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# MATHEMATICAL FORMULATION OF A CONSTITUTIVE LAMBERT-TYPE DIFFERENTIAL EQUATION FOR PREDICTING THE DYNAMIC RESPONSE OF MATERIALS

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**Abstract:** This paper presents a mathematical formulation of a constitutive Lambert-type differential equation on the basis of the stress decomposition theory in order to predict the dynamic behavior of a variety of materials. The expansion of the nonlinear elastic spring force law required in terms of a generalized form of the Newton's binomial function of deformation provided, under relaxation of stress conditions, the time versus deformation variation as a Chapman-Richards-type growth model. Numerical applications carried out demonstrated successfully the ability of the model to reproduce the S-shaped response of viscoelastic materials. It has been shown that an increase of viscoelastic characteristics, increases significantly the sensitivity of the model, which becomes flexible enough for experimental data fitting.

**Keywords:** Chapman-Richards-type model, constitutive Lambert-type equation, Newton's binomial function, nonlinear time dependent deformation, Riccati equation, viscoelasticity.

**2000 AMS Subject Classification:** 47H17; 47H05; 47H09

## 1. Introduction

The time dependent behavior of viscoelastic materials study has been and continues to be one of the most interesting subjects in mechanics due to various uses of these

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materials in biomedical and engineering branches. Viscoelastic materials exhibit both elastic solid and viscous fluid behaviors. They dissipate energy and are characterized by time and history-dependent properties. Thus, their constitutive stress- strain relationship is time dependent, and may be linear or curvilinear. The linear behavior law is only judicious for materials undergoing small deformations when they are subjected to applied forces. The linear viscoelasticity is usually expressed in the Boltzmann single integral equation or in the differential form. It is well known that, from a point of view of mathematics, the integral constitutive representation is more complex to build than the constitutive differential form. The simplest constitutive differential relation in linear viscoelasticity is derived often from combinations of elastic springs and viscous dashpots arranged in series and/or parallel. But, mechanical responses of viscoelastic materials are in general nonlinear so that, the well-known established linear theory of viscoelasticity must be reasonably replaced by nonlinear theories. However, nonlinear models are more difficult to formulate than linear theories, since these models lead frequently to solve nonlinear differential equations that are generally non-integrable. A satisfactory constitutive equation must take into account elastic, viscous and inertial nonlinearities characterizing the viscoelastic material [1, 2]. In this perspective, the simple classical Maxwell and Voigt models or their different combinations can be modified and extended to higher order stress or strain terms in order to account for material nonlinearities. The obtained model according to Alfrey and Doty [3], is interesting since, it estimates the material properties in terms of differential equations that can be solved for a wide variety of transient conditions. For this, several theoretical viscoelastic models with varying complexities have been developed for predicting the nonlinear time dependent behavior of viscoelastic materials [4]. But, most of these models fail to include the inertia of the material studied in the constitutive equations. Moreover, there are only a few theoretical models that are formulated with constant-value material coefficients so that, the material functions are considered as stress, strain or strain rate dependent. A number of recent successful works are shown to be based on the modification and extension of classical linear viscoelastic models to large deformations [5, 6, 7, 8, 9].

The models [5, 6, 7] take into account only the elastic nonlinearity by insertion of a nonlinear elastic spring force in the classical linear viscoelastic models. Thus, these models are insufficient for a satisfactory rheological estimation of materials. By contrast, the model [9] consisting of a simple nonlinear generalized Maxwell fluid model with constant material coefficients, in which a nonlinear spring obeying to a power-law is connected to a nonlinear dashpot obeying also to a power-law, gives account for both elastic and viscous nonlinearities simultaneously.  The model [9] appears useful to represent the nonlinear time dependent behavior of viscoelastic materials. However, in [9] the inertial contribution is also neglected. In the model [8], the elastic, viscous and inertial nonlinearities are simultaneously introduced. The model [8] tried to provide analytically a complete estimate of the viscoelastic material behavior. The model [8] is formulated on the basis of the method presented previously by Bauer [2] for a complete characterization of viscoelastic arterial walls. In effect, Bauer [2], to overcome the above mentioned-shortcomings in viscoelastic modeling, developed a theory based on the classical Voigt model. The Bauer's theory [2] is intended to give satisfactorily and simultaneously account for strong elastic, viscous and inertial nonlinearities characterizing a viscoelastic material. The method consists essentially to decompose the total stress acting on the material as the sum of three components, that is, the elastic, viscous and inertial stresses and to express the pure elastic stress as a nonlinear function of deformation. The pure viscous and inertial stresses are then constructed as a first and second time derivatives of a similar function of deformation to the nonlinear elastic function, respectively. A fundamental theoretical difficulty in the use of the Bauer's theory [2] consists of the determination of appropriate nonlinear elastic spring force function that tends towards the expected linear elastic behavior for small deformations. In the Bauer's study [2], the pure elastic stress is expanded in a power series of strain, the pure viscous stress is developed as a first time derivative of a similar power series of strain, and the pure inertial stress is expressed as a second time derivative of a similar power series of strain. The Bauer's stress decomposition method [2], consisting to express the stress as a sum of three elementary stresses, has been after used by many authors [10, 11, 12] for a complete characterization of arterial behavior. In [12], following the Bauer's method [2], the elastic stress is expanded in a power series of strain. Monsia [8], in regard of the Bauer's method [2], expanding the pure elastic stress in negative powers series of deformation and, expressing the pure viscous stress as a first time derivative of a similar negative powers series of deformation, the pure inertial stress as a second time derivative of a similar negative powers series of deformation, developed a hyperlogistic equation that represents successfully the time-dependent mechanical properties of a variety of viscoelastic materials. Recently, the Bauer stress decomposition theory has been described in a single differential constitutive equation by Monsia [13, 14]. The use of this constitutive equation requires the specification of the judicious nonlinear elastic spring force function $\varphi(\varepsilon)$ where the deformation $\varepsilon(t)$ is a scalar function. In [13] the function $\varphi(\varepsilon)$ is expressed as a simple rational law that led in the absence of exciting stress to represent the time dependent deformation of the material system of interest in terms of a useful hyper-exponential type function. Using also the same rational function as the nonlinear elastic spring force law $\varphi(\varepsilon)$, but with the presence of a constant exciting stress term, Monsia [14] described mathematically and successfully the time dependent nonlinear creep behavior of viscoelastic materials exhibiting simultaneously elastic, viscous and inertial nonlinearities. More recently, Monsia and Kpomahou [15], employing the Monsia formulation [13, 14] of the Bauer's theory [2], with now the use of a Newton's binomial function as the nonlinear elastic spring force law, developed successfully a theoretical nonlinear mechanical model applicable for representing the well known S-shaped time dependent behavior of viscoelastic materials. It is also well known that a variety of differential equations arises in the formulation of mechanical problems. The use of the Monsia formulation [13, 14] of the Bauer's theory [2], leads then, often to a Lambert-type differential equation [8, 13-15]. In this work, following the Monsia formulation [13**,** 14] of the Bauer's theory [2], and using a generalized form of the Newton's binomial function of deformation for the pure elastic constitutive law $\varphi(\varepsilon)$, a constitutive Lambert-type differential equation is developed. A noteworthy feature is that, under relaxation of stress process, this equation is solved in closed form solution, by using change of variable techniques, in elementary functions. The Chapman-Richards-type solution obtained showed that the model can be successfully applied to represent the curvilinear S-shaped response of a variety of viscoelastic materials. Numerical studies are performed to illustrate the effects of rheological parameters action on the material response. It has been observed that an increase of the elastic and viscous factors affects, on the time period considered, the value of strain in opposite directions, and strongly increases the sensitivity of the model, of which the ability for experimental data fitting also increases.

## 2. Formulation of the Mathematical Model

### 2.1. Theoretical Developments

We develop in this part the evolutions equations governing the elastic, viscous and inertial nonlinearities characterizing simultaneously the material system studied. To that end, we use the Monsia formulation [13, 14] of the Bauer's theory [2]. The fundamental constitutive equation for a nonlinear elastic spring force law $\varphi(\varepsilon)$, is then written in the form [13, 14]

$$\ddot{\varepsilon}\frac{d\varphi}{d\varepsilon}+\dot{\varepsilon}^2\frac{d^2\varphi}{d\varepsilon^2}+\frac{b}{c}\dot{\varepsilon}\frac{d\varphi}{d\varepsilon}+\frac{a}{c}\varphi(\varepsilon)=\frac{1}{c}\sigma_t \qquad (1)$$

where the dot over a symbol denotes a differentiation with respect to time and the inertial module $c\neq 0$. The coefficients $a$ and $b$ denote respectively the stiffness and viscosity modules. These all three coefficients $a$, $b$ and $c$ are time independent material parameters. The total exciting stress $\sigma_t$ is a scalar function in the present one-dimensional mechanical model. To progress in this study it is necessary to specify the function $\varphi(\varepsilon)$. In this work, the nonlinear elastic spring force function is assumed described by the following law

$$\varphi(\varepsilon)=(1-\varepsilon^{\alpha})^{-1} \qquad (2)$$

where $\alpha$ is a material constant different from zero. Therefore, using Equation (2), Equation (1) may be written in the form

$$\sigma_t = \alpha c \frac{\ddot{\varepsilon}\varepsilon^{\alpha-1}(1-\varepsilon^{\alpha}) + (\alpha-1)\dot{\varepsilon}^2\varepsilon^{\alpha-2}(1-\varepsilon^{\alpha}) + 2\alpha\dot{\varepsilon}^2\varepsilon^{2(\alpha-1)}}{(1-\varepsilon^{\alpha})^3} + \alpha b\dot{\varepsilon}\varepsilon^{\alpha-1}(1-\varepsilon^{\alpha})^{-2} + a(1-\varepsilon^{\alpha})^{-1}$$

or

$$\ddot{\varepsilon} + (\alpha-1)\frac{\dot{\varepsilon}^2}{\varepsilon} + 2\alpha\varepsilon^{\alpha-1}\frac{\dot{\varepsilon}^2}{1-\varepsilon^{\alpha}} + \frac{b}{c}\dot{\varepsilon} + \frac{a}{\alpha c}\frac{1-\varepsilon^{\alpha}}{\varepsilon^{\alpha-1}} = \frac{(1-\varepsilon^{\alpha})^2\sigma_t}{\alpha c\varepsilon^{\alpha-1}} \quad (3)$$

Equation (3) determines the differential constitutive relationship between the total exciting stress $\sigma_t$ and the induced strain $\varepsilon(t)$. Equation (3) represents a Lambert-type second-order nonlinear ordinary differential equation in $\varepsilon(t)$ for a given total exciting stress $\sigma_t$. It may be turned into a simple form by means of a change of variable. By introducing the auxiliary variable

$$x = 1-\varepsilon^{\alpha} \quad (4)$$

Equation (3) becomes, after adequate algebraic transformations

$$\ddot{x} - 2\frac{\dot{x}^2}{x} + \lambda\dot{x} - (\omega_o^2 - \frac{\sigma_t}{c}x)x = 0 \quad (5)$$

where $\lambda = \frac{b}{c}$, $\omega_o^2 = \frac{a}{c}$, and the coefficient $c$ different from zero. Equation (5), a Lambert-type differential equation, can be solved in closed form solution by using suitable boundary conditions that satisfy the dynamics of the material system studied.

**2.2. Dimensionalization**

Noting that the strain $\varepsilon(t)$ is a dimensionless quantity, the different coefficients that are present in Equation (3) have the following dimensions. Let us denote by $M$, $L$ and $T$ the mass, length and time dimension respectively, the dimension of the stress becomes $ML^{-1}T^{-2}$. Therefore, the dimension of $a$ is given by $ML^{-1}T^{-2}$, that of $b$ varies as $ML^{-1}T^{-1}$, and that of $c$ varies as $ML^{-1}$ (mass per unit length).

**2.3. Solution using a stress $\sigma_t = 0$**

**2.3.1. Evolution Equation of Deformation** $\varepsilon(t)$

In the absence of total exciting stress that is, setting the exciting stress equal to zero ($\sigma_t = 0$), which denotes a relaxation phase where the applied stress is removed after the loading process, the internal dynamics of the material system studied can be represented by the following nonlinear ordinary differential equation

$$\ddot{x} - 2\frac{\dot{x}^2}{x} + \lambda\dot{x} - \omega_o^2 x = 0 \qquad (6)$$

**2.3.2. Solving Time Dependent Deformation Equation**

At first sight, note that in [8, 13-15] a similar equation to (6) is already solved. Thus, setting the following change of variable

$$f = \frac{\dot{x}}{x} \qquad (7)$$

Equation (6) transforms, after a few mathematical manipulations, in the following Riccati ordinary differential equation for the variable $f$ which possesses the strain rate dimension

$$\dot{f} = f^2 - \lambda f + \omega_o^2 \qquad (8)$$

Taking into consideration the following boundary conditions

$t \to 0, \ \lim f(t) = f_o$

and

$t \to +\infty, \ \lim f(t) = 0$

the solution of Equation (6) becomes

$$x(t) = x_{\max}\left[1 + q\exp(-\lambda\delta t)\right]^{-\frac{1}{n}} \qquad (9)$$

in which

$\delta = \sqrt{1 - 4\frac{\omega_o^2}{\lambda^2}}$ , $f_2 = \frac{\lambda}{2}(1-\delta)$, $q = \frac{f_o}{f_2 - f_o}$, $n = \frac{\lambda\delta}{f_2}$, and the condition

$t \to +\infty, \ \lim x(t) = x_{\max} = K$ . By considering Equation (4) the desired time dependent strain may be written

$$\varepsilon(t) = \left[1 - K\left[1 + q\exp(-\lambda\delta t)\right]^{-\frac{1}{n}}\right]^{\frac{1}{\alpha}} \tag{10}$$

Equation (10) provides the strain versus time relationship in the viscoelastic material under study. It predicts analytically the strain versus time variation of the material system considered as a Chapman-Richards-type model, well known to be powerful for reproducing mathematically any S curve shape.

## 3. Numerical Results and Discussion

Some numerical applications in this part are presented to illustrate the predictive capability of the model to reproduce the mechanical response of the material system studied and the effects of rheological parameters action on the material response. In the following of this work the numerical illustrations are investigated at the fixed value $K = -1$. Figure 1 illustrates the typical time dependent strain behavior with an increase until a maximum asymptotical value, obtained from Equation (10) with the fixed value of coefficients at $\alpha = 2$, $\lambda = 4.9275$, $\omega_o = 2$, $f_o = 1$. It can be seen from Figure 1 that the model is capable to represent mathematically and accurately the typical exponential deformation response of viscoelastic materials, to say, soft living tissues, soft soils, etc [9, 13-15]. The strain versus time curve is nonlinear, with a nonlinear beginning initial region. The plotting illustrates then the S-shaped deformation behavior of the viscoelastic material under study.

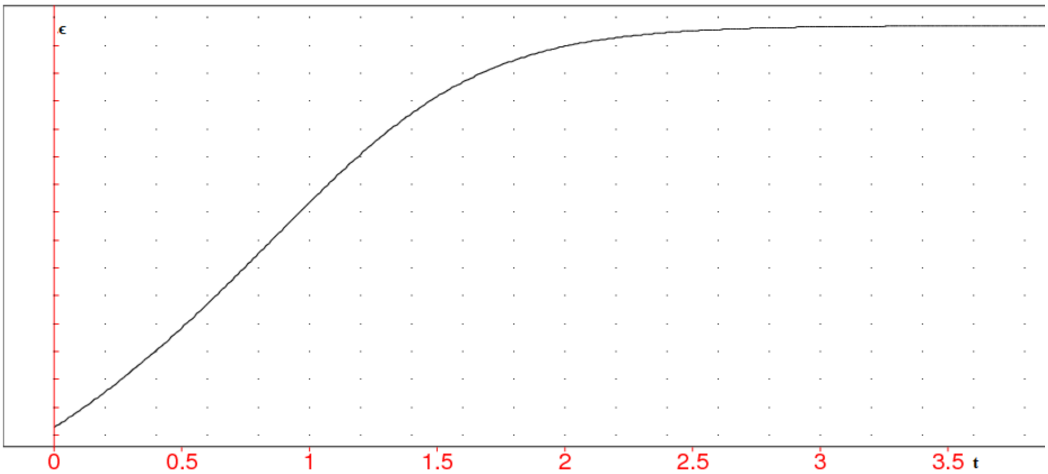


**Figure 1. Typical strain versus time curve showing an asymptotical value.**

Figures 2, 3, 4 and 5 illustrate the effects of material parameters action on the time-strain response. The effects of these parameters are studied by varying one coefficient while the other three are kept constant. As shown in Figure 2, the change of the parameter $\alpha$ has a high effect on the strain versus time curve. An increase $\alpha$, decreases the initial value of the strain and the maximum strain value. But, an increasing $\alpha$ has no significant effect on the time needed to attain the maximum strain. The red line corresponds to $\alpha = 2$, the blue line to $\alpha = 3$, and the green line to $\alpha = 5$. The other parameters are $\lambda = 4.9275$, $\omega_o = 2$, $f_o = 1$.

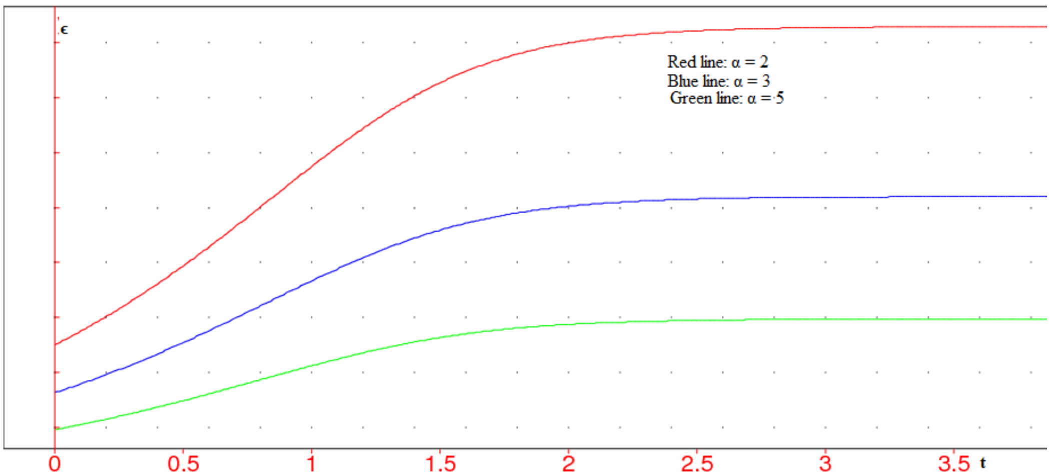


**Figure 2. Strain-time curves showing the effects of the material coefficient $\alpha$.**

Figure 3 shows the effects of the damping coefficient $\lambda$ on the strain-time relationship. An increase $\lambda$, increases the initial value of the strain. The strain value also increases fast in the early time periods with an increasing $\lambda$. But, this influence decreases as time tends to infinity, and the curves tend towards the same asymptotic value of the strain. . The red line corresponds to $\lambda = 4.9275$, the blue line to $\lambda = 5$, and the green line to $\lambda = 6$. The other parameters are $\alpha = 2$, $\omega_o = 2.41397$, $f_o = 1$.

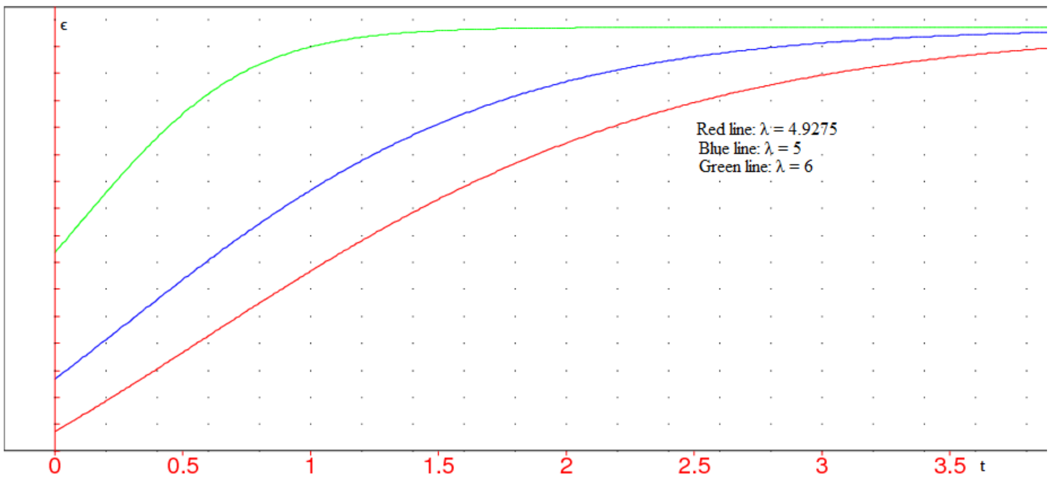


**Figure 3. Curves showing the effects of the damping coefficient $\lambda$.**

From Figure 4, it can be observed the dependence of the strain versus time curve on the natural frequency $\omega_o$. In effect, an increasing $\omega_o$, decreases contrary to the damping coefficient, the initial value of the strain. The strain value decreases also on the time period considered with increase $\omega_o$. The red line corresponds to $\omega_o = 2.4$, the blue line to $\omega_o = 2.45$, and the green line to $\omega_o = 2.46$. The other parameters are $\alpha = 2$, $\lambda = 4.9275$, $f_o = 1$.

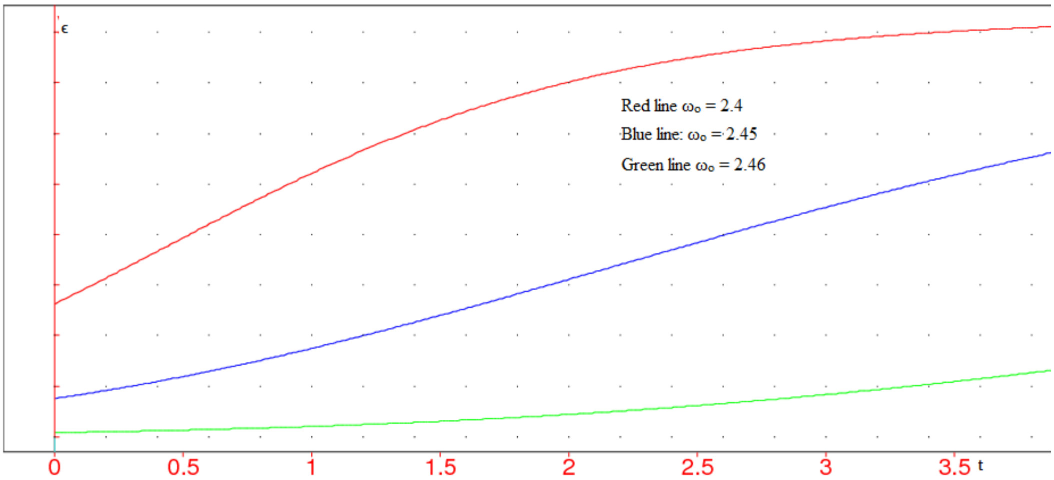


**Figure 4.** **Curves exhibiting the dependence of the strain-time relationship on the natural frequency** $\omega_o$ .

Figure 5 describes the effects of the parameter $f_o$ on the strain versus time curve. We can note that an increasing $f_o$, decreases the initial value of the strain. The strain value decreases also on the early time periods with increase $f_o$. However, this effect decreases as time tends to infinity, and the curves tend towards the same asymptotic strain value. The red line corresponds to $f_o = 0.6$, the blue line to $f_o = 0.8$, and the green line to $f_o = 1$. The other parameters are $\alpha = 2$, $\lambda = 4.9275$, $\omega_o = 2.41397$.

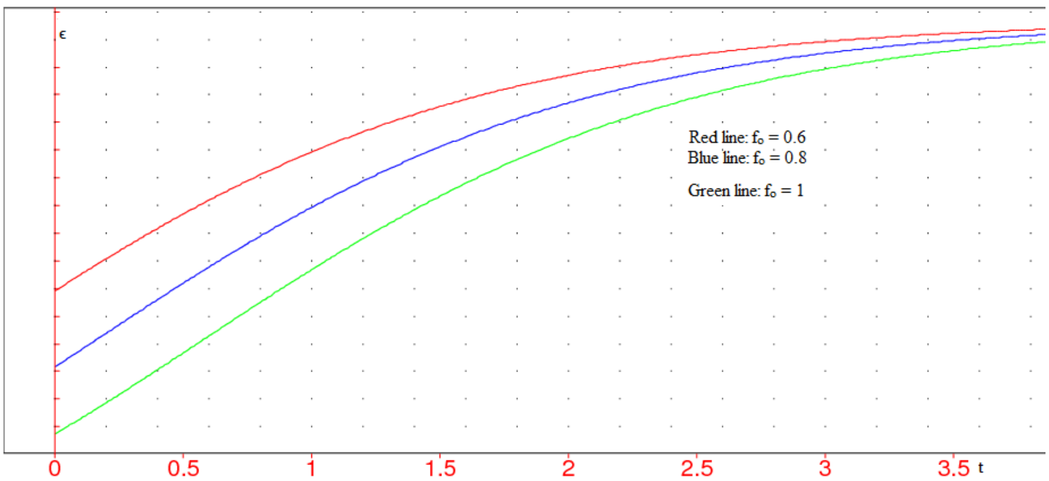


**Figure 5.** **Curves illustrating the effects of the parameter** $f_o$ **action on the strain-time relationship.**

The preceding numerical applications allowed investigating the model properties. The present study provides a nonlinear modified and extended Voigt rheological model. The modification and extension is made by introducing elastic, viscous and inertial nonlinear terms in the classical Voigt model by the use of the Bauer's theory [2]. The application of this approach necessitates an appropriate choice of the pure nonlinear elastic constitutive law $\varphi(\varepsilon)$. In the present model, the elastic spring force law is expressed as a generalized form of the Newton's binomial function. This law is created from the nonlinear elastic constitutive equation proposed first by Popovics [16] and discussed later in [17]. Therefore, the setting $\alpha = 1$, leads $\varphi(\varepsilon)$ to reduce to Newton's binomial function $\varphi(\varepsilon) = (1-\varepsilon)^{-1}$, that agrees very well with the nonlinear polynomial elastic force function used by Bauer's [2], when $\varepsilon$ becomes small, that is to say, when $\varepsilon << 1$. This choice of $\varphi(\varepsilon)$ permitted to describe the time dependent deformation of the material studied, under relaxation of stress conditions, as a Chapman-Richards-type equation, that is well known to be powerful for reproducing any S-shaped data. It is still possible to increase the number of model coefficients by substituting the previous pure elastic spring force $\varphi(\varepsilon) = (1-\varepsilon^{\alpha})^{-1}$ by the following law $(\varepsilon_o + \beta\varepsilon^{\alpha})^{\gamma}$ in order to generalize and eventually to increase the model flexibility. It is worth noting that when $\alpha = 1$, the preceding Chapman-Richards-type model reduces to a hyperlogistic-type growth model. The current model can be also used to study the stress versus time and the stress versus strain relationship of the material system of interest. It is also interesting to note that the proposed model offers the ability, for $K = 1$, to describe mathematically the creep relaxation behavior when the material under study is primarily assumed to be subjected to constant loading. But, these studies will be done as subsequent work.

## 4. Conclusions

A nonlinear constitutive Lambert-type differential equation has been developed. To that end, the stress decomposition theory has been used. The generalized Newton's binomial function chosen as purely nonlinear elastic constitutive law required permitted to represent mathematically the strain versus time relationship of the material system studied, under unloading process, as a Chapman-Richards-type growth model that is useful to reproduce any S-shaped curve. Numerical applications performed shown that an increase of viscoelastic characteristics, increases the sensitivity of the model, of which its flexibility increases significantly for experimental data fitting.

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