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ENDEDGE BLOCK DOMINATION IN GRAPHS

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Abstract. For any graph G (V, E), a block graph $B(G)$ is a graph whose vertices are corresponding to the blocks of G and two vertices in $B(G)$ are adjacent whenever the corresponding blocks contain a common cutvertex in G. An edge dominating set s_e of a block graph $B(G)$ is an endedge block dominating set if s_e contains all endedges of $B(G)$. The endedge block domination number $\gamma'_{eb}(G)$ is the minimum cardinality of an endedge block dominating set . In this paper some bounds for $\gamma'_{eb}(G)$ are obtained in terms of elements of G. Further exact values of $\gamma'_{eb}(G)$ for some standard graphs and relationships with other dominating parameters were obtained.

Keywords: block graph; domination number; endedge domination number; endedge block domination number.
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Introduction

In this paper We follow the notations and terminology of Harary [1]. We consider connected, undirected, finite graphs without loops. Let $G = (V, E)$ be a graph with $|V| = p \; and \; |E| = q$. n denotes number of blocks of G. $N(v)$ and $N[v]$ denote the open and closed neighborhoods of a vertex v respectively in G. The degree of an edge $e = u\,v$ of G is defined by $deg\ e = deg\, u + deg\ v - 2$. The maximum degree of a vertex in G is denoted by $\Delta(G)$ and the minimum degree of a vertex in G is denoted by $\delta(G)$.

A vertex v of V is called a cutvertex if its removal from G increase the number of components of G. A nontrivial connected graph with no cutvertex is called a block. A block incident with exactly one cutvertex is called an endblock. A block incident with more than one cutvertex is called a nonendblock.

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A set D of a graph $G = (V, E)$ is a dominating set if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a minimal dominating set . A set F of edges in a graph $G = (V, E)$ is called an edge dominating set of G if every edge in $E - F$ is adjacent to at least one edge in F. The edge domination number $\gamma^I(G)$ is the minimum cardinality of an edge dominating set of G . Edge domination was introduced by S. Mitchell and S. T. Hedetniemi [2] and is now well studied in graph theory. The edge dominating set is called an endedge dominating set if all endedges belong to edge dominating set of G .

The endedge domination number $\gamma_e^I(G)$ is the minimum cardinality of endedge dominating set of G. Endedge domination is introduced by M.H.Muddebihal and A.R.Sedamkar [3]. A block graph $B(G)$ is a graph whose vertices are corresponding to the blocks of G and two vertices in $B(G)$ are adjacent whenever the corresponding blocks contain a common cutvertex in G. A set D_b of a block graph $B(G) = (H, X)$ is a dominating set if every vertex in $H - D_b$ is adjacent to some vertex in D_b. The domination number $\gamma(B(G))$ is the minimum cardinality of a minimal block dominating set. Block domination is introduced by M. H. Muddebihal, T. Srinivas and Abdul Majeed [4]. We are introducing endedge block domination in this paper and we obtain certain bounds on $\gamma'_{eb}(G)$ in terms of vertices, blocks and other parameters of G.

Results

Initially we begin with endedge block domination number of a graph of some standard graphs, which are straight forward in the following theorem.

Theorem 1 : (i) For any star $K_{1,p}$ with $p \geq 2$, $\gamma^{\mathrm{I}}_{eb}\left(k_{1,P}\right) = \frac{p-1}{2}$, if p is odd .

$= \frac{p}{2}$, if p is even .

(ii) For any path P_p with $p \geq 3$, $\gamma^{\mathrm{I}}_{eb}(P_P) = \frac{p}{3}$, if $p \equiv 0 \ (mod 3)$.

$= \left\lceil \frac{p}{3} \right\rceil$, otherwise.

In the following theorem we obtain upper bound for $\gamma^{\mathrm{I}}_{eb}(G)$ in terms of number of blocks of G.

Theorem 2 : For any connected graph G with $n \geq 2$ blocks , $\gamma^{\mathrm{I}}_{eb}(G) \leq n - 1$. Equality holds for $B(G) \cong K_{1,n-1}$ where n is number of blocks of G .

Proof : Let G be any nontrivial connected graph with $A = \{ B_i \}$, $2 \le i \le n$ blocks. Let $S_e = \{ q_1, q_2, \ldots, q_l \}$, $l < n$ be γ^{I}_{eb}- set in $B(G)$ such that $|S_e| = \gamma^{\mathrm{I}}_{eb}(G)$. We prove the result by induction on number of blocks of G.

Let G be a graph with $n = 2$ blocks. Then $S_e = \{q_j\}, j = 1$ and

$|S_e| = \gamma^{\mathrm{I}}_{eb}(G) = n - 1 = 2 - 1.$

Let G be a graph with $n = 3$ blocks. Then $S_e = \{q_1, q_2\}$ and

$|S_e| = \gamma^{\mathrm{I}}_{eb}(G) = n - 1 = 3 - 1.$

Assume that the result is true for $n = t$ blocks. Then $\gamma^{\mathrm{I}}_{eb}(G) \le t - 1$.

Suppose G has $n = t + 1$ blocks. Then the corresponding block vertex of $(t+1)^{\text{th}}$ block of G is either an endvertex or a nonendvertex incident with either an endblock or a nonend block respectively in $B(G)$.

Then clearly $|S_e| = \gamma^{I}_{eb}(G) \le t - 1 + 1$ gives $\gamma^{\mathrm{I}}_{eb}(G) \le (t + 1) - 1$.

Equality for $B(G) \cong K_{1,n-1}$ is obvious.

Following corollary gives equality for $\gamma^{I}_{eb}(G)$.

Corollary 1 : For any connected graph G with $n \ge 2$ blocks with exactly one cutvertex,

$$\gamma^{\mathrm{I}}_{eb}(G) = \frac{n-1}{2} \text{, if } n \text{ is odd.}$$

$$= \frac{n}{2}\text{, if } n \text{ is even.}$$

Proof : If G has exactly one cutvertex , then $B(G)$ is complete graph and number of vertices of $B(G)$ is n. By Theorem 1 clearly result follows.

In next theorem we obtain upper bound in terms of n for $\gamma^{\mathrm{I}}_{eb}(T)$.

Theorem 3: For any tree with n – blocks $\gamma^{\mathrm{I}}_{eb}(T) \le \left\lfloor \frac{n}{2} \right\rfloor$ if and only if $T \ncong P_4$.

Proof: For necessary condition,

Suppose $\gamma^{\mathrm{I}}_{eb}(T) \le \left\lfloor \frac{n}{2} \right\rfloor$ for any tree T.

If $T \cong P_4$, then $n = 3$ and $\left\lfloor \frac{n}{2} \right\rfloor = 1$. The corresponding $B(T) \cong P_3$ and $\gamma^{\mathrm{I}}_{eb}(T) = 2$.

Then $\gamma^{\mathrm{I}}_{eb}(T) > \left\lfloor \frac{n}{2} \right\rfloor$ a contradiction . Hence $T \ncong P_4$.

For sufficient condition consider $T \ncong P_4$.

Suppose $T \cong P_p$, $p \ne 4$. Then maximum number of blocks in T are $n = p - 1$.

From Theorem 1, $\gamma^{\mathrm{I}}_{eb}(P_p) = \frac{p}{3} \le \left\lfloor \frac{p-1}{2} \right\rfloor$, if p ≡ 0 (mod 3).

$= \left\lceil \frac{p}{3} \right\rceil \leq \left\lfloor \frac{p-1}{2} \right\rfloor$, p ≠ 4 otherwise gives the result.

Theorem 4: For any connected graph G with m endblocks, $\gamma_{eb}^{I}(G) \leq \gamma_{e}^{I}(G) + \left\lfloor \frac{m+1}{2} \right\rfloor$ .The proof of the Theorem 4 requires some lemmas. Before the lemmas we construct some sets in G as well as in $B(G)$ so that to give the proofs of lemmas.

Sets in G are

$E_e = \{ e_1, e_2, \ldots\ldots, e_e\}$ is a set of all endedges.

$E_n = E(G) \setminus E_e$

$E_g = E_n \cup E_e$

$A_1 = \{B_1, B_2, \ldots\ldots, B_i\}$ is a set of endblocks such that each block is adjacent to exactly one block.

$A_2 = \{B_1, B_2, \ldots\ldots, B_j\}$ is a set of endblocks such that each block is adjacent to more than one block.

$A_3 = \{B_1, B_2, \ldots\ldots, B_k\}$ is a set of all nonendblocks.

$A = A_1 \cup A_2 \cup A_3$.

We define a family as $\mathfrak{I} = \{E_e \cup (E(G) \setminus N(E_e))\}$ in G.

Sets in $B(G)$ are as follows.

$H = \{ b_1, b_2, \ldots\ldots, b_n \}$ is a set of all vertices.

$H_1 = \{ b_1, b_2, \ldots\ldots, b_i \}$ is a set of all endvertices of degree 1.

$H_2 = \{ b_1, b_2, \ldots\ldots, b_j \}$ is a set of all nonend noncutvertices.

$H_3 = \{ b_1, b_2, \ldots\ldots, b_k \}$ is a set of all cutvertices.

$X_1 = \{ q_1, q_2, \ldots\ldots, q_t \}$ is a set of all endedges.

$X_2 = E(B(G)) \setminus X_1$

We now define a family $\mathfrak{R} = \{X_1 \cup (X_2 \setminus N(X_1))\}$ in $B(G)$.

We consider $A_1 = \phi$ in Lemma 1, lemma 2 and 3.

Lemma 1: If $A_1 = \phi$ and each block $B_j \in A_2$ has $p \geq 3$ vertices, then , $\gamma_{e}^{I}(G) \cong \gamma^{I}(G)$ and

$\gamma_{eb}^{I}(G) \in X_2$.

Proof: Let $F_1 \subset E_g$. If $A_1 = \phi$ and each block $B_j \in A_2$ has $p \geq 3$ vertices, then for each edge $e \in F_1$, ∃ an edge $e_1 \in \{E_g \setminus F_1\}$ such that $N(e_1) \cap F_1 = \{e\}$. Hence F_1 forms minimal edge dominating set in G. Then $|F_1| = \gamma^{\mathrm{I}}(G)$. Since $A_1 = \phi$ and each block of A_2 has $p \geq 3$ vertices, $\gamma_e^{\mathrm{I}}(G) = \gamma^{\mathrm{I}}(G)$.

In $B(G)$, $H_1 = \phi$. Then $X_1 = \phi$ and $\exists\ X_2^{\mathrm{I}} \subset X_2$ such that every edge in $X_2 \setminus X_2^{\mathrm{I}}$ is adjacent to at least one edge in X_2^{I}. So X_2^{I} forms γ_{eb}^{I} −set and $|X_2^{\mathrm{I}}| = \gamma_{eb}^{\mathrm{I}}(G)$.

Lemma 2: If $A_1 = \phi$, each block $B_j \in A_2$ has exactly two vertices, then $\gamma_e^{\mathrm{I}}(G) \in \mathfrak{I}$ and $\gamma_{eb}^{\mathrm{I}}(G) \in X_2$.

Proof: Let $A_1 = \phi$ and each block $B_j \in A_2$ has exactly two vertices. Then $A_2 \cong E_e$ and $E(G) \setminus A_2 = E_n$. Let $F_2 \subset E_n$ be the minimal edge dominating set of induced Subgraph $\langle E_n \setminus N(A_2)\rangle$. Then $F_2 \cup A_2$ forms γ_e^{I} – set which belongs to $\mathfrak{I}$ and $|F_2 \cup A_2| = \gamma_e^{\mathrm{I}}(G)$.

In $B(G)$, endedge set $X_1 = \phi$ and $\exists\ X_2^{\mathrm{I}} \subset X_2$ such that each $q_i \in X_2 \setminus X_2^{\mathrm{I}}$ is adjacent to at least one edge $q_j \in X_2^{\mathrm{I}}$. Then X_2^{I} forms γ_{eb}^{I} – set and $|X_2^{\mathrm{I}}| = \gamma_{eb}^{\mathrm{I}}(G)$.

Lemma 3: If $A_1 = \phi$ and some blocks $B_j \in A_2$ have exactly two vertices, then $\gamma_e^{\mathrm{I}}(G) \in \mathfrak{I}$ and $\gamma_{eb}^{\mathrm{I}}(G) \in X_2$.

Proof : Let $A_2^{\mathrm{I}} \subset A_2$ be set of endblocks and each block of A_2^{I} has exactly two vertices. Then $A_2^{\mathrm{I}} \cong E_e$ and $E(G) \setminus A_2^{\mathrm{I}} = E_n$. $\exists$ an edge dominating set $F_2 \subset E_n$ of induced subgraph $\langle E_n \setminus N(A_2^{\mathrm{I}})\rangle$ such that $F_2 \cup A_2^{\mathrm{I}}$ forms γ_e^{I} – set which belongs to $\mathfrak{I}$ and $|F_2 \cup A_2^{\mathrm{I}}| = \gamma_e^{\mathrm{I}}(G)$.

In $B(G)$, $X_2^{\mathrm{I}} \subset X_2$ forms γ_{eb}^{I} – set because each edge in $X_2 \setminus X_2^{\mathrm{I}}$ is adjacent to at least one edge in X_2^{I}. Then $|X_2^{\mathrm{I}}| = \gamma_{eb}^{\mathrm{I}}(G)$.

In further **lemma 4, lemma 5 and 6**, we consider $A_2 = \phi$.

Lemma 4: If each block $B_i \in A_1$ has $p \geq 3$ vertices and $A_2 = \phi$, then $\gamma_e^{\mathrm{I}}(G) \cong \gamma^{\mathrm{I}}(G)$ and $\gamma_{eb}^{\mathrm{I}}(G) \in \mathfrak{R}$.

Proof: Let $A_2 = \phi$ and each block $B_i \in A_1$ has $p \geq 3$ vertices. Then $E_e = \phi$ and $\exists$ set of edges $F_1 \subset E_g$ in G such that each edge in $E_g \setminus F_1$ is adjacent to at least one edge of F_1. So F_1 forms minimal edge dominating set. Since $E_e = \phi$, $|F_1| = \gamma^{\mathrm{I}}(G) = \gamma_e^{\mathrm{I}}(G)$.

In $B(G)$, $|H_1| = |X_1|$. Let $X_2^{\mathrm{I}} \subset X_2$ be the minimal edge dominating set of $\langle X_2 \setminus (N(X_1) \cup X_1)\rangle$. Then $X_1 \cup X_2^{\mathrm{I}}$ $forms\ \gamma_{eb}^{\mathrm{I}} - set$ and $|X_1 \cup X_2^{\mathrm{I}}| = \gamma_{eb}^{\mathrm{I}}(G) \in \mathfrak{R}$.

Lemma 5: If each block $B_i \in A_1$ has exactly two vertices and $A_2 = \phi$, then $\gamma_e^{\mathrm{I}}(G) \in \mathfrak{I}$ and $\gamma_{eb}^{\mathrm{I}}(G) \in \mathfrak{R}$.

Proof: Let $A_2 = \phi$, each block $B_i \in A_1$ has $p = 2$ vertices. Then $A_1 = E_e$ and $\exists$ a minimal edge dominating set F_3 of induced subgraph $\langle E_g \setminus (A_1 \cup N(A_1)) \rangle$ such that $A_1 \cup F_3$ forms γ_e^{I}- set. $|A_1 \cup F_3| = \gamma_e^{\mathrm{I}}(G)$ which belongs to $\mathfrak{I}$.

In $B(G)$, $|X_1| = |A_1|$ and X_2^{I} is the minimal edge dominating set of $\langle X_2 \setminus (N(X_1) \cup X_1) \rangle$.Then $X_1 \cup X_2^{\mathrm{I}}$ *forms* $\gamma_{eb}^{\mathrm{I}} - set$ *and* $|X_1 \cup X_2^{\mathrm{I}}| = \gamma_{eb}^{\mathrm{I}}(G)$ which belongs to $\mathfrak{R}$.

Lemma 6: If some blocks $B_i \in A_1$ have exactly two vertices and $A_2 = \phi$, then $\gamma_e^{\mathrm{I}}(G) \in \mathfrak{I}$ and $\gamma_{eb}^{\mathrm{I}}(G) \in \mathfrak{R}$.

Proof : Let some blocks $B_i \in A_1$ have exactly $p = 2$ vertices and $A_2 = \phi$. Then $\exists$ a minimal edge dominating set F_4 of $\langle E(G) \setminus (A_1^{\mathrm{I}} \cup N(A_1^{\mathrm{I}})) \rangle$ where $A_1^{\mathrm{I}} \subset A_1$ is set of all endedges. So $A_1^{\mathrm{I}} \cong E_e$ and $A_1^{\mathrm{I}} \cup F_4$ forms $\gamma_e^{\mathrm{I}} - set$. Then $|A_1^{\mathrm{I}} \cup F_4| = \gamma_e^{\mathrm{I}}(G) \in \mathfrak{I}$.

In $B(G)$, $|X_1| = |A_1|$ and $\exists$ $X_2^{\mathrm{I}} \subset X_2$ such that $X_1 \cup X_2^{\mathrm{I}}$ forms γ_{eb}^{I}- set where X_2^{I} is the minimal edge dominating set of $\langle E(B(G)) \setminus (X_1 \cup N(X_1)) \rangle$.

Then $|X_1 \cup X_2^{\mathrm{I}}| = \gamma_{eb}^{\mathrm{I}}(G)$ which belongs to $\mathfrak{R}$.

In next **lemmas 7, lemma 8 and 9** we consider $A_1 \neq \phi$, $A_2 \neq \phi$.

Lemma 7: If $A_1 \neq \phi$, $A_2 \neq \phi$ and each block $B_i \in A_1$ has $p \geq 3$ vertices, each block $B_j \in A_2$ has $p \geq 3$ vertices then $\gamma_e^{\mathrm{I}}(G) \cong \gamma^{\mathrm{I}}(G)$ and $\gamma_{eb}^{\mathrm{I}}(G) \in \mathfrak{R}$.

Proof : If each block of A_1 and A_2 has 3 or more than three vertices, then $\exists$ a set of edges $F_1 \subset E(G)$ such that every edge in $E(G) \setminus F_1$ is adjacent to at least one edge in F_1. Hence F_1 forms γ^{I}– set in G and since $E_e = \phi$, $|F_1| = \gamma^{\mathrm{I}}(G) = \gamma_e^{\mathrm{I}}(G)$.

In $B(G)$, $|X_1| = |H_1| = |A_1|$ and $\exists$ a minimal edge dominating set $X_2^{\mathrm{I}} \subset X_2$ of induced subgraph $\langle X_2 \setminus (X_1 \cup N(X_1)) \rangle$ such that $X_1 \cup X_2^{\mathrm{I}}$ forms γ_{eb}^{I}- set $\in \mathfrak{R}$.

Then $|X_1 \cup X_2^{\mathrm{I}}| = \gamma_{eb}^{\mathrm{I}}(G)$.

Lemma 8 : If $A_1 \neq \phi$ and each block $B_i \in A_1$ has $p \geq 3$ vertices, $A_2 \neq \phi$ and $A_2^{\mathrm{I}} \subseteq A_2$ where A_2^{I} is set of endedges each has degree ≥ 2, then $\gamma_e^{\mathrm{I}}(G) \in \mathfrak{I}$ and $\gamma_{eb}^{\mathrm{I}}(G) \in \mathfrak{R}$.

Proof: Let $A_2^{\mathrm{I}} \subseteq A_2$ be set of all endedges in G. Let F_2 be the minimal edge dominating set of induced subgraph $\langle E(G) \setminus (A_2^{\mathrm{I}} \cup N(A_2^{\mathrm{I}})) \rangle$. Then $F_2 \cup A_2^{\mathrm{I}}$ forms γ_e^{I} – set in G and $|F_2 \cup A_2^{\mathrm{I}}| = \gamma_e^{\mathrm{I}}(G)$. Since $A_2^{\mathrm{I}} \cong E_e$, $\gamma_e^{\mathrm{I}}(G) \in \mathfrak{I}$.

In $B(G)$, $X_1 \cup X_2^{\mathrm{I}}$ forms γ_{eb}^{I}- set where $X_2^{\mathrm{I}} \subset X_2$ is the minimal edge dominating set of induced subgraph $\langle E(B(G)) \setminus (X_1 \cup N(X_1))\rangle$ which $\in \Re$ and $|X_1 \cup X_2^{\mathrm{I}}| = \gamma_{eb}^{\mathrm{I}}(G)$.

Lemma 9 : If $A_1 \neq \phi$, $A_2 \neq \phi$ and each block $B_j \in A_2$ has $p \geq 3$ vertices, $A_1^{\mathrm{I}} \subseteq A_1$ is set of all endedges with degree 1 then $\gamma_e^{\mathrm{I}}(G) \in \Im$ and $\gamma_{eb}^{\mathrm{I}}(G) \in \Re$.

Proof : Let , $A_1^{\mathrm{I}} \subseteq A_1$ is set of all endedges with degree 1 in G. Then $A_1^{\mathrm{I}} \cong E_e$ and $\exists$ a minimal edge dominating set F_3 of $\langle E(G) \setminus (A_1^{\mathrm{I}} \cup N(A_1^{\mathrm{I}}))\rangle$ such that $F_3 \cup A_1^{\mathrm{I}}$ forms γ_e^{I}– set $\in \Im$. Then $|F_3 \cup A_1^{\mathrm{I}}| = \gamma_e^{\mathrm{I}}(G)$.

In $B(G)$, $|X_1| = |A_1|$ and $\exists$ a minimal edge dominating set $X_2^{\mathrm{I}} \subset X_2$ of induced Subgraph $\langle E(B(G)) \setminus (X_1 \cup N(X_1))\rangle$ such that $X_1 \cup X_2^{\mathrm{I}}$ forms γ_{eb}^{I}- set $\in \Re$.

Then $|X_1 \cup X_2^{\mathrm{I}}| = \gamma_{eb}^{\mathrm{I}}(G)$.

Lemma 10: If $A_1 \neq \phi, A_2 \neq \phi$ and each block of A_1 *and* A_2 has $p = 2$ vertices, then $\gamma_e^{\mathrm{I}}(G) \in \Im$ and $\gamma_{eb}^{\mathrm{I}}(G) \in \Re$.

Proof: Since A_1 and A_2 have all edges, $\{A_1 \cup A_2\} \cong E_e$. Then $\exists$ a minimal edge dominating set F_5 of $\langle E(G) \setminus (E_e \cup N(E_e))\rangle$ such that $E_e \cup F_5$ forms γ_e^{I}– set which belongs to $\Im$. Then $|E_e \cup F_5| = \gamma_e^{\mathrm{I}}(G)$.

In $B(G)$, $|X_1| = |A_1|$ and $\exists$ a minimal edge dominating set $X_2^{\mathrm{I}} \subset X_2$ of induced subgraph $\langle E(B(G)) \setminus (X_1 \cup N(X_1))\rangle$ such that $X_1 \cup X_2^{\mathrm{I}}$ forms γ_{eb}^{I}- set belongs to $\Re$. Then $|X_1 \cup X_2^{\mathrm{I}}| = \gamma_{eb}^{\mathrm{I}}(G)$.

Now we prove **Theorem 4**.

PROOF OF THE THEOREM 4:

Let S_e^{I} be γ_e^{I} – set and S_e be γ_{eb}^{I} - set in G and $B(G)$ respectively . From Lemma 1, lemma 4 and 7 either $A_1 = \phi$ or each block $B_i \in A_1$ has $p \geq 3$ vertices and either $A_2 = \phi$ or each block $B_j \in A_2$ has $p \geq 3$ vertices in G. Then $E_e = \phi$ in G and $\gamma_e^{\mathrm{I}}(G) = \gamma^{\mathrm{I}}(G)$.

In $B(G)$, $|X_1 \cup X_2^{\mathrm{I}}| = \gamma_{eb}^{\mathrm{I}}(G)$ where X_1 is set of all endedges and $X_2^{\mathrm{I}} \subset X_2$ where $X_2 = E(B(G)) \setminus X_1$.

Either $X_1 = \phi$, $X_2^{\mathrm{I}} \subset X_2$ or $X_1 \neq \phi$, $X_2^{\mathrm{I}} \subset X_2$ in $B(G)$. Since $|A_1| = |H_1| = |X_1|$ and $|A_2 \cup A_3| = |H_2 \cup H_3| \geq |X_2|$, Clearly $|X_1 \cup X_2^{\mathrm{I}}| \leq \gamma_e^{\mathrm{I}}(G)$ gives $\gamma_{eb}^{\mathrm{I}}(G) \leq \gamma_e^{\mathrm{I}}(G) + \left\lfloor \frac{m-1}{2} \right\rfloor$.

From lemma 2 and lemma 3, $A_1 = \phi$ and each block $B_j \in A_2^{\mathrm{I}} \subseteq A_2$ has exactly two vertices.

Then $A_2^{\mathrm{I}} \cong E_e \neq \phi$ in G and $S_e^{\mathrm{I}} = A_2^{\mathrm{I}} \cup F_2$ is γ_e^{I} - set where F_2 is minimal edge dominating set of $\langle E(G) \setminus A_2^{\mathrm{I}} \cup N(A_2^{\mathrm{I}})) \rangle$.

In $B(G)$ $X_1 = \phi$ and $S_e = X_2^{\mathrm{I}}$ where $X_2^{\mathrm{I}} \subset X_2$ forms γ_{eb}^{I}- set.

Clearly $\left|X_2^{\mathrm{I}}\right| \leq \left|A_2^{\mathrm{I}} \cup F_2\right|$ gives $\gamma_{eb}^{\mathrm{I}}(G) \leq \gamma_e^{\mathrm{I}}(G) + \left\lfloor\frac{m-1}{2}\right\rfloor$.

From lemma 5, lemma 6 and 9 either each block of $A_1^{\mathrm{I}} \subseteq A_1$ has exactly two vertices and $A_2 = \phi$ or each block of $A_1^{\mathrm{I}} \subseteq A_1$ has exactly two vertices and each block of A_2 has $p \geq 3$ vertices. Then $A_1^{\mathrm{I}} \cong E_e$ and $S_e^{\mathrm{I}} = E_e \cup Q_e$ where Q_e is minimal edge dominating set of $\langle E(G) \setminus (E_e \cup N(E_e)) \rangle$.

In $B(G)$, $X_1 \neq \phi$ *and* $S_e = X_1 \cup X_2^{\mathrm{I}}$ where X_2^{I} is the minimal edge dominating set of induced subgraph $\langle E(B(G)) \setminus (X_1 \cup N(X_1)) \rangle$.

Obviously $|S_e| \leq |S_e^{\mathrm{I}}| + \left\lfloor\frac{m-1}{2}\right\rfloor$ where m is number of end blocks of G.

From Lemma 8, each block $B_i \in A_1$ has $p \geq 3$ vertices and each B_j of $A_2^{\mathrm{I}} \subseteq A_2$ has exactly 2 vertices. Then $A_2^{\mathrm{I}} \cong E_e$ and $S_e^{\mathrm{I}} = E_e \cup Q_e$ where Q_e is minimal edge dominating set of $\langle E(G) \setminus (E_e \cup N(E_e)) \rangle$ in G.

In $B(G)$, $S_e = X_1 \cup X_2^{\mathrm{I}}$ where X_2^{I} is the minimal edge dominating set of induced subgraph $\langle E(B(G)) \setminus (X_1 \cup N(X_1)) \rangle$. Then $|S_e| \leq |S_e^{\mathrm{I}}| \leq |S_e^{\mathrm{I}}| + \left\lfloor\frac{m-1}{2}\right\rfloor$ gives the result.

From lemma 10, each block of A_1 and A_2 has exactly two vertices. Then $A_1 \cup A_2 = E_e$ in G. Hence $S_e^{\mathrm{I}} = E_e \cup Q_e$ where Q_e is minimal edge dominating set of $\langle E(G) \setminus (E_e \cup N(E_e)) \rangle$.

In $B(G)$, $S_e = X_1 \cup X_2^{\mathrm{I}}$ where where $|A_1| = |X_1|$ and $X_2^{\mathrm{I}} \subset X_2$. Clearly, $|S_e| \leq |S_e^{\mathrm{I}}|$ gives $\gamma_{eb}^{\mathrm{I}}(G) \leq \gamma_e^{\mathrm{I}}(G) \leq \gamma_e^{\mathrm{I}}(G) + \left\lfloor\frac{m-1}{2}\right\rfloor$.

Theorem 5 : Every endblock adjacent to exactly one block in G is in every γ_{eb}^{I} – set.

Proof : Set of endblocks, each one is adjacent to exactly one block in G forms a set $H_e \subseteq H$ in $B(G)$ where H_e is set of all end vertices of degree 1 and H is set of all vertices in $B(G)$. Clearly $|H_e| = |X_1|$ where X_1 is set of all endedges belongs to γ_{eb}^{I} – set. Hence the result.

Further theorems provide relations between $\gamma(B(G))$, $\gamma_{eb}^{\mathrm{I}}(G)$ and number of blocks n of G.

Theorem 6: For any connected graph G with $n \geq 2$ blocks, $\gamma(B(G)) \leq \gamma_{eb}^{\mathrm{I}}(G)$.

Proof: Let $H = H_e \cup H_n$ be set of all vertices in $B(G)$ where H_e is set of all endvertices and H_n is set of all nonendvertices. Let X_1 is set of all endedges and $X_2 = E(B(G)) \setminus X_1$ in $B(G)$. We consider the following cases.

Case 1: Suppose $H_e = \emptyset, H_n \neq \emptyset$. Then $X_1 = \emptyset$ .Let $D_b = \{ b_i \}$, $i < n$ be the vertex dominating set of $B(G)$. Let $F \subset X_2^{\mathrm{I}}$ where X_2^{I} is the minimal edge dominating set of $B(G)$ and $X_2^{\mathrm{I}} = F \cup Q_f$ where Q_f is the minimal edge dominating set of $\langle E(B(G)) \setminus (F \cup N(F)) \rangle$ such that F *and* $N(F)$ are incident with $b_i \in D_b$. Since $X_1 = \emptyset$, each vertex of D_b is associated with at least one edge of X_2^{I} clearly $|D_b| \leq |X_2^{\mathrm{I}}|$ gives the result.

Case 2: Suppose $H_e \neq \emptyset$, $H_n \neq \emptyset$. We consider following subcases.

Subcase 2.1: Suppose $B(G)$ has exactly one cutvertex. Then $X_1 \neq \emptyset$, $X_2 = \emptyset$ or
$X_1 \neq \emptyset$, $X_2 \neq \emptyset$ or $X_1 = \emptyset$, $X_2 \neq \emptyset$.

Clearly $|D_b| = 1 \leq |X_1|$ or $|X_1 \cup X_2^{\mathrm{I}}|$ or $|X_2^{\mathrm{I}}|$ gives $\gamma(B(G)) \leq \gamma_{eb}^{\mathrm{I}}(G)$.

Subcase 2.2: Suppose $B(G)$ has more than one cutvertices. Then $X_1 \neq \emptyset$, $X_2 \neq \emptyset$.

Let $N(H_e) = H_e^{\mathrm{I}}$. Then $D_b = H_e^{\mathrm{I}} \cup H_n^{\mathrm{I}}$ where H_n^{I} is γ – set of $\langle H_n \setminus (N(H_e^{\mathrm{I}}) \cup H_e^{\mathrm{I}}) \rangle$ and $X_1 \cup X_2^{\mathrm{I}}$ forms γ_{eb}^{I} – set where X_2^{I} is γ^{I} – set of $\langle X_2 \setminus (X_1 \cup N(X_1)) \rangle$.

Clearly $|H_e^{\mathrm{I}} \cup H_n^{\mathrm{I}}| \leq |X_1 \cup X_2^{\mathrm{I}}|$ gives the result.

Theorem 7: For any connected graph G with $n \geq 2$ blocks, $((B(G)) + \gamma_{eb}^{\mathrm{I}}(G) \leq n$.

Proof: We consider the following cases.

Case 1: Suppose $B(G)$ has endedges. Let $X_1 = \{q_1, q_2, \ldots\ldots, q_m\}$ be set of all endedges in $B(G)$. Let $\{E(B(G)) \setminus X_1\} = X_2$ and $X_2^{\mathrm{I}} \subseteq X_2$ is γ^{I} - set of $\langle X2 \setminus (X_1 \cup N(X_1)) \rangle$. Then $X_1 \cup X_2^{\mathrm{I}}$ forms γ_{eb}^{I} – set in $B(G)$.

Let $X_1^{\mathrm{I}} = \{q_1, q_2, \ldots\ldots, q_i\}$, $i \leq m$ be the set of edges adjacent to X_1. Then $H_2^{\mathrm{I}} = \{b_i\}$, $i \leq m$ denote the γ – set of the induced subgraph $\langle X_1 \cup X_1^{\mathrm{I}} \rangle$.

Further let $E(B(G)) \setminus X_1 \cup X_2^{\mathrm{I}} = X_3$ such that $H_3 = \{ b_j \}$, $j < n$ be the set of vertices incident to the edges of $\langle X_3 \rangle$ but not to the edges of $\langle X_1 \cup X_1^{\mathrm{I}} \rangle$.

Suppose $H_3^{\mathrm{I}} \subseteq H_3$ denotes minimal vertex dominating set of $\langle X_3 \rangle$. Then $H_2^{\mathrm{I}} \cup H_3^{\mathrm{I}}$ is minimal vertex dominating set of $B(G)$.

Clearly, $|X_1 \cup X_2^{\mathrm{I}}| + |H_3^{\mathrm{I}} \cup H_2^{\mathrm{I}}| \leq n$. Hence $\gamma(B(G)) + \gamma_{eb}^{\mathrm{I}}(G) \leq n$.

Case 2 : If $B(G)$ has no endedges, let D_b be the minimal vertex dominating set of $B(G)$. Let $S_e = \{q_1, q_2, \ldots\ldots\ldots, q_l\}$, $l < n$ be the γ_{eb}^{I}- set of $B(G)$.

Suppose $D_2 = \{ b_t \}$, $t < n$ be the set of vertices incident to the edges of S_e.

Assume $\forall\, b_i \in D_b$ are incident with some $q_i \in S_e$, $D_2 \setminus D = D_2^{\mathrm{I}}$ and $D_2^{\mathrm{I}} \cong V - D_b$, then $|D_2^{\mathrm{I}}| + |D_b| = n$ otherwise $|D_2^{\mathrm{I}}| + |D_b| < n$.

Hence from all the cases $\gamma(B(G)) + \gamma_{eb}^{\mathrm{I}}(G) \leq n$.

Theorem 8: For any connected graph G, $\gamma_{eb}^{\mathrm{I}}(G) \leq \left\lfloor \frac{p}{2} \right\rfloor$

Proof : Let $S_e = \{q_1, q_2, \dots\dots\dots, q_l\}$ be the γ_{eb}^{I}- set of $B(G)$ and $X_1 = \{q_1, q_2, \dots\dots\dots, q_m\}$ be the set of all endedges in $B(G)$, $X_2 = E(B(G)) \setminus X_1$. We consider the following cases.

Case 1 : If $X_1 = \phi$, then $\exists$ a set $X_2^{\mathrm{I}} \subseteq X_2$ such that every edge of $X_2 \setminus X_2^{\mathrm{I}}$ is adjacent to at least one edge of X_2^{I}. Hence X_2^{I} forms γ_{eb}^{I}- set of $B(G)$ and since each block of G contains at least two vertices, the result is obvious.

Case 2: If $X_1 \cong S_e$, then clearly $|X_1| \leq \left\lfloor \frac{p}{2} \right\rfloor$ because each $q_i \in X_1$ contains at least two blocks of G and each block of G has at least two vertices.

Case 3: If $X_1 \subset S_e$, then $X_1 \cup X_2^{\mathrm{I}}$ forms γ_{eb}^{I}- set where $X_2^{\mathrm{I}} \subset X_2$ is the dominating set of $\langle X_2 \setminus (X_1 \cup N(X_1)) \rangle$. Clearly $|X_1 \cup X_2^{\mathrm{I}}| = |S_e| \leq \left\lfloor \frac{p}{2} \right\rfloor$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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