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SUZUKI-TYPE FIXED POINT THEOREM IN b-METRIC-LIKE SPACES AND ITS APPLICATION TO INTEGRAL EQUATIONS

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Abstract. Recently, Alghamdi et al. [1] introduced and studied a new generalization of metric-like space and b-metric space which is called a b-metric-like space. In 2013, N. Shobkolaei et al. [16] proved some Suzuki-type fixed point results in the set of metric-like spaces. In this paper, we extend and generalize Suzuki-type fixed point theorem in the set of b-metric-like space and establish certain results as corollaries. Also, many examples and an application to integral equations are presented to verify the effectiveness and applicability of our main results.

Keywords: partial metric space; metric-like space; b-metric space; complete b-metric-like space; fixed point.

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1. Introduction

One of the main tools in fixed point theory is the Banach contraction principle proved by Banach in 1922 [4]. There exists many generalizations of this theorem in the literature.

Several mathematicians have defined and studied various generalizations of metric spaces. In 1994, Matthews [12] introduced the notion of partial metric space and generalized Banach

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contraction mapping theorem in such spaces. After that, many authors have studied fixed point results in partial metric spaces. The notion of b-metric space was established by Bakhtin [3] and Czerwik [5]. Since then several papers have dealt with fixed point theory for single-valued and multi-valued operators in b-metric spaces (see [6], [8], [9], [10], [11], [13], [14], etc). In 2012, Amini-Harandi [2] introduced the concept of metric-like space, which is an interesting generalization of partial metric space. Recently, Alghamdi et al. [1] introduced and studied a new generalization of metric-like space and b-metric space which is called a b-metric-like space.

In 2008, Suzuki [15] introduced an interesting generalization of Banach fixed point theorem in a complete metric space. Since then many authors have extended Suzuki's result in various spaces. In 2013, N. Shobkolaei et al. [16] proved some Suzuki-type fixed point results in the setup of metric-like spaces. The aim of this paper is to extend and generalize Suzuki-type fixed point theorem in the setup of b-metric-like spaces and derive certain results as corollaries. Also, many examples and an application to integral equations are provided in support of our main results. Our fixed point results generalize and improve some well-known results in metric-like spaces and b- metric spaces proved in the literature.

2. Preliminaries

Throughout the article, we denote by $\mathbb{R}$, the set of all real numbers, by $\mathbb{R}^+$, the set of all nonnegative real numbers and by $\mathbb{N}$, the set of all natural numbers.

Definition 2.1. [12] A mapping $p : X \times X \to \mathbb{R}^+$, where X is a nonempty set, is said to be a partial metric on X if for any $x,y,z \in X$ the following conditions hold true:

(P1) $x = y$ if and only if $p(x,x) = p(y,y) = p(x,y)$;

(P2) $p(x,x) \leq p(x,y)$;

(P3) $p(x,y = p(y,x)$;

(P4) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

The pair (X, p) is then called a partial metric space.

Definition 2.2. [2] A mapping $\sigma : X \times X \to \mathbb{R}^+$, where X is a nonempty set, is said to be a metric-like on X if for any $x, y, z \in X$ the following conditions hold true:

(L1) $\sigma(x,y) = 0 \Rightarrow x = y$;

(L2) $\sigma(x,y) = \sigma(y,x)$;

(L3) $\sigma(x,z) \leq \sigma(x,y) + \sigma(y,z)$.

The pair (X, σ) is then called a metric-like space. A metric-like on X satisfies all of the conditions of a metric except that $\sigma(x,x)$ may be positive for $x \in X$.

Remark 2.3. Every partial metric space is a metric-like space but not conversely in general (see [2]).

Definition 2.4. [5] A b-metric on a nonempty set X is a function $d : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$ and a constant $s \geq 1$ the following conditions hold true:

(d1) $d(x,y) = 0 \Leftrightarrow x = y$;

(d2) $d(x,y) = d(y,x)$;

(d3) $d(x,y) \leq s[d(x,z) + d(z,y)]$.

The pair (X, d) is called a b-metric space.

Definition 2.5. [1] A b-metric-like on a nonempty set X is a function $\sigma_b : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$ and a constant $s \geq 1$ the following three conditions hold true:

(B1) $\sigma_b(x,y) = 0 \Rightarrow x = y$;

(B2) $\sigma_b(x,y) = \sigma_b(y,x)$;

(B3) $\sigma_b(x,y) \leq s(\sigma_b(x,z) + \sigma_b(z,y))$.

The pair (X, σ_b) is called a b-metric-like space.

Remark 2.6. Every metric-like space and b-metric space is a b-metric-like space but converse need not be true.

We give the following example in support of above remark.

Example 2.7. [1] Let $X = [0, \infty)$. Define a function $\sigma_b : X^2 \to \mathbb{R}^+$ by

$$\sigma_b(x,y) = (x+y)^2.$$

Clearly, (X,σ_b) is not a b-metric or metric-like space. In fact, for all $x,y,z \in X$,

$$\begin{aligned}\sigma_b(x,y) &= (x+y)^2 \leq (x+z+z+y)^2\\ &= (x+z)^2+(z+y)^2+2(x+z)(z+y)\\ &\leq 2((x+z)^2+(z+y)^2)\\ &= 2(\sigma_b(x,z)+\sigma_b(z,y))\end{aligned}$$

and so (B3) holds. Clearly, (B1) and (B2) hold. Thus, (X,σ_b) is a b-metric-like space with constant $s=2$.

We now give some more examples of b-metric-like space.

Example 2.8. [7] Let $X=\mathbb{R}$. Define a function $\sigma_b : X^2 \to \mathbb{R}^+$ by

$$\sigma_b(x,y) = (x^2+y^2)^2.$$

Then (X,σ_b) is a b-metric-like space with constant $s=2$.

Example 2.9. [1] Let $C_b(X) = \{f : X \to R : sup|f(x)| < \infty\}$.The function $\sigma_b : X\times X \to \mathbb{R}^+$, defined by

$$\sigma_b(f,g) = \sqrt[3]{sup(|f(x)|+|g(x)|)^3}$$

for all $f,g \in C_b(X)$, is a b-metric-like with constant $s=\sqrt[3]{4}$, and so $(X,\sigma_b,\sqrt[3]{4})$ is a b-metric-like space.

For this, note that if x,y are two nonnegative real numbers, then $(x+y)^3 \leq 4(x^3+y^3)$ and $\sqrt[3]{x+y} \leq \sqrt[3]{x}+\sqrt[3]{y}$.

This implies that

$\sigma_b(f,g) \leq \sqrt[3]{4}(\sigma_b(f,h)+\sigma_b(h,g))$ for all $f,g,h \in C_b(X)$.

Example 2.10. [1] Let $X=[0,\infty)$. Define a function $\sigma_b : X^2 \to \mathbb{R}^+$ by

$$\sigma_b(x,y) = (\max\{x,y\})^2.$$

Then (X,σ_b) is a b-metric-like space with constant $s=2$. Clearly, (X,σ_b) is not a b-metric or metric-like space.

Example 2.11. [7] Let $X=\mathbb{R}$ and $c\in\mathbb{R}$. Define a function $\sigma_b : X^2 \to \mathbb{R}^+$ by

$$\sigma_b(x,y) = (|x-c|+|y-c|)^2.$$

Then (X,σ_b) is a b-metric-like space with constant $s=2$.

Now we define convergent and Cauchy sequences in b-metric-like spaces.

Definition 2.12. [1] Let (X,σ_b) be a b-metric-like space with constant $s\geq 1$ and let $\{x_n\}$ be a sequence in X. A point $x\in X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim\limits_{n\to\infty}\sigma_b(x,x_n)=\sigma_b(x,x)$, and we say that the sequence $\{x_n\}$ is convergent to x and denote it by $x_n\to x$ as $n\to\infty$.

Definition 2.13. [1] Let (X,σ_b) be a b-metric-like space with constant $s\geq 1$.

(1) A sequence $\{x_n\}$ is called Cauchy if and only if $\lim\limits_{m,n\to\infty}\sigma_b(x_n,x_m)$ exists and is finite.

(2) A b-metric-like space (X,σ_b) is said to be complete if and only if every Cauchy sequence $\{x_n\}$ in X converges to $x\in X$ so that $\lim\limits_{m,n\to\infty}\sigma_b(x_n,x_m)=\sigma_b(x,x)=\lim\limits_{n\to\infty}\sigma_b(x_n,x)$.

Remark 2.14. In a b-metric-like space, limit of a convergent sequence is not necessarily unique and a convergent sequence need not be a Cauchy sequence.

Example 2.15. Let $X=[0,\infty)$ and $\sigma_b=(\max\{x,y\})^2$ for all $x,y\in X$.

Let $x_n=\begin{cases}0 & \text{if n is odd}\\ 1 & \text{if n is even}\end{cases}$

For any $x\geq 1$, $\lim\limits_{n\to\infty}\sigma_b(x,x_n)=\lim\limits_{n\to\infty}(\max\{x_n,x\})^2=x^2=\sigma_b(x,x)$.

Therefore, it is a convergent sequence and $x_n\to x \quad \forall \quad x\geq 1$.

That is, limit of the sequence is not unique.

Also, $\lim\limits_{m,n\to\infty}\sigma_b(x_m,x_n)$ does not exist. Thus, it is not a Cauchy sequence.

Proposition 2.16. [1] Let (X,σ_b) be a b-metric-like space with constant $s\geq 1$ and let $\{x_n\}$ be a sequence in X such that $\lim\limits_{n\to\infty}\sigma_b(x_n,x)=0$. Then

(A) x is unique;

(B) $\frac{1}{s}\sigma_b(x,y)\leq\lim\limits_{n\to\infty}\sigma_b(x_n,y)\leq s\sigma_b(x,y)$ for all $y\in X$.

3. Main results

Theorem 3.1. *Let (X,σ_b) be a complete b-metric-like space with constant $s\geq 1$. Let $T:X\to X$ be a self map and let $\theta=\theta:[0,1)\to(\frac{1}{s+1},1]$ be defined by*

$$\theta(r) = \begin{cases} 1 & ,0 \le r \le \frac{\sqrt{5}-1}{2} \\ \frac{1-r}{r^2} & ,\frac{\sqrt{5}-1}{2} \le r \le r_s \\ \frac{1}{s+r} & ,r_s \le r < 1 \end{cases}$$

where $r_s = \frac{1-s+\sqrt{1+6s+s^2}}{4}$ *is the positive solution of* $\frac{1-r}{r^2} = \frac{1}{s+r}$*. If there exists* $r \in [0,1)$ *such that for each* $x, y \in X$,

$$\theta(r)\sigma_b(x,Tx) \le \sigma_b(x,y) \Rightarrow \sigma_b(Tx,Ty) \le \frac{r}{s^2}\sigma_b(x,y) \tag{3.1}$$

then T *has a unique fixed point* $z \in X$ *and for each* $x \in X$*, the sequence* $\{T^n x\}$ *converges to* z*.*

Proof. First note that $\theta(r) \le 1$ which implies that

$$\theta(r)\sigma_b(x,Tx) \le \sigma_b(x,Tx) \tag{3.2}$$

Therefore, it follows from (3.1) that for each $x \in X$

$$\sigma_b(Tx,T^2x) \le \frac{r}{s^2}\sigma_b(x,y) \tag{3.3}$$

Let $x_0 \in X$ be arbitrary. Define a sequence $\{x_n\}$ by $x_n = Tx_{n-1} = T^n x_0$ for $n \in \mathbb{N}$.
From (3.3), we have
$\sigma_b(x_n,x_{n+1}) = \sigma_b(Tx_{n-1},T^2x_{n-1}) \le \frac{r}{s^2}\sigma_b(x_{n-1},Tx_{n-1}) = \frac{r}{s^2}\sigma_b(x_{n-1},x_n) \le \ldots \le \frac{r^n}{s^{2n}}\sigma_b(x_0,x_1)$.
For $m,n \in \mathbb{N}, m \ge n$, we have

$$\begin{aligned} \sigma_b(x_n,x_m) &\le s\sigma_b(x_n,x_{n+1}) + s^2\sigma_b(x_{n+1},x_{n+2}) + \ldots + s^{m-n-1}\sigma_b(x_{m-1},x_m) \\ &\le (\frac{sr^n}{s^{2n}} + \frac{s^2r^{n+1}}{s^{2n+2}} + \ldots + \frac{s^{m-n-1}r^{m-1}}{s^{2m-2}})\sigma_b(x_0,x_1) \\ &< \frac{r^n}{s^{2n-1}}(1 + \frac{r}{s} + (\frac{r}{s})^2 + \ldots)\sigma_b(x_0,x_1) \\ &= \frac{r^n}{s^{2n-1}}\frac{1}{1-\frac{r}{s}}\sigma_b(x_0,x_1) \to 0 \quad as \quad n \to \infty. \end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $z \in X$ such that $\lim_{n\to\infty}\sigma_b(x_n,z) = \sigma_b(z,z) = \lim_{m,n\to\infty}\sigma_b(x_m,x_n) = 0$.
That is, $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = z$.
We will prove that $Tz = z$.

Put $x = T^{n-1}z$ in (3.3),we get

$$\sigma_b(T^n z, T^{n+1}z) \leq \frac{r}{s^2}\sigma_b(T^{n-1}z, T^n z) \tag{3.4}$$

holds for each $n \in \mathbb{N}$ (where $T^0 z = z$). It follows by induction that

$$\sigma_b(T^n z, T^{n+1}z) \leq \frac{r^n}{s^{2n}}\sigma_b(z, Tz). \tag{3.5}$$

We now show that

$$\sigma_b(z, Tx) \leq r\sigma_b(z, x). \tag{3.6}$$

holds for each $x \neq z$.

Since $\lim \sigma_b(x_n, Tx_n) = \sigma_b(z, z) \neq 0$ and by Proposition 2.16, $\lim \sigma_b(x_n, x) \neq 0$, therefore there exists n_0 such that $\theta(r)\sigma_b(x_n, Tx_n) \leq \sigma_b(x_n, x)$ holds for every $n \geq n_0$.

Assumption (3.1) implies that for such n,

$$\sigma_b(Tx_n, Tx) \leq \frac{r}{s^2}\sigma_b(x_n, x). \tag{3.7}$$

Now taking limit as $n \to \infty$, we get

$\frac{1}{s}\sigma_b(z, Tx) \leq \lim \sigma_b(x_{n+1}, Tx) \leq \frac{r}{s^2}\lim \sigma_b(x_n, x) \leq \frac{r}{s}\sigma_b(z, x)$

$\Rightarrow \sigma_b(z, Tx) \leq r\sigma_b(z, x)$.

Next we will show that for each $n \in \mathbb{N}$,

$$\sigma_b(T^n z, z) \leq \sigma_b(Tz, z) \tag{3.8}$$

For $n = 1$, this relation is obvious. Suppose that it holds for some $m \in \mathbb{N}$.

If $T^m z = z$, then $T^{m+1}z = Tz$ and $\sigma_b(T^{m+1}z, z) = \sigma_b(Tz, z)$.

If $T^m z \neq z$, then

$\sigma_b(T^{m+1}z, z) \leq r\sigma_b(T^m z, z) \leq r\sigma_b(Tz, z) \leq \sigma_b(Tz, z)$.

The result follows by induction.

Now, in order to prove that $Tz = z$, we suppose on the contrary that $Tz \neq z$ and consider the two possible cases:

Case I: $0 \leq r \leq r_s$ $\left(\theta(r) \leq \frac{1-r}{r^2}\right)$

We will first prove that

$$\sigma_b(T^n z, Tz) \leq \frac{r}{s^2}\sigma_b(Tz, z) \quad for \quad n \geq 2. \tag{3.9}$$

For $n=2$, it follows from (3.4).

Now suppose that (3.9) holds for some $n>2$. Then

$\sigma_b(Tz,z) \le [\sigma_b(z,T^nz)+\sigma_b(T^nz,Tz)]$

$\le s\sigma_b(z,T^nz)+\frac{r}{s}\sigma_b(Tz,z)$

$\le s\sigma_b(z,T^nz)+r\sigma_b(Tz,z)$

$\Rightarrow (1-r)\sigma_b(z,Tz) \le s\sigma_b(z,T^nz)$.

Now using (3.5), we have

$\theta(r)\sigma_b(T^nz,T^{n+1}z) \le \frac{1-r}{r^n}\sigma_b(T^nz,T^{n+1}z)$

$\le \frac{1-r}{s^{2n}}\sigma_b(z,Tz)$

$\le \frac{1}{s^{2n-1}}\sigma_b(z,T^nz) \le \sigma_b(z,T^nz)$

Therefore, by assumption (3.1), we get

$\sigma_b(Tz,T^{n+1}z) \le \frac{r}{s^2}\sigma_b(z,T^nz) \le \frac{r}{s^2}\sigma_b(z,Tz)$.

Hence the claim follows by induction.

Now $Tz \neq z$ and (3.9) implies that $T^nz \neq z$ for each $n \in \mathbb{N}$.

Hence, (3.6) implies that

$\sigma_b(z,T^{n+1}z) \le r\sigma_b(z,T^nz) \le r^2\sigma_b(z,T^{n-1}z) \le \ldots \le r^n\sigma_b(z,T^nz)$.

Hence $\lim\limits_{n\to\infty}\sigma_b(z,T^{n+1}z)=0=\sigma_b(z,z)$.

Thus, $T^nz \to z$.

Using this and Proposition 2.16 in (3.9) we get

$$\frac{1}{s^2}\sigma_b(z,Tz) \le \frac{r}{s^2}\sigma_b(Tz,z) \quad as \quad n\to\infty.$$

That is, $\sigma_b(z,Tz)=0$ which is a contradiction.

Case II: $r_s \le r < 1$ $(\theta(r)=\frac{1}{s+r})$.

We will prove that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\theta(r)\sigma_b(x_{n_k},Tx_{n_k})=\sigma_b(x_{n_k},x_{n_k+1}) \le \sigma_b(x_{n_k},z) \tag{3.10}$$

holds for each $k \in \mathbb{N}$.

Now from (3.3), we know that

$$\sigma_b(x_n,x_{n+1}) \le \frac{r}{s^2}\sigma_b(x_{n-1},x_n) \quad holds \quad for \quad each \quad n\in\mathbb{N}.$$

Suppose that

$$\frac{1}{r+s}\sigma_b(x_{n-1},x_n) > \sigma_b(x_{n-1},z),$$
$$\frac{1}{r+s}\sigma_b(x_n,x_{n+1}) > \sigma_b(x_n,z)$$

holds for some $n \in \mathbb{N}$.

Then,

$\sigma_b(x_{n-1},x_n) \leq s(\sigma_b(x_{n-1},z)+\sigma_b(x_n,z)) < \frac{s}{s+r}\sigma_b(x_{n-1},x_n)+\frac{s}{s+r}\sigma_b(x_n,x_{n+1})$
$\leq \frac{s}{s+r}\sigma_b(x_{n-1},x_n)+\frac{r}{s(s+r)}\sigma_b(x_{n-1},x_n) \leq \sigma_b(x_{n-1},x_n)$, which is not possible.

Hence, either

$$\theta(r)\sigma_b(x_{2n-1},x_{2n}) \leq \sigma_b(x_{2n-1},z)$$
$$or\ \theta(r)\sigma_b(x_{2n},x_{2n+1}) \leq \sigma_b(x_{2n-1},z)$$

holds for each $n \in \mathbb{N}$.

In otherwords, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that (3.10) holds for each $k \in \mathbb{N}$.

But assumption (3.1) implies that

$\sigma_b(Tx_{n_k},Tz) \leq \frac{r}{s^2}\sigma_b(x_{n_k},z)$

ie, $\sigma_b(x_{n_k+1},Tz) \leq \frac{r}{s^2}\sigma_b(x_{n_k},z)$.

Taking limit as $k \to \infty$, we get

$\sigma_b(z,Tz) \leq 0$, which is a contradiction.

Thus, we have $Tz = z$.

That is , z is a fixed point of T.

Uniqueness: Let y,z be two fixed points of T such that $y \neq z$. Then

$\sigma_b(y,z) = \sigma_b(y,Tz) \leq r\sigma_b(y,z)$ (using (3.6)), which is not possible.

Therefore, $y = z$.

Now we give an example to support our result.

Example 3.2. Let $X = [0,\infty)$. Define $\sigma_b : X^2 \to \mathbb{R}^+$ by $\sigma_b(x,y) = (x+y)^2\ \forall \quad x,y \in X$.

Then (X,σ_b) is a complete b-metric-like space with constant $s = 2$. Let $T : X \to X$ be a map defined by $Tx = \ln(1+\frac{x}{8})\ \forall\ x \in X$.

Take $r = 0.8$. Then $\theta(r) = \frac{1}{2.8}$. Now,using the Mean Value Theorem for any $x, y \in X$ with $x \leq y$ and that $Tx \leq \frac{x}{8}$, we have

$\theta(r)\sigma_b(x,Tx) = \frac{1}{2.8}(x+\ln(1+\frac{x}{8}))^2 \leq \frac{1}{2.8}(x+\frac{x}{8})^2 < x^2 \leq (x+y)^2 = \sigma_b(x,y)$.

On the other hand, we have

$\sigma_b(Tx,Ty) = (\frac{x}{8}+\frac{y}{8})^2 \leq \frac{0.8}{4}(x+y)^2 = \frac{r}{s^2}\sigma_b(x,y)$.

Thus T satisfies all the hypothesis of Theorem 3.1 and hence T has a unique fixed point ie, $x = 0$.

Theorem 3.3. *Let (X, σ_b) be a complete b-metric-like space with constant $s \geq 1$. Let $S, T : X \to X$ be two mappings. Suppose that there exists $r \in [0,1)$ such that*

$$\max\{\sigma_b(Sx,TSx), \sigma_b(Tx,STx)\} \leq \frac{r}{s}\min\{\sigma_b(x,Sx), \sigma_b(x,Tx)\} \tag{3.11}$$

for every $x \in X$ and that

$$\alpha(y) = inf\{\sigma_b(x,y) + \min\{\sigma_b(x,Sx), \sigma_b(x,Tx)\} : x \in X\} > 0 \tag{3.12}$$

for every $y \in X$ such that y is not a common fixed point of S and T.Then there exists $z \in X$ such that $z = Sz = Tz$. Moreover, if $u = Su = Tu$, then $\sigma_b(u,u) = 0$.

Proof. Let $x_0 \in X$ be arbitrary and define a sequence $\{x_n\}$ by

$$x_n = \begin{cases} Sx_{n-1} & \text{if n is odd} \\ Tx_{n-1} & \text{if n is even} \end{cases} \tag{3.13}$$

Now if n is odd, we have

$$\begin{aligned}
\sigma_b(x_n,x_{n+1}) &= \sigma_b(Sx_{n-1},Tx_n) = \sigma_b(Sx_{n-1},TSx_{n-1}) \\
&\leq \max\{\sigma_b(Sx_{n-1},TSx_{n-1}), \sigma_b(Tx_{n-1},STx_{n-1})\} \\
&\leq \frac{r}{s}\min\{\sigma_b(x_{n-1},Sx_{n-1}), \sigma_b(x_{n-1},Tx_{n-1})\} \\
&\leq \frac{r}{s}\sigma_b(x_{n-1},Sx_{n-1}) = \frac{r}{s}\sigma_b(x_{n-1},x_n).
\end{aligned}$$

If n is even, then we have

$$\begin{aligned}\sigma_b(x_n,x_{n+1}) &= \sigma_b(Tx_{n-1},Sx_n) = \sigma_b(Tx_{n-1},STx_{n-1})\\ &\leq \max\{\sigma_b(Sx_{n-1},TSx_{n-1}),\sigma_b(Tx_{n-1},STx_{n-1})\}\\ &\leq \frac{r}{s}\min\{\sigma_b(x_{n-1},Sx_{n-1}),\sigma_b(x_{n-1},Tx_{n-1})\}\\ &\leq \frac{r}{s}\sigma_b(x_{n-1},Tx_{n-1}) = \frac{r}{s}\sigma_b(x_{n-1},x_n).\end{aligned}$$

Thus for any $n \in \mathbb{N}$,we have

$$\sigma_b(x_n,x_{n+1}) \leq \frac{r}{s}\sigma_b(x_{n-1},x_n). \tag{3.14}$$

Repeating (3.14), we obtain

$$\sigma_b(x_n,x_{n+1}) \leq \frac{r^n}{s^n}\sigma_b(x_0,x_1).$$

For $m,n \in \mathbb{N}, m \geq n$, we have

$$\begin{aligned}\sigma_b(x_n,x_m) &\leq s\sigma_b(x_n,x_{n+1}) + s^2\sigma_b(x_{n+1},x_{n+2}) + \ldots + s^{m-n-1}\sigma_b(x_{m-1},x_m)\\ &\leq (\frac{sr^n}{s^2} + \frac{s^2r^{n+1}}{s^{n+1}} + \ldots + \frac{s^{m-n-1}r^{m-1}}{s^{m-1}})\sigma_b(x_0,x_1)\\ &< \frac{r^n}{s^{n-1}}(1+r+r^2+\ldots)\sigma_b(x_0,x_1)\\ &= \frac{r^n}{s^{n-1}(1-r)}\sigma_b(x_0,x_1) \to 0 \quad as \quad n \to \infty.\end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence. Since X is complete, ther exists $z \in X$ such that $\lim_{n\to\infty}\sigma_b(x_n,z) = \sigma_b(z,z) = \lim_{m,n\to\infty}\sigma_b(x_m,x_n) = 0$.

If z is not a common point of S and T, then by hypothesis (3.12)

$$\begin{aligned}0 &< inf\{\sigma_b(x,z) + \min\{\sigma_b(x,Sx),\sigma_b(x,Tx)\} : x \in X\}\\ &\leq inf\{\sigma_b(x_n,z) + \min\{\sigma_b(x_n,Sx_n),\sigma_b(x_n,Tx_n)\} : n \in \mathbb{N}\}\\ &\leq inf\{\frac{r^n}{s^{n-1}(1-r)}\sigma_b(x_0,x_1) + \sigma_b(x_n,x_{n+1}) : n \in \mathbb{N}\}\\ &\leq inf\{\frac{r^n}{s^{n-1}(1-r)}\sigma_b(x_0,x_1) + \frac{r^n}{s^n}\sigma_b(x_0,x_1) : n \in \mathbb{N}\} = 0,\end{aligned}$$

which is a contradiction.

Therefore, $z = Sz = Tz$.

If $u = Tu = Su$ for some $u \in X$, then

$$\begin{aligned}\sigma_b(u,u) &= \max\{\sigma_b(Su,TSu),\sigma_b(Tu,STu)\}\\ &\leq \frac{r}{s}\min\{\sigma_b(u,Su),\sigma_b(u,Tu)\}\\ &= \frac{r}{s}\min\{\sigma_b(u,u),\sigma_b(u,u)\}\\ &= \frac{r}{s}\sigma_b(u,u).\end{aligned}$$

$\Rightarrow \sigma_b(u,u) = 0.$

Corollary 3.4. *Let (X,σ_b) be a complete b-metric-like space with constant $s \geq 1$. Let $T : X \to X$ be a mapping. Suppose that there exists $r \in [0,1)$ such that*

$$\sigma_b(Tx,T^2x) \leq \frac{r}{s}\sigma_b(x,Tx)$$

for every $x \in X$ and that

$$\alpha(y) = inf\{\sigma_b(x,y)+\sigma_b(x,Tx) : x \in X\} > 0$$

for every $y \in X$ such that $y \neq Ty$. Then there exists $z \in X$ such that $z = Tz$. Moreover, if $u = Tu$, then $\sigma_b(u,u) = 0$.

Proof. Take $S = T$ in Theorem 3.3 .

Example 3.5. Let $X = [0,\infty)$ and $\sigma_b : X^2 \to \mathbb{R}^+$ be defined as

$$\sigma_b(x,y) = (x+y)^2.$$

Then (X,σ_b) is a complete b-metric-like space with $s = 2$. Define $T : X \to X$ by $Tx = \frac{x}{2}$ and $S : X \to X$ by $Sx = \frac{x}{4}$ for all $x \in X$.

Then, $\max\{\sigma_b(Sx,TSx),\sigma_b(Tx,STx)\} = \max\{\sigma_b(\frac{x}{4},\frac{x}{8}),\sigma_b(\frac{x}{2},\frac{x}{8})\} = \frac{25}{64}x^2$.

Now, $\min\{\sigma_b(x,Sx),\sigma_b(x,Tx)\} = \min\{\sigma_b(x,\frac{x}{4}),\sigma_b(x,\frac{x}{2})\} = \frac{25}{16}x^2$.

Here $r = \frac{1}{2}$ and also $\alpha(y) > 0$ for every $y \in X$ such that y is not a common fixed point of S and T.

Thus all the conditions of Theorem 3.3 are satisfied and $x = 0$ is a common fixed point of S and T.

Theorem 3.6. *Let (X,σ_b) be a complete b-metric-like space with constant $s \geq 1$. Let S,T be mappings from X onto itself. Suppose that there exists $r > s$ such that*

$$\min\{\sigma_b(Sx,TSx),\sigma_b(Tx,STx)\} \geq r\max\{\sigma_b(x,Sx),\sigma_b(x,Tx)\}$$

for every $x \in X$ and that

$$\alpha(y) = inf\{\sigma_b(x,y)+\min\{\sigma_b(x,Sx),\sigma_b(x,Tx)\} : x \in X\} > 0 \tag{3.15}$$

for every $y \in X$ such that y is not a common fixed point of S and T.Then there exists $z \in X$ such that $z = Sz = Tz$. Moreover, if $u = Su = Tu$, then $\sigma_b(u,u) = 0$.

Proof. Let x_0 be arbitrary. Since S is onto, ther is an element x_1 such that $x_1 = S^{-1}x_0$, ie, $Sx_1 = x_0$. Now since T is also onto, there is an element x_2 such that $Tx_2 = x_1$. Proceeding in the same way, we can find x_{2n+1} such that $Sx_{2n+1} = x_{2n}$ and x_{2n+2} such that $Tx_{2n+1} = x_{2n+2}$ for $n = 1,2,3,\dots$

Therefore, $x_{2n} = Sx_{2n+1}$ and $x_{2n+1} = Tx_{2n+2}$ for $n = 0,1,2,\dots$

If $n = 2m$, then

$$\begin{aligned}
\sigma_b(x_{n-1},x_n) = \sigma_b(x_{2m-1},x_{2m}) = \sigma_b(Tx_{2m},Sx_{2m+1}) &= \sigma_b(TSx_{2m+1},Sx_{2m+1}) \\
&\geq \min\{\sigma_b(TSx_{2m+1},Sx_{2m+1}),\sigma_b(STx_{2m+1},Tx_{2m+1})\} \\
&\geq r\max\{\sigma_b(Sx_{2m+1},x_{2m+1}),\sigma_b(Tx_{2m+1},x_{2m+1})\} \\
&\geq r\sigma_b(Sx_{2m+1},x_{2m+1}) = \sigma_b(x_{2m},x_{2m+1}) = \sigma_b(x_n,x_{n+1}).
\end{aligned}$$

If $n = 2m+1$, then

$$\begin{aligned}
\sigma_b(x_{n-1},x_n) = \sigma_b(x_{2m},x_{2m+1}) = \sigma_b(Sx_{2m+1},Tx_{2m+2}) &= \sigma_b(STx_{2m+2},Tx_{2m+2}) \\
&\geq \min\{\sigma_b(TSx_{2m+2},Sx_{2m+2}),\sigma_b(STx_{2m+2},Tx_{2m+2})\} \\
&\geq r\max\{\sigma_b(Sx_{2m+2},x_{2m+2}),\sigma_b(Tx_{2m+2},x_{2m+2})\} \\
&\geq r\sigma_b(Tx_{2m+2},x_{2m+2}) = \sigma_b(x_{2m+1},x_{2m+2}) = \sigma_b(x_n,x_{n+1}).
\end{aligned}$$

Thus for any $n \in \mathbb{N}$, we have

$\sigma_b(x_{n-1},x_n) \geq r\sigma_b(x_n,x_{n+1})$

$\Rightarrow \sigma_b(x_n,x_{n+1}) \leq \frac{1}{r}\sigma_b(x_{n-1},x_n) ... \leq \frac{1}{r^n}\sigma_b(x_0,x_1).$

Let $\alpha = \frac{1}{r}$, then $0 < \alpha < 1$. Therefore,

$\sigma_b(x_n, x_{n+1}) \leq \alpha^n \sigma_b(x_0, x_1)$.

Now if $m, n \in \mathbb{N}, m \geq n$, then

$$\begin{aligned}
\sigma_b(x_n, x_m) &\leq s\sigma_b(x_n, x_{n+1}) + s^2\sigma_b(x_{n+1}, x_{n+2}) + \ldots s^{m-n-1}\sigma_b(x_{m-1}, x_m) \\
&\leq (s\alpha^n + s^2\alpha^{n+1} + \ldots)\sigma_b(x_0, x_1) \\
&< s\alpha^n(1 + s\alpha + (s\alpha)^2 \ldots)\sigma_b(x_0, x_1) \\
&= \frac{\alpha^n}{(1 - s\alpha)}\sigma_b(x_0, x_1) \to 0 \quad as \quad n \to \infty.
\end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence. Since X is complete, ther exists $z \in X$ such that $\lim_{n\to\infty} \sigma_b(x_n, z) = \sigma_b(z, z) = \lim_{m,n\to\infty} \sigma_b(x_m, x_n) = 0$.

If z is not a common point of S and T, then by hypothesis (3.15)

$$\begin{aligned}
0 &< inf\{\sigma_b(x, z) + \min\{\sigma_b(x, Sx), \sigma_b(x, Tx)\} : x \in X\} \\
&\leq inf\{\sigma_b(x_n, z) + \min\{\sigma_b(x_n, Sx_n), \sigma_b(x_n, Tx_n)\} : n \in \mathbb{N}\} \\
&\leq inf\{\frac{\alpha^n}{1 - s\alpha}\sigma_b(x_0, x_1) + \sigma_b(x_n, x_{n+1}) : n \in \mathbb{N}\} \\
&\leq inf\{\frac{\alpha^n}{1 - s\alpha}\sigma_b(x_0, x_1) + \alpha^n\sigma_b(x_0, x_1) : n \in \mathbb{N}\} = 0,
\end{aligned}$$

which is a contradiction.

Therefore, $z = Sz = Tz$.

If $u = Tu = Su$ for some $u \in X$, then

$$\begin{aligned}
\sigma_b(u, u) &= \min\{\sigma_b(Su, TSu), \sigma_b(Tu, STu)\} \\
&\geq r\max\{\sigma_b(u, Su), \sigma_b(u, Tu)\} \\
&= r\max\{\sigma_b(u, u), \sigma_b(u, u)\} \\
&= r\sigma_b(u, u).
\end{aligned}$$

$\Rightarrow \sigma_b(u, u) = 0$.

Corollary 3.7. *Let (X, σ_b) be a complete b-metric-like space with constant $s \geq 1$. Let $T : X \to X$ be an onto mapping. Suppose that there exists $r > s$ such that*

$$\sigma_b(Tx, T^2x) \geq r\sigma_b(x, Tx)$$

for every $x \in X$ and that

$$\alpha(y) = inf\{\sigma_b(x,y) + \sigma_b(x,Tx) : x \in X\} > 0$$

for every $y \in X$ such that $y \neq Ty$. Then there exists $z \in X$ such that $z = Tz$. Moreover, if $u = Tu$, then $\sigma_b(u,u) = 0$.

Proof. Take $S = T$ in Theorem 3.6.

Corollary 3.8. *Let (X, σ_b) be a complete b-metric-like space with constant $s \geq 1$.Let T be a continuous mapping from X onto itself. Suppose that there exists $r > s$ such that*

$$\sigma_b(Tx, T^2x) \geq r\sigma_b(x, Tx)$$

for every $x \in X$.Then there exists $z \in X$ such that $z = Tz$. Moreover, if $u = Tu$, then $\sigma_b(u,u) = 0$.

Proof. Suppose that there exists $y \in X$ with $Ty \neq y$ such that $\alpha(y) = inf\{\sigma_b(x,y) + \sigma_b(Tx,x) : x \in X\} = 0$.

Then there exists a sequence $\{x_n\}$ in X such that

$\lim\{\sigma_b(x_n,y) + \sigma_b(Tx_n,x_n)\} = 0$.

Therefore, $\lim \sigma_b(x_n,y) = 0$ and $\lim \sigma_b(Tx_n,x_n) = 0$.

Now, $\sigma_b(y,y) \leq \sigma_b(y,x_n) + \sigma_b(x_n,y)$

$\Rightarrow \sigma_b(y,y) = 0$.

Also, $\sigma_b(Tx_n,y) \leq \sigma_b(Tx_n,x_n) + \sigma_b(X_n,y) \to 0$ as $n \to \infty$.

Since T is continuous, we have

$Ty = T(\lim x_n) = \lim Tx_n = y$, which is a contradiction.

Hence, if $y \neq Ty$, then $\alpha(y) > 0$. Therefore by Corollary (3.7), there exists $z \in X$ such that $z = Tz$.

Example 3.9. Let $X = [0,\infty)$ and $\sigma_b : X \times X \to \mathbb{R}^+$ be defined by

$$\sigma_b(x,y) = (x+y)^2.$$

Then (X, σ_b) is a complete b-metric-like space with $s = 2$. Define $T : X \to X$ by $Tx = 3x$. Then clearly T is onto and continuous. Also for each $x \in X$, we have

$\sigma_b(Tx^2,Tx) = 144x^2 \geq r16x^2 = r\sigma_b(Tx,x)$ where $r = 3,4,\ldots,9$. Thus T satisfies all the conditions of Corollary (3.8) and $x = 0$ is a fixed point of T.

Corollary 3.10. *Let (X,σ_b) be a complete b-metric-like space with constant $s \geq 1$. Let T be a continuous mapping from X onto itself. Suppose that there exists $r > s$ such that*

$$\sigma_b(Tx,T^2x) \geq r\min\{\sigma_b(x,Tx),\sigma_b(y,Ty),\sigma_b(x,y)\} \tag{3.16}$$

for every $x,y \in X$.Then there exists $z \in X$ such that $z = Tz$. Moreover, if $u = Tu$, then $\sigma_b(u,u) =$ 0.

Proof. Replacing y by Tx in (3.16), we get
$\sigma_b(Tx,T^2x) \geq \min\{\sigma_b(x,Tx),\sigma_b(T^2x,Tx),\sigma_b(Tx,x)\}\ \forall x \quad \in X.$
If $Tx = T^2x$, then Tx is a fixed point of T and we are done.
So, now assume that $Tx \neq T^2x$. Since $r > s \geq 1$, it follows that
$\sigma_b(Tx,T^2x) \geq r\sigma_b(x,Tx)$ for all $x \in X$.
Therefore, by Corollary (3.8), we get that T has a fixed point in X.

Remark 3.11. For $s = 1$, we get the corresponding results proved by N. Shobkolaei et al.[16].

4. Application to the existence of solution of integral equations

In this section, we study the existence of solutions of nonlinear integral equations using Theorem 3.1 in b-metric-like space.
Consider the integral equation

$$u(x) = \int_0^a G(x,t)f(t,u(t))\,dt \quad for\ \ all\ \ x \in [0,a] \tag{4.1}$$

where $a > 0, f : [0,a]\times\mathbb{R} \to \mathbb{R}$ and $G : [0,a]\times[0,a] \to [0,\infty)$ are continuous functions on $[0,a]$. Let $X = C([0,a])$ be the set of real valued continuous functions on $[0,a]$. Define a function σ_b on X as

$$\sigma_b(u,v) = \sup_{x\in[0,a]} (|u(x)|+|v(x)|)^2 \quad forall \quad u,v \in X.$$

Clearly (X, σ_b) is a complete b-metric-like space with constant $s = 2$. Let a function θ be defined as in Theorem 3.1 and the mapping $T : X \to X$ be defined by

$$Tu(x) = \int_0^a G(x,t) f(t,u(t))\, dt \ \ for \ \ all \ \ x \in [0,a].$$

Suppose that there exists $r \in [0,1)$ such that for every $t \in [0,a]$ and $u, v \in X$, the inequality

$$\theta(r)\sigma_b(Tu,u) \leq \sigma_b(u,v) \Rightarrow (|f(t,u(t))| + |f(t,v(t))|)^2 \leq \frac{r^2}{16}(|u(t)| + |v(t)|)^2. \tag{4.2}$$

Also, assume that

$$\sup_{x\in[0,a]} \int_0^a G(x,t) \ \ dt \leq 1. \tag{4.3}$$

Theorem 4.1. *Under assumptions (4.2) and (4.3), the integral equation (4.1) has a unique solution in* $C([0,a])$.

Proof. For all $x \in [0,a]$,

$$\begin{aligned}
(|Su(x)| + |Sv(x)|)^2 &= (|\int_0^a G(x,t) f(t,u(t))\, dt| + |\int_0^a G(x,t) f(t,v(t))\, dt|)^2 \\
&\leq (\int_0^a G(x,t)|f(t,u(t))|\, dt + |\int_0^a G(x,t)|f(t,v(t))|\, dt)^2 \\
&= (\int_0^a G(x,t)(|f(t,u(t))| + |f(t,v(t))|)\, dt)^2 \\
&\leq \frac{r}{4}(\int_0^a G(x,t)(|u(t)| + |v(t)|)\, dt)^2 \\
&\leq \frac{r}{4}(\int_0^a G(x,t)(\sigma_b(u,v))^{\frac{1}{2}}\, dt)^2 \\
&\leq \frac{r}{4}(\sigma_b(u,v))(\int_0^a G(x,t)\, dt)^2 \\
&\leq \frac{r}{4}\sigma_b(u,v).
\end{aligned}$$

$\Rightarrow \sup_{x\in[0,a]} (|Tu(x)| + |Tv(x|)^2 \leq \frac{r}{4}\sigma_b(u,v)$.

That is, $\sigma_b(Tu,Tv) \leq \frac{r}{4}\sigma_b(u,v)$.

Thus all the conditions of Theorem 3.1 holds and therefore T has a unique fixed point u in $X = C([0,a])$.

Hence, u is the solution of integral equation (4.1).

Conflict of Interests

The authors declare that there is no conflict of interests.

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