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LARGE OPEN FILTER BASES FOR FIXED POINTS

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Abstract. A local base of the neighbourhood system associated with a fixed point is called a fixed filter base. The possibilities of constructing fixed filter bases are discussed, which lead to a concept of fixed point at infinity.

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1. Introduction

A point in a metric space or a topological space is identified with its neighbourhood system when topological extensions are studied (see, for example [2]). So, a fixed point of a mapping may also be considered as a local base for the neighbourhood system. It is a filter base of nonempty open sets containing the fixed point. If a local base for a neighbourhood system of a point in a metric space is considered, then the diameter of the members of the local base tends

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to zero, when the local base is considered as a directed set under the inclusion relation. The following convention is to be followed in the first two sections of this article.

Convention 1.1. *A filter base $\mathscr{U}$ of nonempty open subsets of a metric space (X,d) is called a diametrically zero converging open filter base (or, DZCO-filter base) if the net $(diam\, U)_{U\in\mathscr{U}}$ of nonnegative extended real numbers converges to zero over the directed set $\mathscr{U}$ under the inclusion relation. Here $diam\, U = \sup\{d(x,y) : x,y \in U\}$.*

Thus, $\mathscr{U}$ is a *DZCO*-filter base in a metric space (X,d), if

(i) $U \in \mathscr{U} \Longrightarrow U \neq \emptyset$ and U is open;
(ii) $U,V \in \mathscr{U} \Longrightarrow$ there exists $W \in \mathscr{U}$ such that $W \subseteq U \cap V$; and
(iii) $diam\, U \to 0$ as U varies in the directed set $\mathscr{U}$.

Definition 1.2. *Let $f : X \to X$ be a given mapping on a metric space (X,d). A DZCO-filter base $\mathscr{U}$ in X is called a fixed filter base of f in X, if for given $U \in \mathscr{U}$, there is a $V \in \mathscr{U}$ such that $f(V) \subseteq U$.*

If $f : X \to X$ is a mapping on a metric space (X,d), $x \in X$ is a fixed point of f, and f is continuous at x, then $\mathscr{U} = \left\{B\left(x,\frac{1}{n}\right) : n = 1,2,\ldots\right\}$ is a fixed filter base of f in X, when $B\left(x,\frac{1}{n}\right) = \left\{y \in X : d(x,y) < \frac{1}{n}\right\}$. Continuity is essential to ensure that existence of a fixed point implies existence of a fixed filter base.

Example 1.3.

Define $f : R \to R$ by $f(x) = 1+x$ for $x \neq 0$ and $f(0) = 0$, when R is the real line endowed with the usual metric. This function f has the unique fixed point zero at which f is not continuous. But there is no fixed filter base of f in R.

On the other hand the existence of a fixed filter base need not imply the existence of a fixed point.

Example 1.4.

Define $f:(0,1)\to(0,1)$ by $f(x)=\frac{x}{2}$, for $x\in(0,1)$ with the usual metric on $(0,1)$. Then f has a fixed filter base $\mathscr{U}=\left\{\left(0,\frac{1}{n}\right):n=1,2,\dots\right\}$. This function f has no fixed point in $(0,1)$. But a fixed point may be realized in the completion of $(0,1)$.

Proposition 1.5. *Let $f:(X,d)\to(X,d)$ be a function with a fixed filter base $\mathscr{U}$ in a complete metric space (X,d). Suppose to each $U\in\mathscr{U}$, there is a $V\in\mathscr{U}$ such that $\overline{V}$, the closure of V, is contained in U. Then there is a fixed point $x\in X$ such that $\mathscr{U}$ is a local base for the neighbourhood system for x.*

Proof. Since (X,d) is complete and *diam* $\overline{U}\to 0$ as U varies in $\mathscr{U}$, then $\bigcap_{U\in\mathscr{U}}\overline{U}=\{x\}$ for some $x\in X$. By our hypothesis, the equality $\bigcap_{U\in\mathscr{U}}U=\{x\}$ is also true. To each $U\in\mathscr{U}$, there is a $V\in\mathscr{U}$ such that $f(V)\subseteq U$. Thus $f(x)\in U,\forall U\in\mathscr{U}$. This proves that x is a fixed point of f. Since *diam* $U\to 0$ as U varies in $\mathscr{U}$, $x\in U$ and U is open for every $U\in\mathscr{U}$, and $\bigcap_{U\in\mathscr{U}}U=\{x\}$, then the filter base $\mathscr{U}$ should be a local base to the neighbourhood system for x. □

Proposition 1.6. *Let $f:(X,d)\to(X,d)$ be a continuous function with a fixed filter base $\mathscr{U}$ in a compact metric space (X,d). Then there is a fixed point $x\in X$ such that $x\in\overline{U},\forall U\in\mathscr{U}$.*

Proof. Note that $\bigcap_{U\in\mathscr{U}}\overline{U}=\{x\}$, for some $x\in X$. To each $U\in\mathscr{U}$, there is a $V\in\mathscr{U}$ such that $f(V)\subseteq U$; and by the continuity of f, $f(x)\in f(\overline{V})\subseteq\overline{f(V)}\subseteq\overline{U}$ are true. Thus $f(x)\in\bigcap_{U\in\mathscr{U}}\overline{U}=\{x\}$ so that $f(x)=x$. This completes the proof. □

Example 1.7.

Consider the continuous function $f:[0,1]\to[0,1]$ defined by $f(x)=\frac{1}{2}x$, $\forall x\in[0,1]$, when $[0,1]$ is endowed with the usual metric. The filter base $\mathscr{U}=\left\{(0,\frac{1}{n}):n=1,2,\dots\right\}$ is a fixed filter base of f and 0 is a fixed point of f such that 0 is in the closure of $(0,\frac{1}{n})$, for each n. But $(0,\frac{1}{n})$ is not a neighbourhood of 0, for any n.

Convention 1.8. *A DZCO-filter base $\mathscr{U}$ in a metric space is called a DZCOR-filter base, if for each $U\in\mathscr{U}$, there is a $V\in\mathscr{U}$ such that $\overline{V}\in U$.*

Definition 1.9. *A DZCOR-filter base $\mathscr{U}$ in a metric space (X,d) is called a fixed R-filter base of a function $f:X\to X$, if it is a fixed filter base of f in X.*

The filter base $\left\{\left(0,\frac{1}{n}\right) : n = 1,2,\ldots\right\}$ and $\left\{\left[0,\frac{1}{n}\right) : n = 1,2,\ldots\right\}$ are fixed filter bases for the function given in example 1.7. The first one is not a fixed R-filter base, whereas the second one is a fixed R-filter base.

2. Construction of fixed filter bases

Theorem 2.1. *Let $f : X \to X$ be a mapping on a metric space (X,d) such that $d(f(x),f(y)) \leq kd(x,y)\ \forall x,y \in X$ and for some fixed $k \in (0,1)$. If $x_1 \in X$ is fixed, and $x_{n+1} = f(x_n)$, $n = 1,2,\ldots$, are defined, then $\left\{\bigcup_{i\geq n} B(x_i,k^n) : n = 1,2,\ldots\right\}$ is a fixed base of f in X.*

Proof. Recall that $B(x,r) = \{y \in X : d(x,y) < r\}$. Write $B_{n,r} = \bigcup_{i\geq n} B(x_i,r)$, when $x_{n+1} = f(x_n)$, $n = 1,2,\ldots$, and x_1 is fixed. The inequalities

$$d(x_{n+2},x_{n+1}) \leq kd(x_{n+1},x_n) \leq k^2 d(x_n,x_{n-1}) \leq \cdots \leq k^n d(x_2,x_1) \text{ and}$$
$$d(x_n,x_m) \leq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \cdots + d(x_{m-1},x_m)$$

for $m > n$ imply that $\lim_{n\to\infty} diam\{x_n,x_{n+1},\ldots\} = 0$, and hence $0 \leq \limsup_{n\to\infty} diam\, B_{n,r} \leq 2r$. If $x \in B_{n,r}$ then $x \in B(x_i,r)$ for some $i \geq n$, and hence $d(x_{i+1},f(x)) = d(f(x_i),f(x)) \leq kd(x_i,x)$ imply that $f(x) \in B(x_{i+1},kr)$. So,

$$f(B_{n+1,kr}) \subseteq f(B_{n+1,r}) \subseteq f(B_{n,r}) \subseteq B_{n+1,kr} \subseteq B_{n+1,r} \subseteq B_{n,r}.$$

Thus, $\left\{B_{n,k^n} : n = 1,2,\ldots\right\}$ is a filter base of nonempty open sets such that

(i) $\lim_{n\to\infty} diam\, B_{n,k^n} = 0$, and
(ii) $f(B_{n+1,k^{n+1}}) \subseteq B_{n+1,k^{n+1}} \subseteq B_{n,k^n}$, for every $n = 1,2,\ldots$.

So, $\{B_{n,k^n} : n = 1,2,\ldots\}$ is a fixed filter base of f. This completes the proof. □

One may find a continuous extension of f to the completion of f, a fixed point of this extension as the limit of the sequence x_n as in Banach contraction theorem, and thereby a fixed R-filter base of f. When the construction of completion through Cauchy sequences (see comments following Proposition 2.1 in [1]) and the Banach fixed point iteration procedure (see Theorem 3.1 in [1]) are recalled, the following theorem is concluded.

Theorem 2.2. *Let f, X, d, k, and $x_1, x_2, \ldots$ be as in theorem* 2.1. *Suppose further that (X,d) is complete. Then*

$$\left\{\left\{x \in X : \lim_{n\to\infty} d(x_n, x) < \frac{1}{m}\right\} : m = 1, 2, \ldots\right\}$$

is a fixed R-filter base of f in X.

These illustrated constructions reveal the difficulties in constructing fixed filter bases and their dependence on the construction of fixed points. It is expected that some good methods will be developed.

3. Fixed point at infinity

The previous section gives a motivation to propose the next definition.

Definition 3.1. *Let X be a locally compact Hausdorff space which is not compact. Let $f : X \to X$ be a mapping. Then f is said to have a fixed point at infinity, if for each compact subset K_1 of X, there is a compact subset K_2 of X such that $f(X \backslash K_2) \subseteq X \backslash K_1$.*

Theorem 3.2. *Let X and f be as in the definition* 3.1. *Then f has a fixed point at infinity if and only if the closure of inverse image of any compact subset of X is compact.*

Proof. Suppose f has a fixed point at infinity. Let K_1 be a compact subset of X. Then there is a compact subset K_2 of X such that $f(X \backslash K_2) \subseteq X \backslash K_1$. So, $f^{-1}(K_1) \subseteq K_2$, and hence the closure of $f^{-1}(K_1)$ is compact, because K_2 is compact.

Conversely assume that the closure of inverse image of any compact subset of X is compact. Let K_1 be a compact subset of X. Let K_2 be the closure of $f^{-1}(K_1)$ in X. Then K_2 is compact and $f(X \backslash K_2) \subseteq X \backslash K_1$. □

Corollary 3.3. *A homeomorphism from a locally compact Hausdorff space that is not compact onto itself has a fixed point at infinity.*

Proof. This is true because inverse image of a compact set is compact in this case. □

The definition 3.1 agrees with the classical definition of point at infinity for one-point compactification of a locally compact Hausdorff space that is not compact. However, it is possible to extend the definition 3.1 and the theorem 3.2 to metric spaces in the following manner.

Definition 3.4. *Let X be a metric space which is not bounded. Let $f : X \to X$ be a mapping. Then f is said to have a fixed point at infinity, if for every closed and bounded set C_1 in X, there is a closed and bounded set C_2 in X such that $f(X \backslash C_2) \subseteq X \backslash C_1$.*

Theorem 3.5. *Let X and f be as in the definition 3.4. Then f has a fixed point at infinity, if and only if inverse image of any bounded subset of X is bounded in X.*

Proof. Suppose f has a fixed point at infinity. Let C_1 be the closure of a given bounded set B in X. Then C_1 is also a bounded subset of X. Then, there is a closed and bounded set C_2 in X such that $f(X \backslash C_2) \subseteq X \backslash C_1$. Then $f^{-1}(B) \subseteq f^{-1}(C_1) \subseteq C_2$, and hence $f^{-1}(B)$ is bounded, because C_2 is bounded.

Conversely, assume that inverse image of any bounded set is a bounded set in X. Let C_1 be a closed and bounded subset of X. Let C_2 be the closure of $f^{-1}(C_1)$. Then C_2 is closed and bounded and $f(X \backslash C_2) \subseteq X \backslash C_1$. □

Corollary 3.6. *Let $f : (X,d) \to (X,d)$ be a mapping on an unbounded metric space (X,d) such that*

$$d(x,y) \leq kd(f(x), f(y)), \forall x,y \in X,$$

for some $k > 0$. Then f has a fixed point at infinity.

Remark 3.7. *Since bounded sets are also defined in Hausdorff topological vector spaces (see* [3]*), "metric spaces" can be replaced by "nonzero Hausdorff topological vector spaces" in definition* 3.4 *and in theorem* 3.5 *in which unboundedness of topological vector spaces need not be mentioned. The following corollary is the one corresponding to the corollary* 3.6*, but for linear mappings.*

Corollary 3.8. *Let $f : X \to X$ be a linear mapping on a nonzero Hausdorff topological vector space X. Suppose, to each open neighbourhood U of 0 in X, there corresponds an open neighbourhood V of 0 in X such that $f^{-1}(V) \subseteq U$. Then f has a fixed point at infinity.*

Conflict of Interests

The authors declare that there is no conflict of interests.

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