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ON OUTPUT SUBSYSTEMS OF FUZZY MOORE MACHINES

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Abstract. The purpose of this paper is to study fuzzy Moore machines and their (output) subsystems. Apart from usual properties of subsystems of a fuzzy Moore machine, we characterize them using a class of fuzzy sets for fixed strings of input and output. Also a class of subsystems of a given fuzzy Moore machines is obtained with the help of fuzzy points. Cyclic and super cyclic subsystems are also encountered and characterized. The concept of subsystem is generalized to output subsystem. While proving (cartesian) product of output subsystems is an output subsystem, we introduce products of fuzzy Moore machines. These products of fuzzy Moore machines with the help of separability of functions and without separability of functions are analyzed and natural products are introduced.

Keywords: Subsystem; Finite state machine; Fuzzy Moore machine: Restricted product: Wreath product

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1. Preliminaries

In recent studies on fuzzy automaton, various extensions such as, general fuzzy automaton [5, 16], Intutionistic fuzzy automaton [4], Bipolar fuzzy automaton [9], fuzzy pushdown automaton [14, 4] etc are successfully studied. Apart from these extensions various properties of

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fuzzy finite state machines are extended to these extensions [1, 3, 7, 8, 10, 11, 12, 13]. In [10] subsystem of fuzzy finite state machine is introduced and various issues relating to them are discussed. Since many concepts of fuzzy finite state machine are introduced for fuzzy Mealy machine [2, 8, 13, 15]. It is natural to think about the extension for fuzzy Moore machine. In [1] fuzzy Mealy and Moore machines are introduced and discussed comparatively. We use the definition of fuzzy Moore machine given in the [1] and discuss mainly (output) subsystem of fuzzy Moore machines in this paper.

Recall that X^* denote the set of all string of finite length over X, λ denotes the empty string and $|X|$ denotes the length of x.

The basic definitions of fuzzy Mealy and Moore machines are given in [1] as follows:
A fuzzy Mealy machine is a sextuple $S = (Q, \Sigma, \Gamma, I, \mu, \omega)$, where Q is a non-empty finite set of state of S; Σ is a non-empty finite set of inputs of S; Γ is a non-empty finite set of outputs of S; $I : Q \longrightarrow [0,1]$, is called initial fuzzy state in S; $\mu : Q \times \Sigma \times Q \longrightarrow [0,1]$, is called fuzzy **transition function** and $\omega : Q \times \Sigma \times \Gamma \longrightarrow [0,1]$, called fuzzy **output function**.

A fuzzy Moore machine is a sextuple $M = (Q, \Sigma, \Gamma, I, \mu, \delta)$, where $Q, \Sigma, \Gamma, I, \mu$ are similar as in the above definition of fuzzy Mealy Machine and $\delta : Q \times \Gamma \longrightarrow [0,1]$, called fuzzy **output function**. The fuzzy set δ induces the fuzzy set $\delta^\# : Q \times \Sigma^* \times \Gamma^* \longrightarrow [0,1]$ as follows:
$\delta^\#(p,x,\lambda) = \delta^\#(p,\lambda,\alpha) = 0$; $\delta^\#(p,\sigma,\tau) = \bigvee_{t \in Q} \{\mu(p,\sigma,t) \wedge \delta(t,\tau)\}$ and $\delta^\#(p,\sigma x,\tau\alpha)$ $= \bigvee_{t \in Q} \{\mu(p,\sigma,t) \wedge \delta(t,\tau) \wedge [\delta^\#(t,x,\alpha)]\}, \forall\, p \in Q, \sigma \in \Sigma, \tau \in \Gamma, x \in \Sigma^*$ and $\alpha \in \Gamma^*$. Thus, the following results are obvious.

Theorem 1.1. *Let $M = (Q, \Sigma, \Gamma, I, \mu, \delta)$ be fuzzy Moore machine. Then for $p \in Q, \sigma \in \Sigma, x \in \Sigma^*$ and $\alpha \in \Gamma^*$, if $|x| \neq |\alpha|$, then we have $\delta^\#(p,x,\alpha) = 0$.*

Let σ be a fuzzy subset of a nonempty set of X. Then $supp(\sigma) = \{x \in X | \sigma(x) > 0\}$ is the support of σ. Throughout this paper $\wedge$ denotes infimum and $\vee$ denotes supremum of a set. Let $a \in Q$ and $t \in [0,1]$. Then the fuzzy subset q_t of Q is defined by $q_t(q) = t$ and $q_t(r) = 0$, if $q \neq r\, \forall r \in Q$.

2. Fuzzy moore machines and homomorphisms

In this section, we introduce fuzzy Moore machines and discuss various properties of them. Recall. X^* denote the set of all string of finite length over X, λ denotes the empty string and $|x|$ denotes the length of x.

Let σ be a fuzzy subset of a nonempty set of X. Then $supp(\sigma) = \{x \in X | \sigma(x) > 0\}$ is the support of σ. Throughout this paper $\wedge$ denotes infimum and $\vee$ denotes supremum of a set. Let $q \in Q$ and $t \in [0,1]$. Then the fuzzy subset q_t of Q is defined by $q_t(q) = t$ and $q_t(r) = 0$, if $q \neq r \ \forall\, r \in Q$.

Definition 2.1. *Fuzzy Moore machine is a quintuple $M = (Q,X,Y,\delta,\sigma)$ where Q is a finite non-empty set called set of states, X is a finite non-empty set called set of inputs, Y is a finite non-empty set called set of outputs, δ is a fuzzy subset of $Q \times X \times Q$ called the transition function, σ is a fuzzy subset of $Q \times Y$ called the output function and following condition is satisfied:*
$\forall\, q \in Q, a \in X, (\exists\, p \in Q, \delta(q,a,p) > 0)) \Leftrightarrow (\exists\, b \in Y, \sigma(q,b) > 0)$.

Definition 2.2. *Let $M = (Q,X,Y,\delta,\sigma)$ be a fuzzy Moore machine. Then*
(i) define $\delta^* : Q \times X^* \times Q \longrightarrow [0,1]$ as: for all $q, p \in Q, a \in X, x \in X^*$

$$\delta^*(q,\lambda,p) = \begin{cases} 1, & \text{if } q = p, \\ 0, & \text{if } q \neq p, \text{ and} \end{cases}$$

$\delta^*(q,ax,p) = \bigvee_{r\in Q} \{\delta(q,a,r) \wedge \delta^*(r,x,p)\}$
(ii) define $\sigma^\# : Q \times X^* \times Y^* \longrightarrow [0,1]$ as: for all $q \in Q, a \in X, x \in X^*, b \in Y, y \in Y^*$

$$\sigma^\#(q,x,y) = \begin{cases} 1, & \text{if } x = y = \lambda \\ 0, & \text{if } (x = \lambda, y \neq \lambda) \text{ or } (y = \lambda, x \neq \lambda), \end{cases}$$

$\sigma^\#(q,a,b) = \bigvee_{r\in Q} \{\delta(q,a,r) \wedge \sigma(r,b)\}$ *and*
$\sigma^\#(q,ax,by) = \bigvee_{r\in Q} \{\delta(q,a,r) \wedge \sigma(r,b) \wedge \sigma^\#(r,x,y)\}$

The following theorem is independent of the output function and can be found in many references for example [1, 2, 5, 6, 7].

Theorem 2.3. *Let $M=(Q,X,Y,\delta,\sigma)$ be a fuzzy Moore machine. Then $\delta^*(q,xu,p)=\bigvee_{r\in Q}\{\delta^*(q,x,r)\wedge \delta^*(r,u,p)\},\ \forall\, q,p\in Q, and\ x,u\in X^*$.*

The following couple of theorems show that the input and output has same length for the working of fuzzy Moore machines.

Theorem 2.4. *Let $M=(Q,X,Y,\delta,\sigma)$ be a fuzzy Moore machine. If $|x|\neq|y|$, then $\sigma^{\#}(q,x,y)=0,\ \forall\, q\in Q, x\in X^*, y\in Y^*$.*

Proof. Without loss of generality assume that $|x|>|y|$. If $|y|=0$, then $y=\lambda$. Thus by definition of $\sigma^{\#}, \sigma^{\#}(q,x,y)=\sigma^{\#}(q,x,\lambda)=0$. Suppose that the theorem holds for $|y|=n-1$. Let $y=y_1y_2y_3...y_n$. Then $|x|$ is at least $n+1$. Suppose, $x=x_1x_2x_3...x_nx_{n+1}$. Then
$\sigma^{\#}(q,x_1x_2x_3...x_nx_{n+1},y_1y_2y_3...y_n)=\bigvee\{\delta(q,x_1,r_1)\wedge\sigma(r_1,y_1)\wedge[\sigma^{\#}(r_1,x_2x_3...x_n\,x_{n+1},y_2y_3...y_n)]|r_1\in Q\}$ $=\bigvee\{[\delta(q,x_1,r_1)\wedge\sigma(r_1,y_1)]\wedge[\delta(r_1,x_2,r_2)\wedge\sigma(r_2,y_2)]\wedge[\delta(r_2,x_3,r_3)\wedge\sigma(r_3,y_3)]\wedge$
$...\wedge[\delta(r_{n-2},x_{n-1},r_{n-1})\wedge\sigma(r_{n-1},y_{n-1})]\wedge[\delta(r_{n-1},x_n,r_n)\wedge\sigma(r_n,y_n)]\wedge\sigma^{\#}(r_n,x_{n+1},\lambda)|r_i\in Q\}$
$=\bigvee\{[\delta(q,x_1,r_1)\wedge\sigma(r_1,y_1)]\wedge[\delta(r_1,x_2,r_2)\wedge\sigma(r_2,y_2)]\wedge[\delta(r_2,x_3,r_3)\wedge\sigma(r_3,y_3)]\wedge$
$...\wedge[\delta(r_{n-2},x_{n-1},r_{n-1})\wedge\sigma(r_{n-1},y_{n-1})]\wedge[\delta(r_{n-1},x_n,r_n)\wedge\sigma(r_n,y_n)]\wedge 0|r_i\in Q\}=0.$ □

Theorem 2.5. *Let $M=(Q,X,Y,\delta,\sigma)$ be a fuzzy Moore machine, If $|x|=|y|$ then $\sigma^{\#}(q,ax,by)=\bigvee_{r\in Q}\{\sigma^{\#}(q,a,b)\wedge[\delta(q,a,r)\wedge\sigma^{\#}(r,x,y)]\},\ \forall\, q\in Q, x\in X^*, a\in X, y\in Y^*, b\in Y$.*

Proof. By the definition 2.2, $\sigma^{\#}(q,ax,by)=\bigvee_{r\in Q}\{\delta(q,a,r)\wedge\sigma(r,b)\wedge\sigma^{\#}(r,x,y)\}=\bigvee_{r\in Q}\{[\delta(q,a,r)\wedge\sigma(r,b)]\wedge[\delta(q,a,r)\wedge\sigma^{\#}(r,x,y)]\}=\bigvee_{r\in Q}\{\sigma^{\#}(q,a,b)\wedge[\delta(q,a,r)\wedge\sigma^{\#}(r,x,y)]\}$. □

Inductively one can easily prove that for any $q\in Q$ and $x\in X^*$($\exists\, p\in Q$ such that $\delta^*(q,x,p)>0)\Leftrightarrow(\exists\, y\in Y^*$ such that $\sigma^{\#}(q,x,y)>0)$ and $|x|=|y|$. Throughout this paper whenever we talk about δ^* and $\sigma^{\#}$ for strings of input x and output y, we mean it for $|x|=|y|$.

Definition 2.6. *Let $M=(Q,X,Y,\delta,\sigma)$ be a fuzzy Moore machine. Let $q,p\in Q$. Then p is called an immediate successor of q, if $\exists\ a\in X$ and $b\in Y$ such that $\delta(q,a,p)\wedge\sigma(q,b)>0$ and p is called successor of q, if $\exists\ x\in X^*$ and $y\in Y^*$ such that $\delta^*(q,x,p)\wedge\sigma^{\#}(q,x,y)>0$.*

Let $M=(Q,X,Y,\delta,\sigma)$ be a fuzzy Moore machine and $q\in Q$. We shall denote $S(q)$ the set of all successor of q. If $T\subseteq Q$, then set of all successor of T, denoted by $S(T)$, is defined by the set $S(T)=\bigcup\{S(q)\mid q\in T\}$.

Theorem 2.7. *Let $M = (Q,X,Y,\delta,\sigma)$be a fuzzy Moore machine. Define a relation $\sim$ on Q as $p \sim q$ if and only if q is successor of p. Then $\sim$ is reflexive and transitive.*

Clearly $\sim$ is not symmetric.

Theorem 2.8. *Let $M = (Q,X,Y,\delta,\sigma)$be a fuzzy Moore machine. Let $A,B \subseteq Q$*

(1) if $A \subseteq B$ then $S(A) \subseteq S(B)$.

(2) $A \subseteq S(A)$.

(3) $S(S(A)) = S(A)$.

(4) $S(A \cup B) = S(A) \cup S(B)$.

(5) $S(A \cap B) \subseteq S(A) \cap S(B)$.

Proof. The proofs of (1), (2), (4) and (5) are straightforward.
(3) By (2) we have $S(A) \subseteq S(S(A))$. Let $q \in S(S(A))$. Then $q \in S(p)$, for some $p \in S(A)$. Thus $p \in S(r)$, for some $r \in A$. Now, q is successor of p and p is successor of r, hence by Theorem (2.7), q is successor of r. Thus $q \in S(r) \subseteq S(A)$. Hence, $S(S(A)) \subseteq S(A)$. □

Definition 2.9. *Let $M = (Q,X,Y,\delta,\sigma)$ be a fuzzy Moore machine. Let $T \subseteq Q$. Let δ' and σ' be fuzzy subset of $Q \times X \times Q$ and $Q \times X \times Y$ respectively and let $N = (T,X,Y,\delta',\sigma')$. Then N is called a submachine of M, if (1) $\delta' = \delta|_{T\times X\times T}$ and $\sigma' = \sigma|_{T\times Y}$ and (2) $S(T) \subseteq T$.*

Clearly, if K is a submachine of N and N is a submachine of M, then K is a submachine of M.

Definition 2.10. *Let $M = (Q,X,Y,\delta,\sigma)$ be a fuzzy Moore machine. Then M is called strongly connected, if $p \in S(q)$, $\forall\, p,q \in Q$.*

Definition 2.11. *Let $M_1 = (Q_1,X_1,Y_1,\delta_1,\sigma_1)$ and $M_2 = (Q_2,X_2,Y_2,\delta_2,\sigma_2)$ be a fuzzy Moore Machines. A triplet (f,g,h) of mappings, $f : Q_1 \longrightarrow Q_2$, $g : X_1 \longrightarrow X_2$ and $h : Y_1 \longrightarrow Y_2$, is called fuzzy Moore machine homomorphism from M_1 to M_2, denoted by $(f,g,h) : M_1 \longrightarrow M_2$, if (i) $\delta_1(q_1,x_1,p_1) \leq \delta_2(f(q_1),g(x_1),f(p_1))$ (ii) $\sigma_1^{\#}(q_1,x_1,y_1) \leq \sigma_2^{\#}(f(q_1),g(x_1),h(y_1))$, $\forall\, q_1,\, p_1 \in Q_1, x_1 \in X_1^*$ and $y_1 \in Y_1^*$. Fuzzy Moore machine homomorphism (f,g,h) is called strong homomorphism, if $\delta_2(f(q),\, g(x), f(p)) = \delta_1(q,x,p)$ and $\sigma_2^{\#}(f(q),g(x),h(y)) = \sigma_1^{\#}(q,x,y)$, $\forall\, p,q \in Q_1, x \in X_1^*, y \in Y_1^*$.*

Remark 2.12. *In above definition 2.11, if $X_1 = X_2$, $Y_1 = Y_2$ and g,h are identity maps, then we simply write $f : M_1 \longrightarrow M_2$ and say that f is a homomorphism or strong homomorphism accordingly.*

Theorem 2.13. *Let $(f,g,h) : M_1 \longrightarrow M_2$ be a fuzzy Moore machine homomorphism. Then*

(1) if p is a successor of q in M_1, then $f(p)$ is a successor of $f(q)$ in M_2.

(2) $S(f(q)) = f(S(q)),\ \forall\, q \in Q_1$, if (f,g,h) is strong.

Proof. The proof of (1) is straightforward.

(2) $f(p) \in f(S(q)) \Leftrightarrow p \in S(q) \Leftrightarrow \delta_1^*(q,x,p) \wedge \sigma_1^\#(q,x,y) > 0 \Leftrightarrow \delta_1^*(q,x,p) > 0$ and $\sigma_1^\#(q,x,y) > 0 \Leftrightarrow \delta_2^*(f(q),g(x),f(p)) > 0$ and $\sigma_2^\#(f(q),g(x),h(y)) > 0 \Leftrightarrow \delta_2^*\ (f(q),g(x),f(p)) \wedge \sigma_2^\#(f(q), g(x),h(y)) > 0 \Leftrightarrow f(p) \in S(f(q))$. □

Theorem 2.14. *Let $M_1 = (Q_1,X_1,Y_1,\delta_1,\sigma_1)$ and $M_2 = (Q_2,X_2,Y_2,\delta_2,\sigma_2)$ be a fuzzy Moore Machines and let $(f,g,h) : M_1 \longrightarrow M_2$ be onto homomorphism. If M_1 is strongly connected, then M_2 is strongly connected.*

Proof. Let $q_2,q_2' \in Q_2$. Then $\exists\ q_1,q_1' \in Q_1$ such that $f(q_1) = q_2$ and $f(q_1') = q_2'$. Since M_1 is strongly connected, we have $q_1 \in S(q_1')$. Then $f(q_1) \in f(S(q_1'))$. By Theorem 2.13(2) $f(q_1) \in S(f(q_1'))$, that is $q_2 \in S(q_2')$. Hence, M_2 is strongly connected. □

3. Fuzzy subsystems of fuzzy moore machines

In this section the concept of fuzzy subsystem of fuzzy Moore machine is introduced. Its characterization will be discussed through a fuzzy set defined for fixed strings of input and output. For a fixed state and an element of [0,1] a particular class of fuzzy subsystems will be obtained. Towards the end of the section notions of cyclic and super cyclic fuzzy subsystems will be discussed.

Definition 3.1. *Let $M = (Q,X,Y,\delta,\sigma)$ be a Fuzzy Moore Machines. Let μ be a fuzzy subset of Q. Then μ is called a fuzzy subsystem of M, if $\mu(q) \geq \mu(p) \wedge \delta(p,a,q) \wedge \sigma(p,b),\ \forall\, q,p \in Q, a \in X$ and $b \in Y$.*

If $(Q,X,Y,\delta,\sigma,\mu)$ is a fuzzy subsystem of M, then we shall write μ for $(Q,X,Y,\delta,\sigma,\mu)$.

Theorem 3.2. *Let $M=(Q,X,Y,\delta,\sigma)$ be a fuzzy Moore machine. Then μ is a fuzzy subsystem of M if and only if $\mu(q)\geq\mu(p)\wedge\delta^*(p,x,q)\wedge\sigma^\#(p,x,y),\ \forall\, q,p\in Q,x\in X^*,y\in Y^*$.*

Proof. Suppose μ is a fuzzy subsystem of M. Let $q,p\in Q,x\in X^*$ and $y\in Y^*$. We prove the theorem by mathematical induction on $|x|=|y|=n$. If $n=0$, then $x=y=\lambda$. Now if $q=p$, then $\mu(p)\wedge\delta^*(q,\lambda,q)\wedge\sigma^\#(q,\lambda,\lambda)=\mu(q)$. If $q\neq p$, then $\mu(p)\wedge\delta^*(p,\lambda,q)\wedge\sigma^\#(p,\lambda,\lambda)=0\leq\mu(q)$. Thus, the theorem is true for $n=0$. Assume that the theorem is true for all $u\in X^*$ and $v\in Y^*$ such that $|u|=|v|=n-1,n>1$. Let $x=au$ and $y=bv$ where $a\in X,b\in Y$ and $|u|=|v|=n-1$. Then $\mu(p)\wedge\delta^*(p,x,q)\wedge\sigma^\#(q,x,y)=\mu(p)\wedge\delta^*(p,au,q)\wedge\sigma^\#(q,au,bv)=\mu(p)\wedge\{\bigvee_{r\in Q}[\delta(p,a,r)\wedge\delta^*(r,u,q)]\wedge[\delta(p,a,r)\wedge\sigma(r,b)\wedge\sigma^\#(r,u,v)]\}=\mu(p)\wedge\{\bigvee_{r\in Q}[\delta(p,a,r)\wedge\delta^*(r,u,q)]\wedge[\sigma^\#(p,a,b)\wedge\sigma^\#(r,u,v)]\}=\{\bigvee_{r\in Q}[\mu(p)\wedge\delta(p,a,r)\wedge\sigma^\#(p,a,b)]\wedge[\delta^*(r,u,q)\wedge\sigma^\#(r,u,v)]\}\leq\bigvee_{r\in Q}\{\mu(r)\wedge\delta^*(r,u,q)\wedge\sigma^\#(r,u,v)\}\leq\mu(q)$. Hence, $\mu(q)\geq\mu(p)\wedge\delta^*(p,x,q)\wedge\sigma^\#(p,x,y)$. The converse is trivial. □

The following theorem gives a class of constant fuzzy subsystems for M.

Theorem 3.3. *Every constant fuzzy set μ on Q determines a fuzzy subsystem of M.*

Proof. Suppose μ is constant fuzzy set of Q. Then for any $p,q\in Q$, we have $\mu(p)=\mu(q)$. Then for any $a\in X$ and $b\in Y$, clearly $\mu(q)=\mu(p)\geq\mu(p)\wedge\delta(q,a,p)\wedge\sigma(q,b)$. Therefore, μ is a fuzzy subsystem of M. □

Theorem 3.4. *Let $M=(Q,X,Y,\delta,\sigma)$ be a fuzzy Moore machine. Let μ_1 and μ_2 be fuzzy subsystems of M. Then*

(1) $\mu_1\cap\mu_2$ is a fuzzy subsystem of M and

(2) $\mu_1\cup\mu_2$ is a fuzzy subsystem of M.

Proof. Since μ_1 and μ_2 are fuzzy subsystem of M, for $p,q\in Q,x\in X^*,y\in Y^*$ we have $\mu_1(q)\geq\mu_1(p)\wedge\delta^*(p,x,q)\wedge\sigma^\#(p,x,y)$ and $\mu_2(q)\geq\mu_2(p)\wedge\delta^*(p,x,q)\wedge\sigma^\#(p,x,y)$

1. Therefore, $(\mu_1\cap\mu_2)(q)=\mu_1(q)\wedge\mu_2(q)\geq(\mu_1(p)\wedge\mu_2(p))\wedge\delta^*(p,x,q)\wedge\sigma^\#(p,x,y)$. Hence, $(\mu_1\cap\mu_2)$ is a fuzzy subsystem.

2. Therefore, $(\mu_1 \cup \mu_2)(q) = \mu_1(q) \vee \mu_2(q) \geq (\mu_1(p) \vee \mu_2(p)) \wedge \delta^*(p,x,q) \wedge \sigma^{\#}(p,x,y)$. Hence, $(\mu_1 \cup \mu_2)$ is a fuzzy subsystem. □

The following example show that the complement of a fuzzy subsystem is not always a fuzzy subsystem.

Example 3.5. *Let $Q = \{p,q\}, X = \{a\}, Y = \{b\}, \delta(r,a,s) = \frac{1}{3}\ \forall\, r,s \in Q, \sigma(r,b) = \frac{1}{2}\ \forall\, r \in Q$. Let $\mu(q) = \frac{4}{5}$ and $\mu(p) = \frac{1}{2}$. Then $\mu(q) \geq \mu(p) \wedge \delta(p,a,q) \wedge \sigma(p,b)$ and $\mu(p) \geq \mu(q) \wedge \delta(q,a,p) \wedge \sigma(q,b)$. Then, μ is a fuzzy subsystem, but $\mu^c = 1 - \mu$ is not.*

Theorem 3.6. *Let $M_1 = (Q_1, X_1, Y_1, \delta_1, \sigma_1)$ and $M_2 = (Q_2, X_2, Y_2, \delta_2, \sigma_2)$ be fuzzy Moore machines. Let $(f,g,h) : M_1 \longrightarrow M_2$ be onto strong homomorphism. If μ is a fuzzy subsystem of M_1, then $f(\mu)$ is a fuzzy subsystem of M_2.*

Proof. Let $p_2, q_2 \in Q_2$ and $x_2 \in X_2^*, y_2 \in Y_2^*$. Since f is onto, there exist $p_1, q_1 \in Q_1$ be such that $f(p_1) = p_2$ and $f(q_1) = q_2$. Also, g and h are onto, therefore there exists $x_1 \in X_1^*$ and $y_1 \in Y_1^*$ such that $g(x_1) = x_2$ and $h(y_1) = y_2$. Suppose also that there is $r_1 \in Q_1$ be such that $f(r_1) = p_2$. Then, $\delta_1^*(p_1,x_1,q_1) = \delta_2^*(f(p_1),g(x_1),f(q_1)) = \delta_2^*(f(r_1),g(x_1),f(q_1)) = \delta_1^*(r_1,x_1,q_1)$. Similarly $\sigma_1^{\#}(p_1,x_1,y_1) = \sigma_1^{\#}(r_1,x_1,y_1)$.

Now, $f(\mu)(p_2) \wedge \delta_2^*(p_2,x_2,q_2) \wedge \sigma_2^{\#}(p_2,x_2,y_2)$

$= \bigvee\{\mu(r_1) | f(r_1) = p_2\} \wedge \delta_2^*(p_2,x_2,q_2) \wedge \sigma_2^{\#}(p_2,x_2,y_2)$

$= \bigvee\{\mu(r_1) \wedge \delta_2^*(p_2,x_2,q_2) \wedge \sigma_2^{\#}(p_2,x_2,y_2) | f(r_1) = p_2\}$

$= \bigvee\{\mu(r_1) \wedge \delta_2^*(f(p_1),g(x_1),f(q_1)) \wedge \sigma_2^{\#}(f(p_1),g(x_1),h(y_1)) | f(r_1) = p_2\}$

$= \bigvee\{\mu(r_1) \wedge \delta_1^*(p_1,x_1,q_1) \wedge \sigma_1^{\#}(p_1,x_1,y_1) | f(r_1) = p_2\}$

$= \bigvee\{\mu(r_1) \wedge \delta_1^*(r_1,x_1,q_1) \wedge \sigma_1^{\#}(r_1,x_1,y_1) | f(r_1) = p_2\}$

$\leq \bigvee\{\mu(q_1) | f(r_1) = p_2)\}$, since μ is fuzzy subsystem of M_1

$\leq \bigvee\{f(\mu)(q_2) | f(r_1) = p_2)\}$.

$= f(\mu)(q_2)$.

Therefore, $f(\mu)$ is a fuzzy subsystem of M_2 □

The following example show that the ontoness is necessary for the above theorem.

Example 3.7. *Let $Q_1 = \{p,q\}, Q_2 = \{r,s\}, X = \{a\}, Y = \{b\}, \delta_1(q,a,q) = \delta_1(p,a,p) = \delta_1(p,a,q) = \delta_1(q,a,p) = 1, \sigma_1(t,b) = \frac{1}{2}\ \forall t \in Q.$ and $\delta_2(r,a,s) = \frac{1}{4}, \delta_2(s,a,r) = \frac{1}{7}, \delta_2(r,a,r) = 1 = \delta_2(s,a,s) = 1, \sigma_2(r,b) = \frac{1}{2}, \sigma_2(s,b) = \frac{1}{8}$. Let $f : Q_1 \longrightarrow Q_2$ defined by $f(q) = f(p) = r$. Then f is not onto. Clearly, f is strong homomorphism. Let μ_1 be a fuzzy subset of Q_1 such that $\mu_1(p) = \frac{1}{2}, \mu_1(q) = \frac{2}{3}$. Then μ_1 is fuzzy subsystem of M_1, but $f(\mu)$ is not a fuzzy subsystem of M_2.*

Theorem 3.8. *Let $(f,g,h) : M_1 \longrightarrow M_2$ be a strong homomorphism. If μ is the fuzzy subsystem of M_2. Then $f^{-1}(\mu)$ is a fuzzy subsystem of M_1.*

Proof. Let $M_1 = (Q_1,X_1,Y_1,\delta_1,\sigma_1)$ and $M_2 = (Q_2,X_2,Y_2,\delta_2,\sigma_2)$ be fuzzy Moore machines. Let $p_1,q_1 \in Q_1$ and $x_1 \in X_1^*, y_1 \in Y_1^*$. Then $f(p_1), f(q_1) \in Q_2, g(x_1) \in X_2^*, h(y_1) \in Y_2^*$. Now since μ is fuzzy subsystem of M_2, we have, $\mu(f(p_1)) \geq \mu(f(q_1)) \wedge \delta_2(f(q_1),g(x_1),f(p_1)) \wedge \sigma_2(f(q_1),g(x_1),h(y_1))$. Thus, $\mu(f(p_1)) \geq \mu(f(q_1)) \wedge \delta(q_1,x_1,p_1) \wedge \sigma_1(q_1,x_1,y_1)$. That is, $f^{-1}(\mu)(p_1) \geq f^{-1}(\mu)(q_1) \wedge \delta(q_1,x_1,p_1) \wedge \sigma_1(q_1,x_1,y_1)$. Therefore, $f^{-1}(\mu)$ is a fuzzy subsystem of M_1. □

Theorem 3.9. *Let $M = (Q,X,Y,\delta,\sigma)$ be a fuzzy Moore machine and μ be a fuzzy set of Q. Then*

(1) if μ is fuzzy subsystem of M, then $N = (Supp(\mu),X,Y,\delta',\sigma')$ is a submachine of M, where
$\delta' = \delta|_{Supp(\mu)\times X\times Supp(\mu)}$ and $\sigma' = \sigma|_{Supp(\mu)\times Y}$.

(2) if $N_t = (\mu_t,X,Y,\delta_t,\sigma_t)$ is a submachine of M, where,
$\mu_t = \{q \in Q | \mu(q) \geq t\}$, $\delta_t = \delta|_{\mu_t\times X\times \mu_t}$, and $\sigma_t = \sigma|_{\mu_t\times Y}$, $t \in [0,1]$, then μ is a fuzzy subsystem of M.

Proof. 1. Let $p \in S(Supp(\mu))$. Then $p \in S(q)$, for some $q \in Supp(\mu)$. Then $\mu(q) > 0$. Since $p \in S(q)$, $\exists\ x \in X^*, y \in Y^*$ such that $\delta^*(q,x,p) \wedge \sigma^{\#}(q,x,y) > 0$. μ is fuzzy subsystem, we have $\mu(p) \geq \mu(q) \wedge \delta^*(q,x,p) \wedge \sigma^{\#}(q,x,y) > 0$ Thus, $p \in Supp(\mu)$. Therefore $S(Supp(\mu)) \subseteq Supp(\mu)$. Hence, N is a submachine of M.

2. Let $q,p \in Q, x \in X^*, y \in Y^*$. If $\mu(p) = 0$ or $\delta^*(q,x,p) = 0$ or $\sigma^{\#}(q,x,y) = 0$ then $\mu(q) \geq 0 = \mu(p) \wedge \delta^*(p,x,q) \wedge \sigma^{\#}(p,x,y)$. Suppose, $\mu(p) > 0$, $\delta^*(p,x,q) > 0, \sigma^{\#}(p,x,y) > 0$ and let $\mu(p) \wedge \delta^*(p,x,q) \wedge \sigma^{\#}(p,x,y) = t$. Then $p \in \mu_t$. Since N_t is submachine of M, we have

$S(\mu_t)=\mu_t$. Now, $q\in S(p)$ and $S(p)\subseteq S(\mu_t)$ as $p\in\mu_t$. As $S(\mu_t)=\mu_t$, we have $q\in\mu_t$. Hence, $\mu(q)\geq t=\mu(p)\wedge\delta^*(p,x,q)\wedge\sigma^{\#}(p,x,y)$. Thus, μ is fuzzy subsystem. □

The following example show that a fuzzy subsystem of M need not be a submachine of M

Example 3.10. *Let Q,X,Y,δ,σ be defined in Example 3.5. Let $\mu(q)=\frac{4}{5}$ and $\mu(p)=\frac{1}{2}$. Then μ is a fuzzy subsystem. Let $t=\frac{2}{3}$. Let $N_t=(\mu_t,X,Y,\delta_t,\sigma_t)$. Now $\mu(q)\geq t$. Thus, $q\in\mu_t$. Also $\delta(q,a,p)=\frac{1}{3}>0$ and $\sigma(q,b)=\frac{1}{2}>0$. Thus, $\delta(q,a,p)\wedge\sigma(q,b)>0$. Therefore, $p\in S(q)$. Thus $p\in S(\mu_t)$. But $\mu(p)=\frac{1}{2}<t$. Thus, $p\notin\mu_t$. Hence, N_t is not a submachine of M.*

We now define a fuzzy subset μ of Q to characterize it as a fuzzy subsystem for fixed input and output strings as follows:
Let $M=(Q,X,Y,\delta,\sigma)$ be a fuzzy Moore machine and μ be a fuzzy subset of Q. For $x\in X^*, y\in Y^*$ define a fuzzy subset (μxy) of Q by $(\mu xy)(q)=\bigvee_{p\in Q}\{\mu(p)\wedge\delta^*(p,x,q)\wedge\sigma^{\#}(p,x,y)\}$, $\forall\, q\in Q$.

Theorem 3.11. *Let $M=(Q,X,Y,\delta,\sigma)$ be a fuzzy Moore machine and let μ be a fuzzy subset of Q. Then μ is a fuzzy subsystem of M if and only if $\mu xy\subseteq\mu$, $\quad\forall\, x\in X^*, y\in Y^*$.*

Proof. Let μ be a fuzzy subsystem of M. Let $x\in X^*, y\in Y^*, q\in Q$. Then $(\mu xy)(q)=\bigvee_{p\in Q}\{\mu(p)\wedge\delta^*(p,x,q)\wedge\sigma^{\#}(p,x,y)\}\leq\mu(q)$. Hence, $\mu(xy)\subseteq\mu$.
Conversely, let $q\in Q$ and $x\in X^*, y\in Y^*$. Then
$\mu(q)\geq(\mu xy)(q)=\bigvee_{p\in Q}\{\mu(p)\wedge\delta^*(p,x,q)\wedge\sigma^{\#}(p,x,y)\}\geq\mu(p)\wedge\delta^*(p,x,q)\wedge\sigma^{\#}(p,x,y)$, $\forall\, p\in Q$. Hence, μ is a fuzzy subsystem of M. □

Theorem 3.12. *Let $M=(Q,X,Y,\delta,\sigma)$ be a fuzzy Moore machine. Then for all fuzzy subset μ of Q, $(\mu xy)uv=(\mu xu)yv$, $\forall\, u,x\in X^*, v,y\in Y^*$*

Proof. Let μ be a fuzzy finite subset of Q and let $x,u\in X^*$ and $y,v\in Y^*$. We use induction on $|u|=|v|=n$ to prove the theorem.

Case (i) If $n=0$, then $u=v=\lambda$. Let $q\in Q$. Then
$(\mu xy)\lambda\lambda(q)=\bigvee_{p\in Q}\{(\mu xy)(p)\wedge\delta(p,\lambda,q)\wedge\sigma^{\#}(p,\lambda,\lambda)\}=(\mu xy)(q)$. Hence, $\mu xy\lambda\lambda=(\mu xy)=(\mu x\lambda)y\lambda$.

Case (ii) Suppose, that the theorem is true for all $u \in X^*, v \in Y^*$ such that $|u| = |v| = n-1, n > 1$ and for all μ. Let $u' = au \in X^*$ where $a \in X, u' \in X^*$ and $v' = bv \in Y^*$ where $b \in Y, v \in Y^*$ and $|u| = |v| = n-1$. Let $q \in Q$. Then,

$(\mu xu')yv'(q) = (\mu xau)ybv(q) = (\mu(xa)u)(yb)v(q) = \bigvee_{r\in Q}\{(\mu xayb)(r) \wedge \delta^*(r,u,q) \wedge \sigma^{\#}(r,u,v)$
$\} = \bigvee_{r\in Q}\{\bigvee_{p\in Q}\{(\mu xy)(p) \wedge \delta(p,a,r) \wedge \sigma^{\#}(p,a,b)\} \wedge \delta^*(r,u,q) \wedge \sigma^{\#}(r,u,v)\} = \bigvee_{p\in Q}\{(\mu xy)(p) \wedge$
$\{\bigvee_{r\in Q}\{\delta(p,a,r) \wedge \delta^*(r,u,q)\} \wedge \{\sigma^{\#}(p,a,b) \wedge \{\delta(p,a,r) \wedge \sigma^{\#}(r,u,v)\}\}\}\}$
$= \bigvee_{p\in Q}\{(\mu xy)(p) \wedge \delta^*(p,au,q) \wedge \sigma^{\#}(p,au,bv) = \bigvee_{p\in Q}\{(\mu(xy))(p) \wedge \delta^*(p,u',q) \wedge \sigma^{\#}(p,u',v') =$
$(\mu xy)u'v'(q)$.

Hence, $(\mu xu')yv' = (\mu xy)u'v'$. □

Our aim is now to use the characterization Theorem 3.11 to find a particular class of fuzzy subsystems of M, we begin with classes of fuzzy sets

Definition 3.13. *Let $M = (Q,X,Y,\delta,\sigma)$ be a fuzzy Moore machine and μ be a fuzzy subset of Q. Define fuzzy subsets μXY and μX^*Y^* of Q by*

$(\mu XY)(p) = \bigvee_{a\in X, b\in Y, r\in Q}\{\mu(r) \wedge \delta(r,a,p) \wedge \sigma(r,b)\}$ $\forall\, p \in Q$ *and*

$(\mu X^*Y^*)(p) = \bigvee_{u\in X^*, v\in Y^*, r\in Q}\{\mu(r) \wedge \delta^*(r,u,p) \wedge \sigma^{\#}(r,u,v),\}$ $\forall\, p \in Q$.

Note that

(1) $(\mu XY) \subseteq (\mu X^*Y^*)$,

(2) $(\mu XY) = 0$ and $(\mu X^*Y^*) = 0$ if there exists $r \in Q$ such that $\mu(r) = 0$, and

(3) $(\mu xy) \subseteq (\mu X^*Y^*)$ $\forall\ x \in X^*, y \in Y^*$.

Theorem 3.14. *Let $M = (Q,X,Y,\delta,\sigma)$ be a fuzzy Moore machine $t \in [0,1]$ and $q \in Q$. Then $(q_tXY)(p) = \bigvee_{a\in X, b\in Y}\{t \wedge \delta(q,a,p) \wedge \sigma(q,b)\}$, $\forall\, p \in Q$ and $(q_tX^*Y^*)(p) = \bigvee_{u\in X^*, v\in Y^*}\{t \wedge \delta^*(q,u,p) \wedge \sigma^{\#}(q,u,v)\}$ $\forall\, p \in Q$.*

One can note that for arbitrary fuzzy subset of Q, μX^*Y^* is not necessarily a fuzzy subsystem of M, but for $\mu = q_t$ for any $q \in Q$ and $t \in (0,1]$, $(q_tX^*Y^*)$ is a fuzzy subsystem of M. Thus we have following theorem

Theorem 3.15. *Let $M = (Q,X,Y,\delta,\sigma)$ be a fuzzy Moore machine. Let $t \in (0,1]$ and $q \in Q$. Then the following hold*

(1) $q_tX^*Y^*$ *is a fuzzy subsystem of* M*,*

(2) $Supp(q_tX^*Y^*) = S(q)$*.*

Proof. 1. Let $x \in X^*$ and $y \in Y^*$. Then for any $r \in Q$, we have

$$((q_tX^*Y^*)(xy))(r) = \bigvee_{p\in Q}\{(q_tX^*Y^*)(p) \wedge \delta^*(p,x,r) \wedge \sigma^\#(p,x,y)\} =$$
$$= \bigvee_{p\in Q}\{\bigvee_{u\in X^*,v\in Y^*}\{t \wedge \delta^*(q,u,p) \wedge \sigma^\#(q,u,v)\} \wedge \delta^*(p,x,r) \wedge \sigma^\#(p,x,y)\}$$
$$= \bigvee_{p\in Q,u\in X^*,v\in Y^*}\{t \wedge \delta^*(q,u,p) \wedge \sigma^\#(q,u,v) \wedge \delta^*(p,x,r) \wedge \sigma^\#(p,x,y)\}$$
$$= \bigvee_{p\in Q,u\in X^*,v\in Y^*}\{t \wedge \{\delta^*(q,u,p) \wedge \delta^*(p,x,r)\} \wedge \{\sigma^\#(q,u,v) \wedge \{\delta^*(q,u,p) \wedge \sigma^\#(p,x,y)\}\}$$
$$= \bigvee_{u\in X^*,v\in Y^*}\{t \wedge \delta^*(q,ux,r) \wedge \sigma^\#(q,ux,vy)\}$$
$$\leq \bigvee_{u'\in X^*,v'\in Y^*}\{t \wedge \delta^*(q,u',r) \wedge \sigma^\#(q,u',v')\}$$
$$\leq (q_tX^*Y^*)(r).$$

Thus, $((q_tX^*Y^*)(xy)) \subseteq (q_tX^*Y^*)$. Hence, $(q_tX^*Y^*)$ is a fuzzy subsystem of M, by Theorem(3.11).

2. $p \in S(q) \Leftrightarrow \exists\, x \in X^*, y \in Y^*$ such that $\delta^*(q,x,p) \wedge \sigma^\#(q,x,y) > 0 \Leftrightarrow \bigvee_{x\in X^*,y\in Y^*}\{t \wedge \delta^*(q,x,p) \wedge \sigma^\#(q,x,y)\} > 0 \Leftrightarrow (q_tX^*Y^*)(p) > 0 \Leftrightarrow p \in Supp(q_tX^*Y^*)$. □

Theorem 3.16. *Let* $M = (Q,X,Y,\delta,\sigma)$ *be a fuzzy Moore machine. Let* μ *be a fuzzy subset of* Q *and* $q \in Q$*. Then the following are equivalent*

(1) μ *is a fuzzy subsystem of* M*,*

(2) $q_tX^*Y^* \subseteq \mu$, $\forall\, t \in [0,1]$ *such that* $t \leq \mu(q)$*,*

(3) $q_tXY \subseteq \mu$, $\forall\, q_t \subseteq \mu$, $\forall\, t \in [0,1]$ *such that* $t \leq \mu(q)$*.*

Proof. 1. $\Rightarrow$ 2. Let $q \in Q, t \in [0,1]$ such that $t \leq \mu(q)$. Then for $p \in Q$, we have

$$(q_tX^*Y^*)(p) = \bigvee_{u\in X^*,v\in Y^*}\{t \wedge \delta^*(q,u,p) \wedge \sigma^\#(q,u,v)\} \leq \bigvee_{u\in X^*,v\in Y^*}\{\mu(q) \wedge \delta^*(q,u,p) \wedge \sigma^\#(q,u,v)\}$$
$\leq \mu(p)$, since μ is fuzzy subsystem. Hence, $q_tX^*Y^* \subseteq \mu$.

2. $\Rightarrow$ 3. Clear, due to $q_tXY \subseteq q_tX^*Y^*$.

3. $\Rightarrow$ 1. let $p,q \in Q$ and $a \in X, b \in Y$. If $\mu(q) = 0$ or $\delta(q,a,p) = 0$ or $\sigma(q,b) = 0$ then $\mu(p) \geq 0 = \mu(p) \wedge \delta(q,a,p) \wedge \sigma(q,b)$. Suppose $\mu(q) \neq 0$ and $\delta(q,a,p) \neq 0$ and $\sigma(q,b) \neq 0$. Let $\mu(q) = t$. Thus, by the hypothesis, $q_tXY \subseteq \mu$. Then $\mu(p) \geq (q_tXY)(p) = \bigvee_{u\in X,v\in Y}\{t \wedge \delta(q,u,p) \wedge \sigma(q,v)\} \geq t \wedge \delta(q,a,p) \wedge \sigma(q,b) = \mu(q) \wedge \delta^*(q,a,p) \wedge \sigma(q,b)$. Hence, μ is a fuzzy subsystem of M. □

Corollary 3.17. *Let $M = (Q, X, Y, \delta, \sigma)$ be a fuzzy Moore machine and μ be a fuzzy subsystem of M. Then for any $q \in Q$, we have*

(1) $\mu \supseteq q_{\mu(q)}XY$. and

(2) $\mu \supseteq q_{\mu(q)}X^*Y^*$.

Definition 3.18. *Let $M = (Q, X, Y, \delta, \sigma)$ be a fuzzy Moore machine and μ be a fuzzy subsystem of M. Then μ is called cyclic if $\exists\ q \in Q, t \in (0,1]$ with $t \leq \mu(q)$ such that $\mu \leq q_t X^*Y^*$. In this case we call q_t a generator of μ.*

The Theorem 3.16 enable to characterize cyclic fuzzy subsystems as:

Theorem 3.19. *Let $M = (Q, X, Y, \delta, \sigma)$ be a fuzzy Moore machine. and μ be a fuzzy subsystem of M. Then μ is cyclic if and only if $\exists\ q \in Q$ and $t \in (0,1]$ such that $\mu = q_t X^*Y^*$, whenever $t \leq \mu(q)$.*

Theorem 3.20. *Let $M = (Q, X, Y, \delta, \sigma)$ be a fuzzy Moore machine. Suppose the fuzzy subsystem μ of M is cyclic with generator q_t, $q \in Q$ and $t \in (0,1]$. Then*

(1) $\mu(q) = t$,

(2) $\mu(q) \geq \mu(p)$, $\forall\ p \in Q$,

(3) for any fuzzy subsystem μ' of M such that $\mu' \subseteq \mu$, if $\mu'(q) \geq \mu'(r)$, $\forall\ r \in Q$, we have $\mu' = q_{\mu'(q)}X^*Y^*$.

Proof. 1. Since $\mu = q_t X^*Y^*$, we have $\mu(q) = (q_t X^*Y^*)(q) = \bigvee_{x \in X^*, y \in Y^*} \{t \wedge \delta^*(q,x,q) \wedge \sigma^\#(q,x,y)\}$ $= t \wedge (\bigvee_{x \in X^*, y \in Y^*} \{\delta^*(q,x,q) \wedge \sigma^\#(q,x,y)\}) = t \wedge 1 = t$.
2. Let $p \in Q$. Since $\mu = q_t X^*Y^*$, we have $\mu(p) = (q_t X^*Y^*)(p) = \bigvee_{x \in X^*, y \in Y^*} \{t \wedge \delta^*(q,x,p) \wedge \sigma^\#(q,x,y)\} = \bigvee_{x \in X^*, y \in Y^*} \{\mu(q) \wedge \delta^*(q,x,p) \wedge \sigma^\#(q,x,y)\} = \mu(q) \wedge (\bigvee_{x \in X^*, y \in Y^*} \{\delta^*(q,x,p) \wedge \sigma^\#(q,x,y)\})$ $\leq \mu(q)$.
3. Let $p \in Q$. Since $\mu' \subseteq \mu$ we have $\mu'(p) \leq \mu(p)$. Then $\mu'(p) = \mu'(p) \wedge \mu(p)$. Also since $\mu = q_t X^*Y^*$, $\mu(p) = (q_t X^*Y^*)(p) = \bigvee_{x \in X^*, y \in Y^*} \{t \wedge \delta^*(q,x,p) \wedge \sigma^\#(q,x,y)\} = \bigvee_x \in X, y \in Y$ $\{\mu(q) \wedge \delta^*(q,x,p) \wedge \sigma^\#(q,x,y)\}$. Hence, $\mu'(p) = \mu'(p) \wedge \mu(p) = \bigvee_{x \in X^*,}$ $y \in Y^*$ $\{\mu'(p) \wedge \mu(q) \wedge \delta^*(q,x,p) \wedge \sigma^\#(q,x,y)\} = \bigvee_x \in X^*, y \in Y^*$ $\{\mu'(p) \wedge \delta^*(q,x,p) \wedge \sigma^\#(q,x,y)\}$, since

$\mu'(p) \leq \mu'(q) \leq \mu(q) \leq \bigvee_x \in X^*, y \in Y^* \{\mu'(q) \wedge \delta^*(q,x,p) \wedge \sigma^\#(q,x,y)\} = (q_{\mu'(q)}X^*\,Y^*)(p)$. Hence $\mu' \subseteq q_{\mu'(q)}X^*Y^*$. Thus, $\mu' = q_{\mu'(q)}X^*Y^*$, by above corollary. □

Definition 3.21. *Let $M = (Q,X,Y,\delta,\sigma)$ be a fuzzy Moore machine and μ a fuzzy subsystem of M. Then μ is called super cyclic, if $q_{\mu(q)}$ is its generator $\forall\, q \in Q$.*

Theorem 3.22. *Let $M = (Q,X,Y,\delta,\sigma)$ be a fuzzy Moore machine and μ a fuzzy subsystem of M. Then μ is called super cyclic if and only if $\mu = q_{\mu(q)}X^*Y^*, \quad \forall\, q \in Q$.*

Theorem 3.23. *If μ is super cyclic, then μ is constant.*

Proof. Since μ is super cyclic, for any $p \in Q$ we have $\mu = p_{\mu(p)}X^*Y^*$. Also, we have $\mu(p) \geq \mu(r)$, $\forall\, r \in Q$. This implies that $\mu(p) = \mu(r)$, $\forall\, p,r \in Q$. Therefore, μ is constant. □

Corollary 3.24. *Every super cyclic fuzzy subsystem of a fuzzy Moore machine M is cyclic.*

The following example show that a constant fuzzy subsystem μ of M need not be (super) cyclic fuzzy subsystem.

Example 3.25. *Let $Q = \{p,q\}$, $X = \{a\}$, $Y = \{b\}$, $\delta(q,a,q) = \delta(p,a,p) = \frac{1}{2}$, $\delta(p,a,q) = \delta(q,a,p) = \frac{1}{3}$, $\sigma(r,b) = 1 \ \ \forall\ \ r \in Q$. Let $\mu(q) = \mu(p) = \frac{3}{4}$. Then $\mu(q) \geq \mu(p) \wedge \delta(p,a,q) \wedge \sigma(p,b)$ and $\mu(p) \geq \mu(q) \wedge \delta(q,a,p) \wedge \sigma(q,b)$. Hence, μ is a fuzzy subsystem and μ is constant. Now,*

$(q_1X^*Y^*)(p) = \bigvee_{x\in X, y\in Y} \{1 \wedge \delta^*(q,x,p) \wedge \sigma^\#(q,x,y)\} = \frac{1}{3} < \frac{3}{4} = \mu(p)$. *Therefore, μ is not cyclic.*

Theorem 3.26. *Let $M = (Q,X,Y,\delta,\sigma)$ be a fuzzy Moore machine and μ be a fuzzy subsystem of M. Suppose $Supp(\mu) = Q$. If μ is super cyclic, then M is strongly connected.*

Proof. Let $p,q \in Q$. Then $(q_{\mu(q)}X^*Y^*)(p) = \bigvee_{x\in X, y\in Y} \{\mu(q) \wedge \delta^*(q,x,p) \wedge \sigma^\#(q,x,y)\ \} > 0$, since μ is super cyclic $\mu = (q_{\mu(q)}X^*Y^*)$ and $Supp(\mu) = Q$. Hence, $\delta^*(q,x,p) \wedge \sigma^\#(q,x,y)\} > 0$, for some $x \in X^*, y \in Y^*$. Thus $p \in S(q)$. Hence, M is strongly connected. □

Theorem 3.27. *Let $M = (Q,X,Y,\delta,\sigma)$ be a fuzzy Moore machine and μ a fuzzy subsystem of M. Then μ is super cyclic if and only if $\forall\, p,q \in Q$, $\exists\ \ x \in X^*, y \in Y^*$ such that $\delta^*(p,x,q) \wedge \sigma^\#(p,x,y) \geq \mu(p)$.*

Proof. Suppose that μ is super cyclic. Then μ is constant by Theorem (3.23). Suppose $\exists\ p,q \in Q$, $\forall\, x \in X^*, y \in Y^*, \delta^*(p,x,q) \wedge \sigma^{\#}(p,x,y) < \mu(p)$. Then

$$(p_{\mu(p)}X^*Y^*)(q) = \bigvee_{x \in X, y \in Y} \{\mu(p) \wedge \delta^*(p,x,q) \wedge \sigma^{\#}(p,x,y)\} < \mu(p).$$

Thus, $p_{\mu(p)}X^*Y^* \neq \mu$. which is contradiction to μ is super cyclic. Conversely, Suppose that $\forall\ p,q \in Q,\ \exists\ x \in X^*, y \in Y^*$ such that $\delta^*(p,x,q) \wedge \sigma^{\#}(p,x,y) \geq \mu(p)$. Then $\forall\ p,q \in Q,\ \exists\ x \in X^*, y \in Y^*$ such that $\mu(q) \geq \mu(p) \wedge \delta^*(p,x,q) \wedge \sigma^{\#}(p,x,y) = \mu(p)$. Similarly $\mu(p) \geq \mu(q)$. Hence, μ is constant. Now,
$(p_{\mu(p)}X^*Y^*)(q) = \bigvee_{x \in X, y \in Y} \{\mu(p) \wedge \delta^*(p,x,q) \wedge \sigma^{\#}(p,x,y)\} = \mu(p) = \mu(q)$. Thus, $p_{\mu(p)}X^*Y^*$ $= \mu$. Hence, μ is super cyclic. □

4. Output fuzzy subsystems of fuzzy Moore machines

In this section we introduce output fuzzy subsystem of a fuzzy Moore machine and show that it is more specific than the fuzzy subsystem defined in previous section. Moreover it satisfies all the results of fuzzy subsystems. We introduce product of output fuzzy subsystems and prove that it is actually output fuzzy subsystem of various products of fuzzy Moore machines.

Definition 4.1. *Let $M = (Q,X,Y,\delta,\sigma)$ be a Fuzzy Moore Machines. Let μ be a fuzzy subset of Q. Then $(Q,X,Y,\delta,\sigma,\mu)$ is called a output fuzzy subsystem of M, if $\mu(q) \geq \mu(p) \wedge \sigma(p,x,y)$ whenever $\delta(p,x,q) > 0$, for all $q,p \in Q, x \in X^*, y \in Y^*$.*

As before, if $(Q,X,Y,\delta,\sigma,\mu)$ is a output fuzzy subsystem of M, then we shall write μ for $(Q,X,Y,\delta,\sigma,\mu)$. Note that constant fuzzy set μ is an output fuzzy subsystem of M.
The following theorem established the relation between output fuzzy subsystem of M and fuzzy subsystem of M.

Theorem 4.2. *Let $M = (Q,X,Y,\delta,\sigma)$ be a fuzzy Moore machine and let μ be a fuzzy subset of Q. If μ is a output fuzzy subsystem of M, then μ is a fuzzy subsystem of M.*

Proof. Since μ is output fuzzy subsystem of M, we have $\mu(q) \geq \mu(p) \wedge \sigma(p,x,y)$ whenever $\delta(p,x,q) > 0$, for $q,p \in Q, x \in X^*, y \in Y^*$. Obviously $\mu(q) \geq \mu(p) \wedge \delta(p,x,q) \wedge \sigma(p,x,y)$. □

The following example rules out the possibility of the converse of the above theorem.

Example 4.3. *Let $Q=\{p,q\}, X=\{a\}, Y=\{b\}, \delta(r,a,s)=\frac{1}{3}\ \forall\, r,s\in Q, \sigma^{\#}(r,x,y)=0.9\ \forall\ r\in Q, x\in X^*, y\in Y^*$. Let $\mu(q)=0.8$ and $\mu(p)=0.9$. Then $\mu(q)\geq\mu(p)\wedge\delta(p,a,q)\wedge\sigma(p,b)$ and $\mu(p)\geq\mu(q)\wedge\delta(q,a,p)\wedge\sigma(q,b)$. Thus, μ is a fuzzy subsystem. Now, $\delta(p,x,q)=\frac{1}{3}>0$, but $\mu(q)=0.8\ngeq\mu(p)\wedge\sigma^{\#}(p,x,y)=0.9$. Hence, μ is not a output fuzzy subsystem.*

Theorem 4.4. *Let $M=(Q,X,Y,\delta,\sigma)$ be a fuzzy Moore machine. Let μ_1 and μ_2 be output fuzzy subsystems of M. Then*

(1) $\mu_1\cap\mu_2$ is a output fuzzy subsystem of M.

(2) $\mu_1\cup\mu_2$ is a output fuzzy subsystem of M.

Proof. Since, μ_1 and μ_2 are an output fuzzy subsystems of M_1 and M_2, for $p,q\in Q\ \exists\ x\in X^*, y\in Y^*$ and $\delta(p,x,q)>0$. We have, $\mu_1(q)\geq\mu_1(p)\wedge\sigma^{\#}(p,x,y)$ and $\mu_2(q)\geq\mu_2(p)\wedge\sigma^{\#}(p,x,y)$.
1. Hence, $(\mu_1\cap\mu_2)(q)=\mu_1(q)\wedge\mu_2(q)\geq\mu_1(p)\wedge\mu_2(p)\wedge\sigma^{\#}(p,x,y)$, which means that $(\mu_1\cap\mu_2)$ is an output fuzzy subsystem.
2. Hence, $(\mu_1\cup\mu_2)(q)=\mu_1(q)\vee\mu_2(q)\geq\mu_1(p)\vee\mu_2(p)\wedge\sigma^{\#}(p,x,y)$, which means that $(\mu_1\cup\mu_2)$ is an output fuzzy subsystem. □

Definition 4.5. *Let μ_1 and μ_2 be two fuzzy subset of Q_1 and Q_2 respectively. Define $\mu_1\times\mu_2: Q_1\times Q_2\longrightarrow[0,1]$ by $(\mu_1\times\mu_2)(q_1,q_2)=\mu_1(q_1)\wedge\mu_2(q_2)$, $\forall\,(q_1,q_2)\in(Q_1\times Q_2)$. This $\mu_1\times\mu_2$ is called the cartesian product of μ_1 and μ_2.*

We now keep a goal to show that the product of two output fuzzy subsystems is an output fuzzy subsystem. Clearly if both are output fuzzy subsystems from the same fuzzy Moore machine, then the product, which is actually the intersection, is an output fuzzy subsystem, by Theorem 4.4 (1). The problem arises only when output fuzzy subsystems are from different fuzzy Moore machines. To analyze the problem, we define various products of fuzzy Moore machines and discuss that the product of output fuzzy subsystems are actually an output fuzzy subsystem of those products. We begin with definitions of products of fuzzy Moore machines.

Definition 4.6. *Let $M_1=(Q_1,X_1,Y_1,\delta_1,\sigma_1)$ and $M_2=(Q_2,X_1,Y_1,\delta_2,\sigma_2)$ be fuzzy Moore machines. Then the machine $M_1\odot M_2=(Q,X,Y,\delta_1\odot\delta_2,\sigma_1\odot\sigma_2)$ is called*

(1) restricted direct product of M_1 and M_2, symbolically represented as $M_1 \odot_\wedge M_2$, if $Q = Q_1 \times Q_2, X = X_1 = X_2, Y = Y_1 = Y_2,\ \delta_1 \odot \delta_2((q_1,q_2), a, (p_1,p_2)) = \delta_1(q_1,a,p_1) \wedge \delta_2(q_2,a,p_2)$ *and* $\sigma_1 \odot \sigma_2((q_1,q_2), b) = \sigma_1(q_1,b) \wedge \sigma_2(q_2,b)\ \ \forall\ (q_1,q_2),(p_1,p_2) \in (Q_1 \times Q_2),\ a \in X,\ b \in Y.$

(2) full direct product of M_1 and M_2, symbolically represented as $M_1 \odot_\times M_2$, if $Q = Q_1 \times Q_2, X = X_1 \times X_2, Y = Y_1 \times Y_2,\ \delta_1 \odot \delta_2((q_1,q_2),(a_1,a_2),(p_1,p_2)) = \delta_1(q_1,a_1,p_1) \wedge \delta_2(q_2,a_2,p_2)$ *and* $\sigma_1 \odot \sigma_2((q_1,q_2),(b_1,b_2)) = \sigma_1(q_1,b_1) \wedge \sigma_2(q_2,b_2)\ \forall (q_1,q_2),(p_1,p_2) \in (Q_1 \times Q_2),(a_1,a_2) \in (X_1 \times X_2),(b_1,b_2) \in (Y_1 \times Y_2).$

Remark 4.7. *Restricted direct product of fuzzy Moore machines is a particular case of their full direct product, when the set of all inputs and outputs are respectively same in each machines under diagonal mapping.*

Theorem 4.8. *Let* $M_1 = (Q_1, X, Y, \delta_1, \sigma_1)$ *and* $M_2 = (Q_2, X, Y, \delta_2, \sigma_2)$ *be fuzzy Moore machines. Then*

(1) $M_1 \odot_\wedge M_2$ *is restricted direct product of M_1 and M_2 if and only if* $(\delta_1 \odot \delta_2)^*((q_1,q_2),x,(p_1,p_2)) = \delta_1^*(q_1,x,p_1) \wedge \delta_2^*(q_2,x,p_2)$ *and* $(\sigma_1 \odot \sigma_2)^\#((q_1,q_2),x,\ y) = \sigma_1^\#(q_1,x,y) \wedge \sigma_2^\#(q_2,x,y)\ \ \forall\ (q_1,q_2),(p_1,\ p_2) \in (Q_1 \times Q_2), x \in X^*, y \in Y^*.$

(2) $M_1 \odot_\times M_2$ *full direct product of M_1 and M_2, if and only if* $(\delta_1 \odot \delta_2)^*((q_1,q_2),\ (x_1,x_2),(p_1,p_2)) = \delta_1^*(q_1,x_1,p_1) \wedge \delta_2^*(q_2,x_2,p_2)$ *and* $(\sigma_1 \odot \sigma_2)^\#((q_1,q_2),(x_1,x_2),\ (y_1,y_2)) = \sigma_1^\#(q_1,x_1,y_1) \wedge \sigma_2^\#(q_2,x_2,y_2)\ \ \ \forall\ (q_1,q_2),(p_1,p_2) \in (Q_1 \times Q_2),\ (x_1,x_2) \in (X_1^* \times X_2^*),\ (y_1,y_2) \in (Y_1^* \times Y_2^*).$

Proof. Proofs of $(\delta_1 \odot \delta_2)^*$ of both the cases (1) and (2) can be found in [4, 7]

1. Let $(q_1,q_2) \in (Q_1 \times Q_2), x \in X^*, y \in Y^*$. We prove the theorem by mathematical induction on $|x| = |y| = n$.

Case (i) If $n = 0$, then $x = \lambda$ and $y = \lambda$. Clearly by definition,
$(\sigma_1 \odot \sigma_2)^\#((q_1,q_2),\lambda,\lambda)) = 1 = \sigma_1^\#(q_1,\lambda,\lambda) \wedge \sigma_2^\#(q_2,\lambda,\lambda)$. Thus, the theorem is true for n = 0.

Case (ii) Suppose that the theorem is true for $\forall\, u \in X^*, v \in Y^*$ such that $|u| = |v| = n-1, n > 1$. Let $x = au$ and $y = bv$, where $a \in X$ and $b \in Y$ and $|u| = |v| = n-1$. Then,

$(\sigma_1 \odot \sigma_2)^{\#}((q_1,q_2),x,y) = (\sigma_1 \odot \sigma_2)^{\#}((q_1,q_2),au,bv) =$
$= \bigvee\{(\delta_1 \odot \delta_2)((q_1,q_2),a,(r_1,r_2)) \wedge (\sigma_1 \odot \sigma_2)((r_1,r_2),b) \wedge (\sigma_1 \odot \sigma_2)^{\#}((r_1,r_2),u,v) \mid (r_1,r_2) \in (Q_1 \times Q_2)\} = \bigvee\{[\delta_1(q_1,a,r_1) \wedge \delta_2(q_2,a,r_2)] \wedge [\sigma_1(r_1,b) \wedge \sigma_2(r_2,b)] \wedge [\sigma_1^{\#}(r_1,u,v) \wedge \sigma_2^{\#}(r_2,u,v)] | r_1 \in Q_1, r_2 \in Q_2\} = \bigvee\{\delta_1(q_1,a,r_1) \wedge \sigma_1(r_1,b) \wedge \sigma_1^{\#}(r_1,u,v) | r_1 \in Q_1\} \wedge \bigvee\{\delta_2(q_2,a,r_2) \wedge \sigma_2(r_2,b) \wedge \sigma_2^{\#}(r_2,u,v) | r_2 \in Q_2\} = \sigma_1^{\#}(q_1,au,bv) \wedge \sigma_2^{\#}(q_2,au,bv) = \sigma_1^{\#}(q_1,x,y) \wedge \sigma_2^{\#}(q_2,x,y)$.

2. Let $(q_1,q_2) \in (Q_1 \times Q_2), (x_1,x_2) \in (X_1^* \times X_2^*), (y_1,y_2) \in (Y_1^* \times Y_2^*)$. We prove the theorem by mathematical induction on $|x_i| = |y_i| = n$ for $i = 1,2$.

Case (i) If $n = 0$, then $x_1 = x_2 = \lambda$ and $y_1 = y_2 = \lambda$. Clearly by definition $(\sigma_1 \odot \sigma_2)^{\#}((q_1,q_2),(\lambda,\lambda),(\lambda,\lambda)) = 1 = \sigma_1^{\#}(q_1,\lambda,\lambda) \wedge \sigma_2^{\#}(q_2,\lambda,\lambda)$. Thus, the theorem is true for n = 0.

Case (ii) Suppose that the theorem is true for $\forall\, u_1,u_2 \in X^*, v_1,v_2 \in Y^*$ such that $|u_i| = |v_i| = n-1, n > 1$ for $i = 1,2$. Let $x_1 = a_1u_1, x_2 = a_2u_2$ and $y_1 = b_1v_1, y_2 = b_2v_2$, where $a_1 \in X_1, a_2 \in X_2$, $b_1 \in Y_1, b_2 \in Y_2$, and $|u_i| = |v_i| = n-1$, for $i = 1,2$. Then,
$(\sigma_1 \odot \sigma_2)^{\#}((q_1,q_2),(x_1,x_2),(y_1,y_2)) = (\sigma_1 \odot \sigma_2)^{\#}((q_1,q_2),(a_1u_1,a_2u_2),(b_1v_1,b_2v_2)) =$
$= \bigvee\{(\delta_1 \odot \delta_2)((q_1,q_2),(a_1,a_2),(r_1,r_2)) \wedge (\sigma_1 \odot \sigma_2)((r_1,r_2),(b_1,b_2))$
$\wedge(\sigma_1 \odot \sigma_2)^{\#}((r_1,r_2),(u_1,u_2),(v_1,v_2)) | (r_1,r_2) \in (Q_1 \times Q_2)\} = \bigvee\{[\delta_1(q_1,a_1,r_1) \wedge \delta_2(q_2,a_2,r_2)] \wedge [\sigma_1(r_1,b_1) \wedge \sigma_2(r_2,b_2)] \wedge [\sigma_1^{\#}(r_1,u_1,v_1) \wedge \sigma_2^{\#}(r_2,u_2,v_2)] | r_1 \in Q_1, r_2 \in Q_2\} = \bigvee\{\delta_1(q_1,a_1,r_1) \wedge \sigma_1(r_1,b_1) \wedge \sigma_1^{\#}(r_1,u_1,v_1) | r_1 \in Q_1\} \wedge \bigvee\{\delta_2(q_2,a_2,r_2) \wedge \sigma_2(r_2,b_2) \wedge \sigma_2^{\#}(r_2,u_2,v_2) | r_2 \in Q_2\} = \sigma_1^{\#}(q_1,a_1u_1,b_1v_1) \wedge \sigma_2^{\#}(q_2,a_2u_2,b_2v_2) = \sigma_1^{\#}(q_1,x_1,y_1) \wedge \sigma_2^{\#}(q_2,x_2,y_2)$. □

The following theorem show that $\mu_1 \times \mu_2$ is an output fuzzy subsystem of each of the above products of fuzzy Moore machines.

Theorem 4.9. *Let $M_i = (Q_i, X_i, Y_i, \delta_i, \sigma_i)$ be a fuzzy Moore machines, $i = 1,2$. Let μ_1 and μ_2 be an output fuzzy subsystems of M_1 and M_2 respectively. Then $\mu_1 \times \mu_2$ is a an output fuzzy subsystem of fuzzy Moore machine $M_1 \odot_{\wedge} M_2$ and $M_1 \odot_{\times} M_2$.*

Proof. 1. Let $(q_1,q_2),(p_1,p_2) \in (Q_1 \times Q_2), x \in X^*$ and $y \in Y^*$. Let $(\delta_1 \odot \delta_2)^*$ $((q_1,q_2),x,(p_1,p_2)) > 0$. Then $\delta_1^*(q_1,x,p_1)) > 0$ and $\delta_2^*(q_2,x,p_2)) > 0$. Now, $(\mu_1 \times \mu_2)$ $((q_1,q_2))$ $\wedge(\sigma_1 \wedge \sigma_2)^{\#}((q_1,q_2),x,y) = (\mu_1(q_1) \wedge \mu_2(q_2)) \wedge (\sigma_1^{\#}(q_1,x,y) \wedge \sigma_2^{\#}(q_2,x,y)) = [\mu_1(q_1) \wedge \sigma_1^{\#}(q_1,x,y)] \wedge [\mu_2(q_2)\ \wedge\sigma_2^{\#}(q_2,x,y)] \leq \mu_1(p_1)\ \wedge\mu_1(p_2)$. Hence, $\mu_1 \odot_{\wedge} \mu_2$ is output fuzzy subsystems. 2.

Let $(q_1,q_2),(p_1,p_2) \in (Q_1 \times Q_2),(x_1,x_2) \in (X_1 \times X_2)^*$ and $(y_1,y_2) \in (Y_1 \times Y_2)^*$. Let $(\delta_1 \odot \delta_2)^*((q_1,q_2),(x_1,x_2),(p_1,p_2)) > 0$. Then $\delta_1^*(q_1,x_1,p_1)) > 0$ and $\delta_2^*(q_2,x_2,p_2)) > 0$. Now, $(\mu_1 \times \mu_2)$ $((q_1,q_2))$ $\wedge(\sigma_1 \odot \sigma_2)^{\#}$ $((q_1,q_2),$ $(x_1,x_2),$ $(y_1,y_2))$ = $(\mu_1(q_1) \wedge \mu_2(q_2)) \wedge (\sigma_1^{\#}(q_1,x_1, y_1) \wedge \sigma_2^{\#}(q_2,x_2,y_2)) = [\mu_1(q_1) \wedge \sigma_1^{\#}(q_1,x_1,y_1)] \wedge [\mu_2(q_2) \wedge \sigma_2^{\#}(q_2,x_2,y_2)] \leq \mu_1(p_1) \wedge \mu_2(p_2)$. Hence, $\mu_1 \odot_{\times} \mu_2$ is output fuzzy subsystems. □

We now show that $\mu_1 \times \mu_2$ is an output fuzzy subsystem of the cascade product $M_1 \odot_{\omega} M_2$ and the wreath product $M_1 \odot_{\circ} M_2$ in two different approaches. In the first approach, $M_1 \odot_{\omega} M_2$ and $M_1 \odot_{\circ} M_2$ are defined analogous to the definitions of $M_1 \odot_{\times} M_2$ and $M_1 \odot_{\wedge} M_2$. In these cases we have $\omega = (\omega_1, \omega_2)$, where ω_1, ω_2 are crisp functions. Input and output sets of $M_1 \odot_{\circ} M_2$ are respectively $X_1^{Q_2} \times X_2$ and $Y_1^{Q_2} \times Y_2$. In order to show that $\mu_1 \times \mu_2$ is an fuzzy subsystem of $M_1 \odot_{\omega} M_2$ and $M_1 \odot_{\circ} M_2$, we have to use the concept of separable function. The separability of a function was introduced by Malik, Mordeson and Sen in [11]. In our opinion this idea of separability of functions is *not natural*, even though it helps in proving $\mu_1 \times \mu_2$ is an fuzzy subsystem of $M_1 \odot_{\omega} M_2$ and $M_1 \odot_{\circ} M_2$. (see Theorem 4.11). However, in the second approach, we redefine $M_1 \odot_{\omega} M_2$ by extending ω_1 and ω_2 as fuzzy sets rather than crisp functions and we will obtain natural extension of ω_1 and ω_2. These extensions will helps in avoiding unnatural separability concept for proving $\mu_1 \times \mu_2$ is an output fuzzy subsystem of $M_1 \odot_{\omega} M_2$. (see Theorem 4.18). Similarly, considering input and output sets of $M_1 \odot_{\circ} M_2$ as combination of set of fuzzy sets with X_2 and Y_2, we will obtain natural extension of $M_1 \odot_{\circ} M_2$. This will help us in showing $\mu_1 \times \mu_2$ is fuzzy subsystem of $M_1 \odot_{\circ} M_2$, without using separability concept. (see Theorem 4.18).

We begin with first approached of defining $M_1 \odot_{\omega} M_2$, $M_1 \odot_{\circ} M_2$ and proving $\mu_1 \times \mu_2$ is an output fuzzy subsystem of $M_1 \odot_{\omega} M_2$ and $M_1 \odot_{\circ} M_2$ with the help of separability of functions.

Definition 4.10. *Let $M_1 = (Q_1, X, Y, \delta_1, \sigma_1)$ and $M_2 = (Q_2, X, Y, \delta_2, \sigma_2)$ be fuzzy Moore machines. Then the machine $M_1 \odot M_2 = (Q, X, Y, \delta_1 \odot \delta_2, \sigma_1 \odot \sigma_2)$ is called*

(1) cascade product of M_1 and M_2, symbolically represented as $M_1 \odot_{\omega} M_2$, if $Q = Q_1 \times Q_2, X = X_2, Y = Y_2$, $\delta_1 \odot \delta_2((q_1,q_2),a_2,(p_1,p_2)) = \delta_1(q_1,\omega_1(q_2,a_2),p_1) \wedge \delta_2(q_2,a_2,p_2)$ and $\sigma_1 \odot \sigma_2((q_1,q_2),b_2) = \sigma_1(q_1,\omega_2(q_2,b_2)) \wedge \sigma_2(q_2,b_2)$ $\forall\,(q_1,q_2),(p_1,p_2) \in (Q_1 \times Q_2), a_2 \in X_2, b_2 \in Y_2$ and $\omega_1 : Q_2 \times X_2 \longrightarrow X_1, \omega_2 : Q_2 \times Y_2 \longrightarrow Y_1$.

(2) wreath product of M_1 and M_2, symbolically represented as $M_1 \odot_\circ M_2$, if $Q = Q_1 \times Q_2, X = X_1^{Q_2} \times X_2, Y = Y_1^{Q_2} \times Y_2$, $\delta_1 \odot \delta_2((q_1,q_2),(f,a_2),(p_1,p_2)) = \delta_1(q_1,f(q_2),p_1) \wedge \delta_2(q_2,a_2,p_2)$ and $\sigma_1 \odot \sigma_2((q_1,q_2),(g,b_2)) = \sigma_1(q_1,g(q_2)) \wedge \sigma_2(q_2,b_2)\ \forall (q_1,q_2),(p_1,p_2) \in (Q_1 \times Q_2)$, $a_2 \in X_2$, $b_2 \in Y_2$, and $X_1^{Q_2} = \{f : Q_2 \longrightarrow X_1\}$ and $Y_1^{Q_2} = \{g : Q_2 \longrightarrow Y_1\}$.

We, now define separable functions $\delta_1 \odot \delta_2$ and $\sigma_1 \odot \sigma_2$ in both the products $M_1 \odot_\omega M_2$ and $M_1 \odot_\circ M_2$ as follows:

The functions, $\delta_1 \odot \delta_2$ and $\sigma_1 \odot \sigma_2$ of $M_1 \odot_\omega M_2$ are called separable, if $\forall (q_1,q_2),(p_1,p_2) \in (Q_1 \times Q_2), x_2 = x_{21}x_{22}x_{23} \ldots x_{2n} \in X_2, y_2 = y_{21}y_{22}y_{23} \ldots y_{2n} \in Y_2$, $(\delta_1\ \omega\ \delta_2)^*((q_1,q_2),x_2,(p_1,p_2)) = \delta_1^*(q_1,\omega_1(q_2,x_{21})\ \omega_1(q_2^{(1)},x_{22})\ \omega_1(q_2^{(2)},x_{23}) \ldots \omega_1(q_2^{(n-1)},x_{2n}),p_1) \wedge \delta_2^*(q_2,x_2,p_2)$ and $(\sigma_1\ \omega\ \sigma_2)^{\#}((q_1,q_2),x_2,y_2)) = \sigma_1^{\#}(q_1,\omega_1(q_2,x_{21})\ \omega_1(q_2^{(1)},x_{22})\omega_1(q_2^{(2)},x_{23}) \ldots \omega_1(q_2^{(n-1)},x_{2n}), \omega_2(q_2,y_{21})\ \omega_2(q_2^{(1)},y_{22})\ \omega_2(q_2^{(2)},y_{23}) \ldots \omega_2(q_2^{(n-1)},y_{2n})) \wedge \sigma_2^{\#}(q_2,x_2,y_2)$ for some $q_2^{(i)} \in Q_2$, $i = 1,2,3,\ldots,n-1$.

The functions, $\delta_1 \odot \delta_2$ and $\sigma_1 \odot \sigma_2$ of $M_1 \odot_\circ M_2$ are called separable, if $\forall (q_1,q_2),(p_1,p_2) \in (Q_1 \times Q_2)$ and $\forall (f_1,x_{21}),(f_2,x_{22}),(f_3,x_{23}),\ldots,(f_n,x_{2n}) \in (X_1^{Q_2} \times X_2)$ and $\forall (g_1,y_{21}),(g_2,y_{22}),(g_3,y_{23}),\ldots,(g_n,y_{2n}) \in (Y_1^{Q_2} \times Y_2)$, $(\delta_1 \odot \delta_2)^*((q_1,q_2),(f_1,x_{21})\ (f_2,x_{22}) \ldots (f_n,x_{2n}),(p_1,p_2)) = \delta_1^*(q_1,f_1(q_2)\ f_2(q_2^{(1)}) \ldots f_n(q_2^{(n-1)}),p_1) \wedge \delta_2^*(q_2,x_{21}\ x_{22} \ldots x_{2n},p_2)$ and $(\sigma_1 \odot \sigma_2)^{\#}((q_1,q_2),(f_1,x_{21})\ (f_2,x_{22}) \ldots (f_n,x_{2n}),(g_1,y_{21})\ (f_2,y_{22}) \ldots (g_n,y_{2n})) = \sigma_1^{\#}(q_1,f_1(q_2)\ f_2(q_2^{(1)}) \ldots f_n(q_2^{(n-1)}), g_1(q_2)\ g_2(q_2^{(1)}) \ldots g_n(q_2^{(n-1)})) \wedge \sigma_2^{\#}(q_2,x_{21}x_{22} \ldots x_{2n}, y_{21}y_{22} \ldots y_{2n})$ for some $q_2^{(i)} \in Q_2$, $i = 1,2,3,\ldots,n-1$.

Theorem 4.11. *Let $M_1 = (Q_1,X,Y,\delta_1,\sigma_1)$ and $M_2 = (Q_2,X,Y,\delta_2,\sigma_2)$ be fuzzy Moore machines with $\delta_1 \odot \delta_2$ and $\sigma_1 \odot \sigma_2$ are separable functions in the products $M_1 \odot_\omega M_2$ and $M_1 \odot_\circ M_2$. Then*

(1) $M_1 \odot_\omega M_2$ cascade product of M_1 and M_2, if and only if $(\delta_1 \odot \delta_2)^*((q_1,q_2),x_2,(p_1,p_2)) = \delta_1^*(q_1,\omega_1(q_2,x_2),p_1) \wedge \delta_2^*(q_2,x_2,p_2)$ and $(\sigma_1 \odot \sigma_2)^{\#}((q_1,q_2),x_2,y_2) = \sigma_1^{\#}(q_1,\omega_1(q_2,x_2),\omega_2(q_2,y_2)) \wedge \sigma_2^{\#}(q_2,x_2,y_2)\ \ \forall (q_1,q_2),(p_1,p_2) \in (Q_1 \times Q_2), x_2 \in X_2^*, y_2 \in Y_2^*$ and $\omega_1 : Q_2 \times X_2^* \longrightarrow X_1^*, \omega_2 : Q_2 \times Y_2^* \longrightarrow Y_1^*$.

(2) $M_1 \odot_\circ M_2$ wreath product of M_1 and M_2, if and only if $(\delta_1 \odot \delta_2)^*((q_1,q_2),(f,x_2),(p_1,p_2)) = \delta_1^*(q_1,f(q_2),p_1) \wedge \delta_2^*(q_2,x_2,p_2)$ and $(\sigma_1 \odot \sigma_2)^{\#}((q_1,q_2),(f,x_2),(g,y_2)) =$

$\sigma_1^{\#}(q_1, f(q_2),\, g(q_2)) \wedge \sigma_2^{\#}(q_2, x_2, y_2) \quad \forall\, (q_1,q_2),(p_1,p_2) \in (Q_1 \times Q_2), x_2 \in X_2^*, y_2 \in Y_2^*,$ *and* $X_1^{Q_2} = \{f : Q_2 \longrightarrow X_1^*\}$ *and* $Y_1^{Q_2} = \{g : Q_2 \longrightarrow Y_1^*\}$.

Proof. Proofs of $\delta_1 \odot \delta_2$ of both the cases (1) and (2) can be found in [4, 7].

1. Let $(q_1,q_2) \in (Q_1 \times Q_2), x_2 \in X_2^*, y_2 \in Y_2^*$. We prove the theorem by mathematical induction on $|x_2| = |y_2| = n$.

Case (i) If $n = 0$, then $x_2 = \lambda$ and $y_2 = \lambda$. Now by definition,
$(\sigma_1 \odot \sigma_2)^{\#}((q_1,q_2),\lambda,\lambda)) = 1$ and $\sigma_1^{\#}(q_1, \omega_1(q_2,\lambda), \omega_2(r_2,\lambda)) \wedge \sigma_2^{\#}(q_2,\lambda,\lambda) = \sigma_1^{\#}(q_1,\, \lambda, \lambda) \wedge \sigma_2^{\#}(q_2,\lambda,\lambda) = 1 \wedge 1 = 1$. Thus, the theorem is true for n = 0.

Case(ii) Suppose the theorem is true for $\forall\, u_2 \in X_2^*, v_2 \in Y_2^*$ such that $|u_2| = |v_2| = n-1, n > 1$. Let $x_2 = a_2u_2$ and $y_2 = b_2v_2$, where $a_2 \in X_2$, $b_2 \in Y_2$ and $|u_2| = |v_2| = n-1$. Then ,

$(\sigma_1 \odot \sigma_2)^{\#}((q_1,q_2),x_2,y_2) = (\sigma_1 \odot \sigma_2)^{\#}((q_1,q_2),a_2u_2,b_2v_2) =$
$= \bigvee\{(\sigma_1 \odot \sigma_2)^{\#}((q_1,q_2),a_2,b_2) \wedge [(\delta_1 \odot \delta_2)((q_1,q_2),a_2,(r_1,r_2)) \wedge (\sigma_1 \odot \sigma_2)^{\#}((r_1,r_2),u_2,v_2)]|$
$(r_1,r_2) \in (Q_1 \times Q_2)\} = \bigvee\{[\sigma_1^{\#}(q_1,\omega_1(q_2,a_2),\omega_2(q_2,b_2)) \wedge \sigma_2^{\#}(q_2,\, a_2,b_2)] \wedge [\delta_1(q_1,\omega_1(q_2,a_2),$
$r_1) \wedge \delta_2(q_2,a_2,r_2)] \wedge [\sigma_1^{\#}(r_1,\omega_1(r_2,u_2),\omega_2(r_2,v_2)) \wedge \sigma_2^{\#}(r_2,u_2,v_2)]|\,(r_1,r_2) \in (Q_1 \times Q_2)\}$
$= \bigvee\{[\sigma_1^{\#}(q_1,\omega_1(q_2,a_2),\omega_2(q_2,b_2)) \wedge \delta_1(q_1,\omega_1(q_2,\, a_2),r_1) \wedge \sigma_1^{\#}(r_1,\omega_1(r_2,u_2),\omega_2(r_2,v_2))] \wedge$
$[\sigma_2^{\#}(q_2,a_2,b_2) \wedge \delta_2(q_2,a_2,\, r_2) \wedge \sigma_2^{\#}(r_2,u_2,v_2)]|\,(r_1,r_2) \in (Q_1 \times Q_2)\} =$
$\sigma_1^{\#}(q_1,\omega_1(q_2,a_2)\omega_1(r_2,u_2),\omega_2(q_2,\, b_2)\omega_2(r_2,v_2)) \wedge \sigma_2^{\#}(q_2,a_2u_2,b_2v_2) =$
$\sigma_1^{\#}(q_1,\, \omega_1(q_2,a_2u_2),\omega_2(q_2,b_2v_2)) \wedge \sigma_2^{\#}(q_2,\, a_2u_2,b_2v_2) =$
$\sigma_1^{\#}(q_1,\, \omega_1(q_2,x_2),\omega_2(q_2,y_2)) \wedge \sigma_2^{\#}(q_2,\, x_2,y_2)$.

2. Let $(q_1,q_2) \in (Q_1 \times Q_2), x_2 \in X_2^*, y_2 \in Y_2^*$. We prove the theorem by mathematical induction on $|x_2| = |y_2| = n$.

Case (i) If $n = 0$, then $x_2 = \lambda$ and $y_2 = \lambda$. Now by definition,
$(\sigma_1 \odot \sigma_2)^{\#}((q_1,q_2),(f,\lambda),(g,\lambda)) = 1$ and $\sigma_1^{\#}(q_1,f(q_2),g(q_2)) \wedge \sigma_2^{\#}(q_2,\lambda,\lambda) = 1$. Thus, the theorem is true for n = 0.

Case (ii) Suppose that the theorem is true for $\forall\, u_2 \in X_2^*, v_2 \in Y_2^*$ such that $|u_2| = |v_2| = n-1, n > 1$. Let $x_2 = a_2u_2$ and $y_2 = b_2v_2$, where $a_2 \in X_2$, $b_2 \in Y_2$ and $|u_2| = |v_2| = n-1$. Then $(\sigma_1 \odot \sigma_2)^{\#}((q_1,q_2),(f,x_2),(g,y_2)) = (\sigma_1 \odot \sigma_2)^{\#}\,((q_1,q_2),(f,a_2u_2),(g,b_2v_2)) = \bigvee\{(\sigma_1 \odot \sigma_2)^{\#}\,((q_1,q_2),(f,a_2),(g,b_2)) \wedge [(\delta_1 \odot \delta_2)\,((q_1,q_2),(f,a_2),(r_1,r_2)) \wedge (\sigma_1 \odot \sigma_2)^{\#}\,((r_1,r_2),(f,u_2),(g,v_2))]\,|\;(r_1,r_2) \in (Q_1 \times Q_2)\} = \bigvee\{[\sigma_1^{\#}\,(q_1,\, f(q_2),\, g(q_2)) \wedge \sigma_2^{\#}\,(q_2,\, a_2,b_2)] \wedge [\delta_1$

$(q_1, f(q_2), r_1) \wedge \delta_2\ (q_2, a_2, r_2)] \wedge [\sigma_1^{\#}\ (r_1, f(r_2), g(r_2)) \wedge \sigma_2^{\#}\ (r_2, u_2, v_2)] \mid (r_1, r_2) \in (Q_1 \times Q_2)\}$ $= \bigvee \{[\sigma_1^{\#}\ (q_1, f(q_2)\ , g(q_2)) \wedge \delta_1\ (q_1, f(q_2), r_1) \wedge \sigma_1^{\#}\ (r_1, f(r_2), g(r_2)\ )] \wedge [\sigma_2^{\#}\ (q_2, a_2, b_2) \wedge \delta_2\ (q_2, a_2, r_2) \wedge \sigma_2^{\#}\ (r_2, u_2, v_2)] \mid (r_1, r_2) \in (Q_1 \times Q_2)\} = \sigma_1^{\#}\ (q_1, f(q_2)f(r_2), g(q_2)g(r_2)) \wedge \sigma_2^{\#}\ (q_2, a_2u_2, b_2v_2) = \sigma_1^{\#}(q_1, f(q_2), g(q_2)) \wedge \sigma_2^{\#}\ (q_2, x_2, y_2)$. □

Theorem 4.12. *Let $M_i = (Q_i, X_i, Y_i, \delta_i, \sigma_i)$ be a fuzzy Moore machines, $i = 1,2$. Let μ_1 and μ_2 be an output fuzzy subsystems of M_1 and M_2 respectively. Then $\mu_1 \times \mu_2$ is a an output fuzzy subsystem of fuzzy Moore machine $M_1 \odot_\omega M_2$ and $M_1 \odot_\circ M_2$ provided, $\sigma_1 \odot \sigma_2$ is separable in both the products.*

Proof. 1. Let $(q_1,q_2),(p_1,p_2) \in (Q_1 \times Q_2), x_2 \in X_2^*$ and $y_2 \in Y_2^*$. Let $(\delta_1 \odot \delta_2)^*((q_1,q_2), x_2, (p_1,p_2)) > 0$. Then $\delta_1^*(q_1, \omega_1(q_2,x_2), p_1) > 0$ and $\delta_2^*(q_2,x_2,p_2) > 0$. Then $\mu_1(p_1) \geq \mu_1(q_1) \wedge \sigma_1^{\#}(q_1, \omega_1(q_2,x_2), \omega_2(q_2,y_2)$ and $\mu_2(p_2) \geq \mu_2(q_2) \wedge \sigma_2^{\#}(q_2, x_2, y_2)$. Thus, $(\mu_1 \times \mu_2)((q_1,q_2)) \wedge (\sigma_1 \odot \sigma_2)^{\#}((q_1,q_2), x_2, y_2) = (\mu_1(q_1) \wedge \mu_2(q_2)) \wedge (\sigma_1^{\#}(q_1, \omega_1(q_2, x_2), \omega_2(q_2, y_2)) \wedge \sigma_2^{\#}(q_2, x_2, y_2)) = [\mu_1(q_1) \wedge \sigma_1^{\#}(q_1, \omega_1(q_2, x_2), \omega_2(q_2,y_2))] \wedge [\mu_2(q_2) \wedge \sigma_2^{\#}(q_2, x_2, y_2)] \leq \mu_1(p_1) \wedge \mu_1(p_2) = (\mu_1 \times \mu_2)((p_1,p_2))$. Hence, $\mu_1 \times \mu_2$ is an output fuzzy subsystems of $M_1 \odot_\omega M_2$.
2. Let $(q_1,q_2),(p_1,p_2) \in (Q_1 \times Q_2), x_2 \in X_2^*$ and $y_2 \in Y_2^*$. Let $(\delta_1 \odot \delta_2)^*((q_1,q_2), (f,x_2), (p_1,p_2)) > 0$. Then $\delta_1^*(q_1, f(q_2), p_1) > 0$ and $\delta_2^*(q_2,x_2,p_2) > 0$. Then $\mu_1(p_1) \geq \mu_1(q_1) \wedge \sigma_1^{\#}(q_1, f(q_2), g(q_2)$ and $\mu_2(p_2) \geq \mu_2(q_2) \wedge \sigma_2^{\#}(q_2,x_2,y_2)$. Thus, $(\mu_1 \times \mu_2)((q_1,q_2)) \wedge (\sigma_1 \odot \sigma_2)^{\#}((q_1,q_2),x_2,y_2) = (\mu_1(q_1) \wedge \mu_2(q_2)) \wedge (\sigma_1^{\#}(q_1, f(q_2), g(q_2)) \wedge \sigma_2^{\#}(q_2,x_2,y_2)) = [\mu_1(q_1) \wedge \sigma_1^{\#}(q_1, f(q_2), g(q_2))] \wedge [\mu_2(q_2) \wedge \sigma_2^{\#}(q_2,x_2,y_2)] \leq \mu_1(p_1) \wedge \mu_1(p_2) = (\mu_1 \times \mu_2)((p_1,p_2))$. Hence, $\mu_1 \times \mu_2$ is an output fuzzy subsystems of $M_1 \odot_\circ M_2$. □

As we have mention earlier concept of separability is *not natural*, we now try for natural extension $M_1 \odot_\omega M_2$ and $M_1 \odot_\circ M_2$. We end this section and paper by proving $\mu_1 \times \mu_2$ is output fuzzy subsystem of $M_1 \odot_\omega M_2$ and $M_1 \odot_\circ M_2$ without the separability concept in this second approach.

Definition 4.13. *Let $M_i = (Q_i, X_i, Y_i, \delta_i, \sigma_i)$ be a fuzzy Moore machines, $i = 1,2$. Let $\omega_1 : Q_2 \times X_2 \times X_1 \longrightarrow [0,1]$ and $\omega_2 : Q_2 \times Y_2 \times Y_1 \longrightarrow [0,1]$. Define $M_1 \odot_\omega M_2 = (Q_1 \times Q_2, X_2, Y_2, \delta_1 \odot \delta_2, \sigma_1 \odot \sigma_2)$, where $\delta_1 \odot \delta_2((q_1,q_2), a_2, (p_1,p_2)) = \bigvee\{\delta_1\ (q_1, a_1, p_1) \wedge \omega_1\ (q_2, a_2, a_1) \wedge \delta_2$*

$(q_2,a_2,p_2)|\ a_1 \in X_1\}$ *and* $\sigma_1 \odot \sigma_2\ ((q_1,q_2),b_2) = \bigvee \{\omega_2(q_2,\ b_2,b_1) \wedge \sigma_1\ (q_1,b_1) \wedge \sigma_2(q_2,b_2)|$ $b_1 \in Y_1\}$.

The fuzzy sets ω_1 and ω_2 are now extended naturally as follows:

$\omega_1^{\#} : Q_2 \times X_2^* \times X_1^* \to [0,1]$ defined by

$$\omega_1^{\#}(q_2,x_2,x_1) = \begin{cases} 1, & \text{if } x_2 = x_1 = \lambda; \\ 0, & \text{if } x_2 \neq \lambda = x_1 \text{ or } x_2 = \lambda \neq x_1. \end{cases}$$

$\omega_1^{\#}(q_2,a_2,a_1) = \bigvee\{\delta_2(q_2,a_2,r_2) \wedge \omega_1(r_2,a_2,a_1)|r_2 \in Q_2\}$ and

$$\omega_1^{\#}(q_2,a_2x_2,a_1x_1) = \bigvee\{\omega_1(q_2,a_2,a_1) \wedge \delta_2(q_2,a_2,r_2) \wedge \omega_1^{\#}(r_2,x_2,x_1)|r_2 \in Q_2\}.$$

Now, $\omega_2^{\#} : Q_2 \times X_2^* \times Y_2^* \times Y_1^* \longrightarrow [0,1]$ defined by

$$\omega_2^{\#}(q_2,x_2,y_2,y_1) = \begin{cases} 1, & \text{if } x_2 = y_2 = y_1 = \lambda; \\ 0, & \text{otherwise.} \end{cases}$$

$\omega_2^{\#}(q_2,a_2,b_2,b_1) = \bigvee\{\delta_2(q_2,a_2,r_2) \wedge \omega_2^{\#}(r_2,b_2,b_1)|r_2 \in Q_2\}$ and

$\omega_2^{\#}(q_2,a_2x_2,b_2y_2,b_1y_1) = \bigvee\{\delta_2(q_2,a_2,r_2) \wedge \omega_2(r_2,b_2,b_1) \wedge \omega_2^{\#}(r_2,x_2,y_2,y_1)|r_2 \in Q_2\}$. The extensions of $\delta_1 \odot \delta_2$ and $\sigma_1 \odot \sigma_2$ in $M_1 \odot M_2$ takes the following form

$(\delta_1 \odot \delta_2)^* : (Q_1 \times Q_2) \times X_2^* \times (Q_1 \times Q_2) \longrightarrow [0,1]$ defined by

$(\delta_1 \odot \delta_2)^*((q_1,q_2),a_2x_2,(p_1,p_2)) = \bigvee\{[\delta_1 \odot \delta_2((q_1,q_2),a_2,(r_1,r_2)) \wedge (\delta_1 \odot \delta_2)^*(r_1,r_2),x_2,(p_1, p_2)]|(r_1,r_2) \in Q_1 \times Q_2\}$ and $(\sigma_1 \odot \sigma_2)^{\#} : (Q_1 \times Q_2) \times X_2^* \times Y_2^* \longrightarrow [0,1]$ defined by

$$(\sigma_1 \odot \sigma_2)^{\#}((q_1,q_2),x_2,y_2) = \begin{cases} 1, & \text{if } x_2 = y_2 = \lambda; \\ 0, & x_2 = \lambda \neq y_2 \text{ or } x_2 \neq \lambda = y_2. \end{cases}$$

$(\sigma_1 \odot \sigma_2)^{\#}((q_1,q_2),a_2,b_2) = \bigvee\{[\delta_1 \odot \delta_2((q_1,q_2),a_2,(r_1,r_2)) \wedge (\sigma_1 \odot \sigma_2)((r_1,r_2), b_2)]|(r_1,r_2) \in Q_1 \times Q_2\}$ and

$(\sigma_1 \odot \sigma_2)^{\#}((q_1,q_2),a_2x_2,b_2y_2) = \bigvee\{[\delta_1 \odot \delta_2((q_1,q_2),a_2,(r_1,r_2)) \wedge (\sigma_1 \odot \sigma_2)((r_1,r_2),b_2) \wedge (\sigma_1 \odot \sigma_2)^{\#}((r_1,r_2),x_2,y_2)]|\ (r_1,r_2) \in Q_1 \times Q_2\}$

Clearly, by induction on $|u_2|$ one can easily prove that

$(\delta_1 \odot \delta_2)^*((q_1,q_2),u_2x_2,(p_1,p_2)) = \bigvee\{[(\delta_1 \odot \delta_2)^*((q_1,q_2),u_2,(r_1, r_2)) \wedge (\delta_1 \odot \delta_2)^*(r_1,r_2),x_2,(p_1,\ p_2)]|(r_1,r_2) \in Q_1 \times Q_2\}$.

Theorem 4.14. *Let $M_1 = (Q_1,X,Y,\delta_1,\sigma_1)$ and $M_2 = (Q_2,X,Y,\delta_2,\sigma_2)$ be fuzzy Moore machines. Then the output function of the cascade product of M_1 and M_2 satisfies*

$(\sigma_1 \odot \sigma_2)^{\#}((q_1,q_2),x_2,y_2) = \bigvee\{[\sigma_1^{\#}(q_1,x_1,y_1) \wedge \omega_1^{\#}(q_2,x_2,x_1) \wedge \omega_2^{\#}(q_2,x_2,y_2,y_1) \wedge \sigma_2^{\#}(q_2,x_2, y_2)]|(x_1,y_1) \in X_1^* \times Y_1^*\}$.

Proof. Let $(q_1,q_2) \in (Q_1 \times Q_2), x_2 \in X_2^*, y_2 \in Y_2^*$. We prove the theorem by mathematical induction on $|x_2| = |y_2| = n$.

Case (i) If $n = 0$, then $x_1 = x_2 = \lambda$, $y_1 = y_2 = \lambda$. Thus
$(\sigma_1 \odot \sigma_2)^{\#}((q_1,q_2),\lambda,\lambda) = 1$ and
$\bigvee\{[\sigma_1^{\#}(q_1,\lambda,\lambda) \wedge \omega_1^{\#}(q_2,\lambda,\lambda) \wedge \omega_2^{\#}(q_2,\lambda,\lambda,\lambda) \wedge \sigma_2^{\#}(q_2,\lambda,\lambda)]|(x_1,y_1) \in X_1^* \times Y_1^*\} = 1 \wedge 1 = 1$.

Case (ii) If $n = 1$, then $(\sigma_1 \odot \sigma_2)^{\#}((q_1,q_2), a_2, b_2) = \bigvee\{(\delta_1 \odot \delta_2)((q_1,q_2), a_2,(r_1,r_2)) \wedge ((\sigma_1 \odot \sigma_2)((r_1,r_2),b_2))|\ (r_1,r_2) \in Q_1 \times Q_2\} = \bigvee [\bigvee \{\delta_1(q_1,a_1,r_1) \wedge \omega_1(q_2,a_2,a_1) \wedge \delta_2(q_2,a_2,r_2)|a_1 \in X_1\} \wedge \bigvee \{\omega_2(r_2,b_2,b_1) \wedge \sigma_1(r_1,b_1) \wedge \sigma_2(r_2,b_2)\ |b_1 \in Y_1\}\ ]|(r_1,r_2) \in Q_1 \times Q_2\} = \bigvee\{\sigma_1^{\#}(q_1,a_1, b_1) \wedge \omega_1(q_2, a_2,a_1) \wedge \omega_2^{\#}(q_2, a_2,b_2,b_1) \wedge \sigma_2^{\#}(q_2, a_2,b_2)\}\ |(a_1,b_1) \in X_1 \times Y_1$.

Suppose that the theorem is true for $\forall\, x_2 \in X_2^*, y_2 \in Y_2^*$ such that $|x_2| = |y_2| = n-1, n > 1$. Let $u_2 = a_2x_2$ and $v_2 = b_2y_2$, where $a_2 \in X_2$, $b_2 \in Y_2$ and $|x_2| = |y_2| = n-1$. Then,
$(\sigma_1 \odot \sigma_2)^{\#}((q_1,q_2),u_2,v_2) = (\sigma_1 \odot \sigma_2)^{\#}((q_1,q_2),a_2x_2,b_2y_2) =$
$=\bigvee\{[(\delta_1 \odot \delta_2)((q_1,q_2),a_2,(r_1,r_2)) \wedge (\sigma_1 \odot \sigma_2)((r_1,r_2),b_2) \wedge (\sigma_1 \odot \sigma_2)^{\#}((r_1,r_2),x_2,y_2)]|\ (r_1,r_2) \in Q_1 \times Q_2\}$
$= \bigvee[\{\bigvee(\delta_1(q_1,a_1,r_1) \wedge \omega_1(q_2,a_2,a_1) \wedge \delta_2(q_2,a_2,r_2))|a_1 \in X_1\} \wedge \{\bigvee(\omega_2(r_2,b_2,b_1) \wedge \sigma_1(r_1,b_1) \wedge \sigma_2(r_2,b_2))|b_1 \in Y_1\} \wedge \{\bigvee(\sigma_1^{\#}(r_1,x_1,y_1) \wedge \omega_1^{\#}(r_2,x_2,x_1) \wedge \omega_2^{\#}(r_2,x_2,y_2,y_1) \wedge \sigma_2^{\#}(r_2,x_2,y_2))|x_1 \in X_1^*, y_1 \in Y_1^*\}|\ (r_1,r_2) \in Q_1 \times Q_2]$
$= \bigvee\{[\sigma_1^{\#}(q_1,u_1,v_1) \wedge \omega_1^{\#}(q_2,u_2,u_1) \wedge \omega_2^{\#}(q_2,u_2,v_2,v_1) \wedge \sigma_2^{\#}(q_2,u_2,v_2)]|(u_1,v_1) \in X_1^* \times Y_1^*\}$. □

Theorem 4.15. *Let $M_i = (Q_i,X_i,Y_i,\delta_i,\sigma_i)$ be a fuzzy Moore machines, $i = 1,2$. Let μ_1 and μ_2 be an output fuzzy subsystems of M_1 and M_2 respectively. Then $\mu_1 \times \mu_2$ is a an output fuzzy subsystem of fuzzy Moore machine $M_1 \odot_\omega M_2$.*

Proof. Let $(\delta_1 \odot \delta_2)^*((q_1,q_2),x_2,(p_1,p_2)) > 0$. Then,

$$\bigvee\{[\delta_1^*(q_1,x_1,p_1) \wedge \omega_1^{\#}(q_2,x_2,x_1) \wedge \delta_2^*(q_2,x_2,p_2)]|x_1 \in X_1^*\} > 0.$$

Therefore, $\delta_1(q_1,x_1,p_1) > 0$ and $\delta_2(q_2,x_2,p_2) > 0$, for some $x_1 \in X_1^*$

Since, μ_1 and μ_2 are an output fuzzy subsystems of M_1 and M_2, we have $\mu_1(p_1) \geq \mu_1(q_1) \wedge$

$\sigma_1^{\#}(q_1,x_1,y_1)$ and $\mu_2(p_2) \geq \mu_2(q_2) \wedge \sigma_2^{\#}(q_2,x_2,y_2)$

Therefore, $(\mu_1 \times \mu_2)(p_1,p_2) \geq (\mu_1 \times \mu_2)(q_1,q_2) \wedge \sigma_1^{\#}(q_1,x_1,y_1) \wedge \sigma_2^{\#}(q_2,x_2,y_2)$

$\geq (\mu_1 \times \mu_2)(q_1,q_2) \wedge \sigma_1^{\#}(q_1,x_1,y_1) \wedge \sigma_2^{\#}(q_2,x_2,y_2) \wedge \omega_1^{\#}(q_2,x_2,x_1) \wedge \omega_2^{\#}(q_2,x_2,y_2,y_1)$

$= (\mu_1 \times \mu_2)(q_1,q_2) \wedge (\sigma_1 \circ \sigma_2)^{\#}((q_1,q_2),x_2,y_2)$.

Therefore, $(\mu_1 \times \mu_2)$ is an output fuzzy subsystem of $M_1 \odot_\omega M_2$. □

Definition 4.16. *Let $M_i = (Q_i, X_i, Y_i, \delta_i, \sigma_i)$ be a fuzzy Moore machines, $i = 1,2$. Define $M_1 \odot_\circ M_2 = (Q_1 \times Q_2, F(X_1^{Q_2}) \times X_2, F(Y_1^{Q_2}) \times Y_2, \delta_1 \odot \delta_2, \sigma_1 \odot \sigma_2)$, where $\delta_1 \odot \delta_2((q_1,q_2),(f,a_2),(p_1,p_2)) = \bigvee\{\delta_1(q_1,a_1,p_1) \wedge f(q_2,a_1) \wedge \delta_2(q_2,a_2,p_2) | a_1 \in X_1\}$ and $\sigma_1 \odot \sigma_2((q_1,q_2),(g,b_2)) = \bigvee\{\sigma_1(q_1,b_1) \wedge g(q_2,b_1) \wedge \sigma_2(q_2,b_2) | b_1 \in Y_1\}$.*

We have $F(X_1^{Q_2}) = \{f | f : Q_2 \times X_1 \longrightarrow [0,1]\}$ and $F(Y_1^{Q_2}) = \{g | g : Q_2 \times Y_1 \longrightarrow [0,1]\}$.

Now every $(f,a_2) \in F(X_1^{Q_2}) \times X_2$ is extended to $(f^*,x_2) \in F((X_1^*)^{Q_2}) \times X_2^*$ where, $f^* : Q_2 \times X_1^* \longrightarrow [0,1]$, by $f^*(q_2,\lambda) = 1, f^*(q_2,a_1) = f(q_2,a_1)$ and $f^*(q_2,a_1x_1) = \bigvee\{f(q_2,a_1) \wedge \delta_2(q_2,a_2,r_2) \wedge f^*(r_2,x_1) | r_2 \in Q_2\}$, and every $(g,b_2) \in F(Y_1^{Q_2}) \times Y_2$ is extended to $(g^*,b_2) \in F((Y_1^*)^{Q_2}) \times Y_2$, where $g^* : Q_2 \times Y_1^* \times X_2^* \longrightarrow [0,1]$, by

$$g^*(q_2,y_1,x_2) = \begin{cases} 1, & y_1 = x_2 = \lambda; \\ 0, & y_1 = \lambda \neq x_2 \text{ or } y_1 \neq \lambda = x_2. \end{cases}$$

$g^*(q_2,b_1,a_2) = \bigvee\{\delta_2(q_2,a_2,r_2) \wedge g(r_2,b_1) | r_2 \in Q_2\}$ and $g^*(q_2,b_1y_1,a_2x_2) = \bigvee\{\delta_2(q_2,a_2,r_2) \wedge g(r_2,b_1) \wedge g^*(r_2,y_1,x_2) | r_2 \in Q_2\}$.

$(\delta_1 \odot \delta_2)^* : (Q_1 \times Q_2) \times (F((X_1^*)^{Q_2}) \times X_2^*) \times (Q_1 \times Q_2) \longrightarrow [0,1]$

$(\delta_1 \odot \delta_2)^*((q_1,q_2), (f^*,a_2x_2),(p_1,p_2)) = \bigvee \{(\delta_1 \odot \delta_2)((q_1,q_2), (f,a_2), (r_1,r_2)) \wedge ((\delta_1 \odot \delta_2)^*((r_1,r_2), (f^*,x_2), (p_1,p_2))) | (r_1,r_2) \in Q_1 \times Q_2\}$ and $(\sigma_1 \odot \sigma_2)^{\#} : (Q_1 \times Q_2) \times (F((X_1^*)^{Q_2}) \times X_2^*) \times (F((Y_1^*)^{Q_2}) \times Y_2^*) \longrightarrow [0,1]$ by

$$(\sigma_1 \odot \sigma_2)^{\#}((q_1,q_2), (f,x_2),(g,y_2)) = \begin{cases} 1, & x_2 = y_2 = \lambda; \\ 0, & x_2 = \lambda \neq y_2 \text{ or } x_2 \neq \lambda = y_2. \end{cases}$$

$(\sigma_1 \odot \sigma_2)^{\#}((q_1,q_2),(f,a_2),(g,b_2)) = \bigvee\{(\delta_1 \odot \delta_2)((q_1,q_2),(f,a_2),(r_1,r_2)) \wedge ((\sigma_1 \odot \sigma_2)((r_1,r_2),(g,b_2))) | (r_1,r_2) \in Q_1 \times Q_2\}$ and

$(\sigma_1 \odot \sigma_2)^{\#}((q_1,q_2),(f^*,a_2x_2),(g^*,b_2y_2)) = \bigvee\{(\delta_1 \odot \delta_2)((q_1,q_2),(f,a_2), (r_1,r_2)) \wedge ((\sigma_1 \odot \sigma_2)((r_1,r_2),(g,b_2))) \wedge (\sigma_1 \odot \sigma_2)^{\#}((r_1,r_2),(f^*,x_2),(g^*,y_2)) | (r_1,r_2) \in Q_1 \times Q_2\}$.

Clearly, by induction on $|u_2|$ one can easily prove that

$(\delta_1 \odot \delta_2)^*((q_1,q_2), (f^*,u_2x_2), (p_1,p_2)) = \bigvee\{(\delta_1 \odot \delta_2)^*((q_1,q_2), (f^*,u_2),(r_1,r_2)) \wedge((\delta_1 \odot \delta_2)^*((r_1,r_2), (f^*,x_2), (p_1,p_2)))| (r_1,r_2) \in Q_1 \times Q_2\}$.

Theorem 4.17. *Let $M_1 = (Q_1,X,Y,\delta_1,\sigma_1)$ and $M_2 = (Q_2,X,Y,\delta_2,\sigma_2)$ be fuzzy Moore machines. Then the output function of the wreath product of M_1 and M_2 satisfies*
$(\sigma_1 \odot\sigma_2)^{\#}((q_1,q_2), (f^*,x_2),(g^*,y_2)) = \bigvee\{ [\sigma_1^{\#}(q_1,x_1,y_1) \wedge f^*(q_2,x_1) \wedge g^*(q_2,y_1,x_2) \wedge \sigma_2^{\#}(q_2,x_2,y_2)]| (x_1,y_1) \in X_1^* \times Y_1^*\}$.

Proof. Let $(q_1,q_2) \in (Q_1 \times Q_2), f^* \in F((X_1^*)^{Q_2}), g^* \in F((Y_1^*)^{Q_2}), x_2 \in X_2^*, y_2 \in Y_2^*$. We prove the theorem by mathematical induction on $|x_2| = |y_2| = n$.

Case (i) If $n = 0$, then $x_1 = x_2 = \lambda$, $y_1 = y_2 = \lambda$. Theorem is true by definition itself.

Case (ii) If $n = 1$, then $(\sigma_1 \odot \sigma_2)^{\#}((q_1,q_2), (f^*,a_2), (g^*,b_2)) = \bigvee\{(\delta_1 \odot \delta_2)((q_1,q_2), (f,a_2), (r_1,r_2)) \wedge((\sigma_1 \odot \sigma_2) ((r_1,r_2), (g,b_2)))|(r_1,r_2) \in Q_1 \times Q_2\} = \bigvee[\bigvee\{\delta_1(q_1,a_1,r_1) \wedge f(q_2,a_1) \wedge\delta_2(q_2,a_2,r_2) |a_1 \in X_1\} \wedge\bigvee\{g(r_2,b_1) \wedge\sigma_1(r_1,b_1) \wedge\sigma_2(r_2,b_2) |b_1 \in Y_1\}]| (r_1,r_2) \in Q_1 \times Q_2\} = \bigvee\{\sigma_1^{\#}(q_1,a_1,b_1) \wedge f^*(q_2,a_1) \wedge g^*(q_2, b_1,a_2)|(a_1,b_1) \in x_1 \times Y_1 \wedge \sigma_2^{\#}(q_2,a_2,b_2)\}$. Suppose that the theorem is true for $\forall x_2 \in X_2^*, y_2 \in Y_2^*$ such that $|x_2| = |y_2| = n-1, n > 1$. Let $u_2 = a_2x_2$ and $v_2 = b_2y_2$, where $a_2 \in X_2$, $b_2 \in Y_2$ and $|x_2| = |y_2| = n-1, n > 1$. Then $(\sigma_1 \odot\sigma_2)^{\#}((q_1,q_2),(f^*,u_2),(g^*,v_2)) = (\sigma_1 \odot\sigma_2)^{\#}((q_1,q_2),(f^*,a_2x_2), (g^*,b_2y_2)) = \bigvee\{[(\delta_1 \odot\delta_2)((q_1,q_2), (f,a_2),(r_1,r_2)) \wedge(\sigma_1 \odot \sigma_2) ((r_1,r_2), (g,b_2)) \wedge(\sigma_1 \odot \sigma_2)^{\#}((r_1,r_2), (f^*,x_2), (g^*,y_2))]| (r_1,r_2) \in Q_1 \times Q_2\} = \bigvee[\{\bigvee(\delta_1(q_1,a_1,r_1) \wedge f(q_2,a_1) \wedge \delta_2(q_2,a_2,r_2))|a_1 \in X_1\} \wedge \{\bigvee(g(r_2,b_1) \wedge \sigma_1(r_1,b_1) \wedge\sigma_2(r_2,b_2))|b_1 \in Y_1\} \wedge \{\bigvee(\sigma_1^{\#}(r_1,x_1,y_1) \wedge f^*(r_2,x_1) \wedge g^*(r_2,y_1,x_2) \wedge \sigma_2^{\#}(r_2,x_2,y_2))|x_1 \in X_1^*, y_1 \in Y_1^*\} |(r_1,r_2) \in Q_1 \times Q_2] = \bigvee\{[\sigma_1^{\#}(q_1,a_1x_1,b_1y_1) \wedge f^*(q_2,a_1x_1) \wedge g^*(q_2, b_1y_1,a_2x_2)] \wedge\sigma_2^{\#}(q_2,a_2x_2, b_2y_2)|(a_1x_1,b_1y_1) \in X_1^* \times Y_1^*\}$. □

The above theorem enable us to prove that $\mu_1 \times \mu_2$ is an output fuzzy subsystem of $M_1 \odot_\circ M_2$.

Theorem 4.18. *Let $M_i = (Q_i,X_i,Y_i,\delta_i,\sigma_i)$ be a fuzzy Moore machines, $i = 1,2$. Let μ_1 and μ_2 be an output fuzzy subsystems of M_1 and M_2 respectively. Then $\mu_1 \times \mu_2$ is a an output fuzzy subsystem of the fuzzy Moore machine $M_1 \odot_\circ M_2$.*

Proof. Let $(p_1,p_2),(q_1,q_2) \in Q_1 \times Q_2$, $(f^*,x_2) \in F(X_1^*)^{Q_2}$ and $(g^*,y_2) \in F(Y_1^*)^{Q_2}$. Let $(\delta_1 \odot \delta_2)^*((q_1,q_2),(f^*,x_2),(p_1,p_2)) > 0$. Then, $\bigvee\{[\delta_1^*(q_1,x_1,p_1) \wedge f^*(q_2,x_2) \wedge \delta_2^*(q_2,x_2,p_2)]|x_1 \in$

$X_1^*\} > 0$. Therefore,$\delta_1(q_1,x_1,p_1) > 0$ and $\delta_2(q_2,x_2,p_2) > 0$, for some $x_1 \in X_1^*$. Since, μ_1 and μ_2 are an output fuzzy subsystems of M_1 and M_2, we have $\mu_1(p_1) \geq \mu_1(q_1) \wedge \sigma_1^{\#}(q_1,x_1,y_1)$ and $\mu_2(p_2) \geq \mu_2(q_2) \wedge \sigma_2^{\#}(q_2,x_2,y_2)$. Therefore, $(\mu_1 \times \mu_2)(p_1,p_2) \geq (\mu_1 \times \mu_2)(q_1,q_2) \wedge \sigma_1^{\#}(q_1,x_1,y_1) \wedge \sigma_2^{\#}(q_2,x_2,y_2) \geq (\mu_1 \times \mu_2)(q_1,q_2) \wedge \sigma_1^{\#}(q_1,x_1,y_1) \wedge \sigma_2^{\#}(q_2,x_2,y_2) \wedge f^*(q_2,x_1) \wedge g^*(q_2,y_1,x_2) = (\mu_1 \times \mu_2)(q_1,q_2) \wedge (\sigma_1 \odot \sigma_2)^{\#}((q_1,q_2),(f^*,x_2),(g^*,y_2))$. Therefore, $\mu_1 \times \mu_2$ is an output fuzzy subsystem of $M_1 \odot_\circ M_2$. □

5. Conclusion

In this paper the results of fuzzy finite state machine are successfully extended for fuzzy Moore machines. We introduced successor, submachines, subsystem, homomorphism and (super) cyclic subsystems for Fuzzy Moore machines. Along with various properties, we have characterized subsystems and (super) cyclic subsystems. Three classes, based on constants fuzzy sets, fuzzy input-output sets and fuzzy points, of subsystems are also obtained. Subsystem of Fuzzy Moore machine is then extended to output subsystem. It is also proved that the cartesian product of output subsystems is an subsystem for four kinds of products of fuzzy Moore machines. Motivation to introduce extension of output function in fuzzy Moore machine is taken from the already known separability concept of functions [7]. Following are the main results of this paper.

(1) Let $M = (Q,X,Y,\delta,\sigma)$ be a fuzzy Moore machine. If input and output strings has different length, then degree of the input-output function is zero, at each state.

(2) Image of the successor set of a state, under homomorphism, is a successor set of the image of the state.

(3) Let $M = (Q,X,Y,\delta,\sigma)$ be a fuzzy Moore machine and let μ be a fuzzy subset subset of Q. Then μ is a subsystem of M if and only if $\mu xy \subseteq \mu, \quad \forall x \in X^*, y \in Y^*$.

(4) Let $M = (Q,X,Y,\delta,\sigma)$ be a fuzzy Moore machine. Let $t \in (0,1]$ and $q \in Q$. Then the following hold

  (a) $q_t X^* Y^*$ is a subsystem of M,

  (b) $Supp(q_t X^* Y^*) = S(q)$.

(5) Let $M = (Q, X, Y, \delta, \sigma)$ be a fuzzy Moore machine and μ a subsystem of M. Then μ is called super cyclic if and only if $\mu = q_{\mu(q)} X^* Y^*, \ \forall q \in Q$.

(6) Let $M = (Q, X, Y, \delta, \sigma)$ be a fuzzy Moore machine and μ a subsystem of M. Then μ is super cyclic if and only if $\forall p, q \in Q, \exists x \in X^*, y \in Y^*$ such that $\delta^*(p, x, q) \wedge \sigma^{\#}(p, x, y) \geq \mu(p)$.

(7) Product of two output subsystems of fuzzy Moore machines is an output subsystem of the following products: restricted direct , direct, cascade and wreath products.

References

[1] Chaudhari S.R. and Desai A. S., On fuzzy Mealy and Moore machines, Bull. Pure Appl. Math. 4 (2010), 35-384.

[2] Chaudhari S.R. and Morye S.A., Mealy-type fuzzy finite state machines: A Fyzzy Approach, Int. J. Math. Sci. Eng. Appl. 4 (2010), 407-426.

[3] Chaudhari S. R. , Komejwar D. D. , On fuzzy pushdown automata and their covering, Int. J. Copmput. Appl. 46 (2012), 27-33.

[4] Choubey Alka and Ravi K. M. , Intuitionistic fuzzy automata and intuitionistic fuzzy regular expressions, J. Appl. Math. Info. 27 (2009), 409-417.

[5] Doostfatemeh M. and Kremer S. C. , New directions in fuzzy automata, Int. J. Appro. Reasoning, 38 (2005), 175-214.

[6] Eilenburg S., Automata, Languages and Machines, Academic press, London, Vol. A, 1974.

[7] Hwang S. Y. , Fuzzy submachines of a fuzzy finite state machine, J. Appl. Math. Comput. 19 (2005), 457-466.

[8] Liu J., Mo Z. and Wang Y., Products of Mealy-type fuzzy finite state machines, Fuzzy Sets and Systems, 160 (2009), 2401-2415.

[9] Jun Y. B., Kavikumar J. , Bipolar fuzzy finite state machines, Bull. Malays, Sci. Soc. 34 (2011), 181-188.

[10] Malik D. S. , Mordenson J. N. and Sen M. K., On subsystem of a fuzzy finite state machine, Fuzzy Sets and Systems, 68 (1994), 83-92.

[11] Malik D. S. , Mordenson J. N. and Sen M. K., Submachines of fuzzy finite state machine, J. Fuzzy Math. 2 (1994), 781-792.

[12] Malik D. S., Mordenson J. N. and Sen M. K., Product of fuzzy finite state machine, J. Fuzzy Math. 92 (1997), 95-102.

[13] Mordeson J. N. and Malik D. S., Fuzzy Automata and Language : Theory and Applications, Boca Raton London, Chapman Hall/CRC, 2002.

[14] Xing H., Fuzzy Pushdown automata, Fuzzy Sets and Systems, 158 (2007), 1437-1449.

[15] Yongming Li., Pedrycz Witold , The equivalence between fuzzy Mealy and fuzzy Moore machines, Soft Computing 10 (2006), 953-959.

[16] Zahedi M. M., Horry M. and Abolpor K., Bifuzzy (General) topology on max-min general fuzzy automata, Adv. Fuzzy Math. 3 (2008), 51-60.