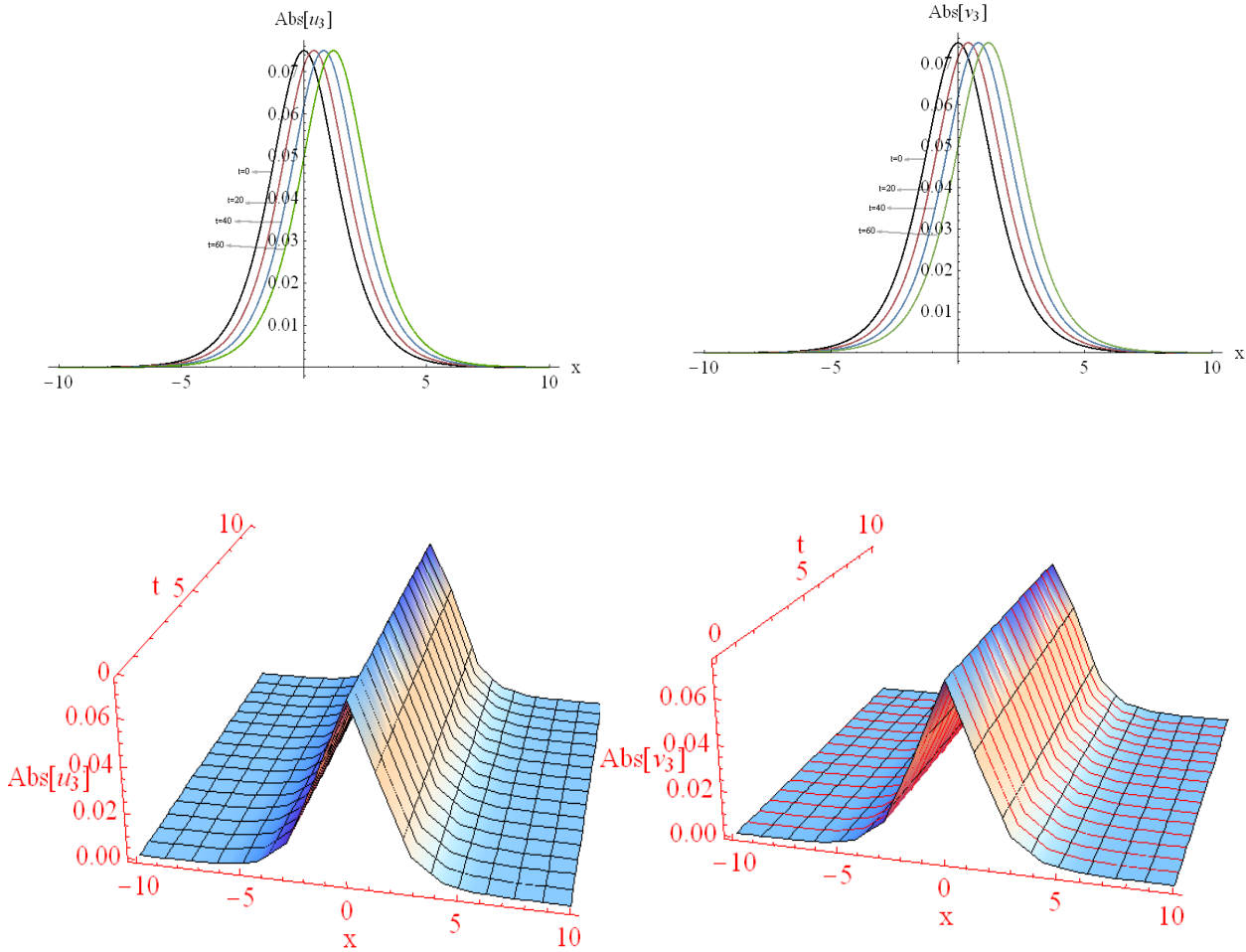
Fig(5) $C_1 = 2$, $C_2 = 1$, $\lambda = 5$, and $c = 1/50$,

4. Conclusion

In this letter the $(\frac{G'(\xi)}{G(\xi)})$-expansion method was applied successfully for solving some solitary wave equations in one and two dimensions. Four equations which are the MRLW, IKdV ,2D KdV and CEWE equations have been solved exactly. As a result, many exact solutions are obtained which include the hyperbolic functions, trigonometric functions and rational functions. It is worthwhile to mention that the proposed method is reliable and effective and gives more solutions. The method can also be efficiently used to construct new and more exact solutions for some other generalized nonlinear wave equations arising in mathematical physics.

Conflict of Interests

The authors declare that there is no conflict of interests.

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