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STUDY OF A NEW CLASS FOR HIGHER-ORDER DERIVATIVES OF HARMONIC MULTIVALENT FUNCTIONS

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Abstract: In this paper, we define and investigate a new class for higher-order derivatives of harmonic multivalent functions. We obtain coefficient inequalities, distortion bounds, extreme points, convex combination. Our results extend corresponding previously known results.

Keywords: harmonic multivalent functions; extreme points; convex combination; higher-order derivatives.

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1. INTRODUCTION

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain $\mathbb{C}$, if both u and v are real harmonic in $\mathbb{C}$. In any simply connected domain $D \subset \mathbb{C}$, we can write $f = h + \overline{g}$, where h and g are analytic in D . We call h the analytic part and g the co-

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analytic part of f . A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ in D (see Clunie and Sheil-Small [3]).

Denote by $K_{\mathcal{H}}(p)$ the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense-preserving in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$. For $f = h + \overline{g} \in K_{\mathcal{H}}(p)$, we may express the analytic functions h and g as

$$h(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n , \qquad g(z) = \sum_{n=p}^{\infty} b_n z^n \ , |b_p| < 1 . \tag{1.1}$$

Also denote by $W_{\mathcal{H}}(p)$ the subclass of $K_{\mathcal{H}}(p)$ containing of all functions $f = h + \overline{g}$, where h and g are given by

$$h(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n , \qquad g(z) = -\sum_{n=p}^{\infty} b_n z^n \ , (a_n \geq 0, b_n \geq 0, |b_p| < 1) . \tag{1.2}$$

We denote by $AK_{\mathcal{H}}(p, q, \gamma, \alpha, \beta)$ the class of all functions of the form (1.1) that satisfy the condition:

$$Re\left\{\frac{zf^{(q+2)}(z) + f^{(q+1)}(z)}{f^{(q+1)}(z) + \gamma z f^{(q+2)}(z)}\right\} > \beta \left|\frac{zf^{(q+2)}(z) + f^{(q+1)}(z)}{f^{(q+1)}(z) + \gamma z f^{(q+2)}(z)} - 1\right| + \alpha, \tag{1.3}$$

where $p \in N = \{1,2,\ldots\}$, $q \in N_o = N \cup \{0\}$, $p > q$, $0 \leq \alpha < p - q$, $\beta \geq 0$, $0 \leq \gamma < 1$ and for each $f = h + \bar{g} \in K_H(p)$, we have

$$f^{(q)}(z) = \delta(p,q) z^{p-q} + \sum_{n=p+1}^{\infty} \delta(n,q) a_n z^{n-q} + \sum_{n=p}^{\infty} \delta(n,q) b_n (\bar{z})^{n-q},$$

$$\delta(p,q) = \frac{p!}{(p-q)!} = \begin{cases} 1 & (q = 0) \\ p(p-1) \ldots (p-q+1) & (q \neq 0) \end{cases}.$$

Let $AW_{\mathcal{H}}(p, q, \gamma, \alpha, \beta)$ be the subclass of $AK_{\mathcal{H}}(p, q, \gamma, \alpha, \beta)$, where

$$AW_{\mathcal{H}}(p, q, \gamma, \alpha, \beta) = W_{\mathcal{H}}(p) \cap AK_{\mathcal{H}}(p, q, \gamma, \alpha, \beta).$$

Remark 1.1.

(1) If $p = 1$ and $q = 0$, we have $AW_{\mathcal{H}}(1,0, \gamma, \alpha, \beta) = KW_{\mathcal{H}}(\gamma, \alpha, \beta)$ which was studied by Atshan and Wanas [2].

(2) If $p = 1$ and $q = \beta = 0$, we have $AW_{\mathcal{H}}(1,0, \gamma, \alpha, 0) = C(\gamma, \alpha)$ which was studied by Mostafa [4] for analytic part.

(3) If $p = 1$ and $q = \gamma = \beta = 0$, we have $AW_{\mathcal{H}}(1,0,0,\alpha,0) = C(\alpha)$ which was studied by Silverman [5] for analytic part.

Lemma 1.1. (Aqlan, [1]) Let $w = u + iv$ and μ be a real number. Then $Re(w) \geq \mu$ if and only if

$$|w + (1-\mu)| \geq |w - (1+\mu)|.$$

Lemma 1.2. (Aqlan, [1]) Let $w = u + iv$ and μ, t be a real number. Then $Re(w) \geq \mu|w-1| + t$ if and only if

$$Re\{w(1+\mu e^{i\theta}) - \mu e^{i\theta}\} \geq t \qquad (-\pi \leq \theta \leq \pi).$$

2. MAIN RESULT

In our first theorem, we introduce a sufficient coefficient bound for harmonic function in $AK_{\mathcal{H}}(p,q,\gamma,\alpha,\beta)$.

Theorem 2.1. Let $f = h + \overline{g}$ with h and g are given by (1.1). If

$$\sum_{n=p+1}^{\infty} \frac{n!}{(n-q-1)!}[(n-q-1)(\beta(1-\gamma)-\alpha\gamma)+n-q-\alpha](|a_n|+|b_n|)$$
$$\leq \frac{p!}{(p-q-1)!}\,[(p-q-1)(\beta(1-\gamma)-\alpha\gamma)-\alpha](1-|b_p|), \qquad (2.1)$$

where $p \in N$, $q \in N_0$, $p > q$, $0 \leq \alpha < p-q$, $\beta \geq 0$, $0 \leq \gamma < 1$, then f is harmonic multivalent sense preserving in U and $f \in AK_{\mathcal{H}}(p,q,\gamma,\alpha,\beta)$.

Proof. For proving $f \in AK_{\mathcal{H}}(p,q,\gamma,\alpha,\beta)$, we must show that (1.3) holds true. by using Lemma 1.2 , it is sufficient to show that

$$Re\left\{\frac{zf^{(q+2)}(z)+f^{(q+1)}(z)}{f^{(q+1)}(z)+\gamma zf^{(q+2)}(z)}(1+\beta e^{i\theta})-\beta e^{i\theta}\right\} > \alpha \quad (-\pi \leq \theta \leq \pi),$$

or equivalently

$$Re\left\{\frac{(1+\beta e^{i\theta})(zf^{(q+2)}(z)+f^{(q+1)}(z))-\beta e^{i\theta}(f^{(q+1)}(z)+\gamma zf^{(q+2)}(z))}{f^{(q+1)}(z)+\gamma zf^{(q+2)}(z)}\right\} > \alpha.$$

If we put

$$L(z) = (1+\beta e^{i\theta})(zf^{(q+2)}(z)+f^{(q+1)}(z)) - \beta e^{i\theta}(f^{(q+1)}(z)+\gamma zf^{(q+2)}(z))$$

and

$$M(z)=f^{(q+1)}(z)+\gamma z f^{(q+2)}(z).$$

In view of Lemma 1.1, we only need to prove that

$$|L(z)+(1-\alpha)M(z)|-|L(z)-(1+\alpha)M(z)|\geq 0.$$

Now,

$$|L(z)+(1-\alpha)M(z)|$$

$$=\Bigg|(1+\beta e^{i\theta})\Bigg(\frac{p!}{(p-q-2)!}\,z^{p-q-1}+\sum_{n=p+1}^{\infty}\frac{n!}{(n-q-2)!}a_n z^{n-q-1}+\sum_{n=p}^{\infty}\frac{n!}{(n-q-2)!}b_n(\overline{z})^{n-q-1}$$

$$+\frac{p!}{(p-q-1)!}z^{p-q-1}+\sum_{n=p+1}^{\infty}\frac{n!}{(n-q-1)!}a_n z^{n-q-1}+\sum_{n=p}^{\infty}\frac{n!}{(n-q-1)!}b_n(\overline{z})^{n-q-1}\Bigg)$$

$$-\beta e^{i\theta}\Bigg(\frac{p!}{(p-q-1)!}z^{p-q-1}+\sum_{n=p+1}^{\infty}\frac{n!}{(n-q-1)!}a_n z^{n-q-1}+\sum_{n=p}^{\infty}\frac{n!}{(n-q-1)!}b_n(\overline{z})^{n-q-1}$$

$$+\frac{\gamma p!}{(p-q-2)!}z^{p-q-1}+\sum_{n=p+1}^{\infty}\frac{\gamma n!}{(n-q-2)!}a_n z^{n-q-1}+\sum_{n=p}^{\infty}\frac{\gamma n!}{(n-q-2)!}b_n(\overline{z})^{n-q-1}\Bigg)$$

$$+(1-\alpha)\Bigg(\frac{p!}{(p-q-1)!}z^{p-q-1}+\sum_{n=p+1}^{\infty}\frac{n!}{(n-q-1)!}a_n z^{n-q-1}+\sum_{n=p}^{\infty}\frac{n!}{(n-q-1)!}b_n(\overline{z})^{n-q-1}$$

$$+\frac{\gamma p!}{(p-q-2)!}z^{p-q-1}+\sum_{n=p+1}^{\infty}\frac{\gamma n!}{(n-q-2)!}a_n z^{n-q-1}+\sum_{n=p}^{\infty}\frac{\gamma n!}{(n-q-2)!}b_n(\overline{z})^{n-q-1}\Bigg)\Bigg|$$

$$=\Bigg|\frac{p!}{(p-q-1)!}\Big[\Big((p-q-1)\big(\beta e^{i\theta}(1-\gamma)+(1-\alpha)\gamma\big)+p-q+1-\alpha\Big)\Big]z^{p-q-1}$$

$$+\sum_{n=p+1}^{\infty}\frac{n!}{(n-q-1)!}\big[(n-q-1)\big(\beta e^{i\theta}(1-\gamma)+(1-\alpha)\gamma\big)+n-q+1-\alpha\big]a_n z^{n-q-1}$$

$$+\sum_{n=p}^{\infty}\frac{n!}{(n-q-1)!}\big[(n-q-1)\big(\beta e^{i\theta}(1-\gamma)+(1-\alpha)\gamma\big)+n-q+1-\alpha\big]b_n(\overline{z})^{n-q-1}\Bigg|$$

$$\geq\frac{p!}{(p-q-1)!}[(p-q-1)(\beta(1-\gamma)+(1-\alpha)\gamma)+p-q+1-\alpha]|z|^{p-q-1}$$

$$-\sum_{n=p+1}^{\infty}\frac{n!}{(n-q-1)!}[(n-q-1)(\beta(1-\gamma)+(1-\alpha)\gamma)+n-q+1-\alpha]|a_n||z|^{n-q-1}$$

$$+\sum_{n=p}^{\infty}\frac{n!}{(n-q-1)!}[(n-q-1)(\beta(1-\gamma)+(1-\alpha)\gamma)+n-q+1-\alpha]|b_n||z|^{n-q-1}.$$

Similarly

$$|L(z)-(1+\alpha)M(z)|$$

$$=\Bigg|\frac{-p!}{(p-q-1)!}\left[(p-q-1)\left(\beta e^{i\theta}(\gamma-1)+(1+\alpha)\gamma\right)-p+q+1+\alpha\right]z^{p-q-1}$$

$$+\sum_{n=p+1}^{\infty}\frac{n!}{(n-q-1)!}\left[(n-q-1)\left(\beta e^{i\theta}(1-\gamma)+(1+\alpha)\gamma\right)+n-q-1-\alpha\right]a_n z^{n-q-1}$$

$$+\sum_{n=p}^{\infty}\frac{n!}{(n-q-1)!}\left[(n-q-1)\left(\beta e^{i\theta}(1-\gamma)-(1+\alpha)\gamma\right)+n-q-1-\alpha\right]b_n(\overline{z})^{n-q-1}\Bigg|$$

$$\leq\frac{p!}{(p-q-1)!}[(p-q-1)(\beta(\gamma-1)+(1+\alpha)\gamma)-p+q+1+\alpha]|z|^{p-q-1}$$

$$+\sum_{n=p+1}^{\infty}\frac{n!}{(n-q-1)!}[(n-q-1)(\beta(1-\gamma)-(1+\alpha)\gamma)+n-q-1-\alpha]\,|a_n||z|^{n-q-1}$$

$$+\sum_{n=p}^{\infty}\frac{n!}{(n-q-1)!}[(n-q-1)(\beta(1-\gamma)-(1+\alpha)\gamma)+n-q+1-\alpha]|b_n||\overline{z}|^{n-q-1}.$$

Then

$$|L(z)+(1-\alpha)M(z)|-|L(z)-(1+\alpha)M(z)|$$

$$\geq\frac{2p!}{(p-q-1)!}[(p-q-1)(\beta(1-\gamma)-\alpha\gamma)+p-q-\alpha]$$

$$-\sum_{n=p+1}^{\infty}\frac{2n!}{(n-q-1)!}[(n-q-1)(\beta(1-\gamma)-\alpha\gamma)+n-q-\alpha]|a_n|$$

$$-\sum_{n=p}^{\infty}\frac{2n!}{(n-q-1)!}[(n-q-1)(\beta(1-\gamma)-\alpha\gamma)+n-q-\alpha]|b_n|\geq 0.$$

The harmonic multivalent function

$$f(z)=\ z^p+\sum_{n=p+1}^{\infty}\frac{(n-q-1)!\,x_n}{n!\,[(n-q-1)(\beta(1-\gamma)-\alpha\gamma)+n-q-\alpha]}z^n$$

$$+\sum_{n=p}^{\infty}\frac{(n-q-1)!\,\overline{y}_n}{n!\,[(n-q-1)(\beta(1-\gamma)-\alpha\gamma)+n-q-\alpha]}(\overline{z})^n\quad,\qquad(2.2)$$

where $\sum_{n=p+1}^{\infty}|x_n|+\sum_{n=p}^{\infty}|y_n|=\frac{p!}{(p-q-1)!}[(p-q-1)(\beta(1-\gamma)-\alpha\gamma)+p-q-\alpha]$, show that the coefficient bound given by (2.1) is sharp. The functions of the form (2.2) are in the class $AK_{\mathcal{H}}(p,q,\gamma,\alpha,\beta)$, because

$$\sum_{n=p+1}^{\infty}\frac{n!}{(n-q-1)!}[(n-q-1)(\beta(1-\gamma)-\alpha\gamma)+n-q-\alpha]\times$$
$$\times\frac{(n-q-1)!\,|x_n|}{n!\,[(n-q-1)(\beta(1-\gamma)-\alpha\gamma)+n-q-\alpha]}$$
$$+\sum_{n=p}^{\infty}\frac{n!}{(n-q-1)!}[(n-q-1)(\beta(1-\gamma)-\alpha\gamma)+n-q-\alpha]\times$$
$$\times\frac{(n-q-1)!\,|y_n|}{n!\,[(n-q-1)(\beta(1-\gamma)-\alpha\gamma)+n-q-\alpha]}$$
$$=\sum_{n=p+1}^{\infty}|x_n|+\sum_{n=p}^{\infty}|y_n|=\frac{p!}{(p-q-1)!}[(p-q-1)(\beta(1-\gamma)-\alpha\gamma)+p-q-\alpha].$$

The restriction placed in Theorem 2.1 on the moduli of the coefficients of $f=h+\overline{g}$ enables us to conclude for arbitrary rotation of the coefficients of f that the resulting functions would still be harmonic multivalent and $f\in AK_{\mathcal{H}}(p,q,\gamma,\alpha,\beta)$.

In the following theorem, it is shown that the condition (2.1) is also necessary for functions in $AW_{\mathcal{H}}(p,q,\gamma,\alpha,\beta)$.

Theorem 2.2. Let $f=h+\overline{g}$ with h and g are given by (1.2) . Then $f\in AW_{\mathcal{H}}(p,q,\gamma,\alpha,\beta)$ if and only if

$$\sum_{n=p+1}^{\infty}\frac{n!}{(n-q-1)!}[(n-q-1)(\beta(1-\gamma)-\alpha\gamma)+n-q-\alpha]a_n$$
$$+\sum_{n=p}^{\infty}\frac{n!}{(n-q-1)!}[(n-q-1)(\beta(1-\gamma)-\alpha\gamma)+n-q-\alpha]b_n$$
$$\leq\frac{p!\,[(p-q-1)(\beta(1-\gamma)-\alpha\gamma)+p-q-\alpha]}{(p-q-1)!}\,, \tag{2.3}$$

where $p\in N,\ q\in N_0,\ p>q,\ \ 0\leq\alpha<p-q\,,\beta\geq 0\,,0\leq\gamma<1.$

Proof. Since $AW_{\mathcal{H}}(p,q,\gamma,\alpha,\beta) \subset AK_{\mathcal{H}}(p,q,\gamma,\alpha,\beta)$, we only need to prove the "only if " part of the theorem. Assume that $f \in AW_{\mathcal{H}}(p,q,\gamma,\alpha,\beta)$. Then by (1.3), we have

$$Re\left\{\frac{zf^{(q+2)}(z)+f^{(q+1)}(z)}{f^{(q+1)}(z)+\gamma zf^{(q+2)}(z)}\left(1+\beta e^{i\theta}\right)-\beta e^{i\theta}\right\}>\alpha,$$

This is equivalent to

$$Re\left\{\frac{\left(1+\beta e^{i\theta}\right)\left(zf^{(q+2)}(z)+f^{(q+1)}(z)\right)-\beta e^{i\theta}\left(f^{(q+1)}(z)+\gamma zf^{(q+2)}(z)\right)}{f^{(q+1)}(z)+\gamma zf^{(q+2)}(z)}\right\}$$

$$= Re\left\{\frac{\frac{p!}{(p-q-1)!}\left[(p-q-1)\left(\beta e^{i\theta}(1-\gamma)-\alpha\gamma\right)+p-q-\alpha\right]z^{p-q-1}}{\frac{p!}{(p-q-1)!}\left(1+\gamma(p-q-1)\right)z^{p-q-1}-\sum_{n=p+1}^{\infty}\frac{n!}{(n-q-1)!}\left(1+\gamma(p-q-1)\right)a_n z^{n-q-1}}\right.$$

$$\frac{-\sum_{n=p+1}^{\infty}\frac{n!}{(n-q-1)!}\left[(n-q-1)\left(\beta e^{i\theta}(1-\gamma)-\alpha\gamma\right)+n-q-\alpha\right]a_n z^{n-q-1}}{-\sum_{n=p}^{\infty}\frac{n!}{(n-q-1)!}\left(1+\gamma(p-q-1)\right)b_n(\bar{Z})^{n-q-1}}$$

$$\left.\frac{-\sum_{n=p}^{\infty}\frac{n!}{(n-q-1)!}\left[(n-q-1)\left(\beta e^{i\theta}(1-\gamma)-\alpha\gamma\right)+n-q-\alpha\right]b_n(\bar{Z})^{n-q-1}}{}\right\}\geq 0. \qquad (2.4)$$

The above required condition (2.4) must hold for all values of z in U. Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$, we must have

$$Re\left\{\frac{\frac{p!}{(p-q-1)!}[p-q-\alpha-\alpha\gamma(p-q-1)]r^{p-q-1}-\sum_{n=p+1}^{\infty}\frac{n!}{(n-q-1)!}[n-q-\alpha-\alpha\gamma(n-q-1)]a_n r^{n-q-1}}{\frac{p!}{(p-q-1)!}\left(1+\gamma(p-q-1)\right)r^{p-q-1}-\sum_{n=p+1}^{\infty}\frac{n!}{(n-q-1)!}\left(1+\gamma(n-q-1)\right)a_n r^{n-q-1}}\right.$$

$$\frac{-\sum_{n=p}^{\infty}\frac{n!}{(n-q-1)!}[n-q-\alpha-\alpha\gamma(n-q-1)]b_n r^{p-q-1}-\beta e^{i\theta}\left[\frac{p!}{(p-q-1)!}(p-q-1)(\gamma-1)r^{p-q-1}\right.}{-\sum_{n=p}^{\infty}\frac{n!}{(n-q-1)!}\left(1+\gamma(n-q-1)\right)b_n r^{n-q-1}}$$

$$\left.\frac{\left.-\sum_{n=p+1}^{\infty}\frac{n!}{(n-q-1)!}(n-q-1)(1-\gamma)a_n r^{p-q-1}+\sum_{n=p}^{\infty}\frac{n!}{(n-q-1)!}(n-q-1)(1-\gamma)b_n r^{p-q-1}\right]}{}\right\}\geq 0.$$

Since $Re(\text{-}e^{i\theta}) \geq -\left|e^{i\theta}\right| = -1$, and letting $r \longrightarrow 1^-$, the above inequality reduces to

$$\frac{\frac{p!}{(p-q-1)!}[(p-q-1)(\beta(1-\gamma)-\alpha\gamma)+p-q-\alpha]-\sum_{n=p+1}^{\infty}\frac{n!}{(n-q-1)!}[(n-q-1)(\beta(1-\gamma)-\alpha\gamma)+n-q-\alpha]a_n}{\frac{p!}{(p-q-1)!}\left(1+\gamma(p-q-1)\right)-\sum_{n=p+1}^{\infty}\frac{n!}{(n-q-1)!}\left(1+\gamma(n-q-1)\right)a_n-\sum_{n=p}^{\infty}\frac{n!}{(n-q-1)!}\left(1+\gamma(n-q-1)\right)b_n}$$

$$\frac{-\sum_{n=p+1}^{\infty}\frac{n!}{(n-q-1)!}[(n-q-1)(\beta(1-\gamma)-\alpha\gamma)+n-q-\alpha]b_n}{} \geq 0.$$

This gives (2.3), and the poof is complete.

Next, we establish the distortion bounds for the function in $AW_{\mathcal{H}}(p,q,\gamma,\alpha,\beta)$ which yields a covering result for this class.

Theorem 2.3. Let $f \in AW_{\mathcal{H}}(p,q,\gamma,\alpha,\beta)$. Then for $|z| = r < 1$, we have

$$|f(z)| \geq (1-b_p)r^p - \frac{(p-q)[(p-q-1)(\beta(1-\gamma)-\alpha\gamma)+p-q-\alpha](1-b_p)}{(p+1)[(p-q)(\beta(1-\gamma)+1-\alpha\gamma)+1-\alpha]} \qquad (2.5)$$

and

$$|f(z)| \leq (1+b_p)r^p + \frac{(p-q)[(p-q-1)(\beta(1-\gamma)-\alpha\gamma)+p-q-\alpha](1-b_p)}{(p+1)[(p-q)(\beta(1-\gamma)+1-\alpha\gamma)+1-\alpha]}. \qquad (2.6)$$

Proof. Assume that $f \in AW_{\mathcal{H}}(p,q,\gamma,\alpha,\beta)$. Then by (2.3), we get

$$|f(z)| = \left| z^p - \sum_{n=p+1}^{\infty} a_n z^n - \sum_{n=p}^{\infty} b_n (\overline{z})^n \right| \geq (1-b_p)r^p - \sum_{n=p+1}^{\infty}(a_n+b_n)r^n$$

$$\geq (1-b_p)r^p - \sum_{n=p+1}^{\infty}(a_n+b_n)r^{p+1}$$

$$= (1-b_p)r^p - \frac{(p-q)!}{(p+1)!\,[(p-q)(\beta(1-\gamma)+1-\alpha\gamma)+1-\alpha]} \times$$

$$\times \sum_{n=p+1}^{\infty} \frac{(p+1)!\,[(p-q)(\beta(1-\gamma)+1-\alpha\gamma)+1-\alpha]}{(p-q)!}(a_n+b_n)r^{p+1}$$

$$\geq (1-b_p)r^p - \frac{(p-q)!}{(p+1)!\,[(p-q)(\beta(1-\gamma)+1-\alpha\gamma)+1-\alpha]}$$

$$\sum_{n=p+1}^{\infty}\frac{n!}{(n-q-1)!}[(n-q-1)(\beta(1-\gamma)-\alpha\gamma)+n-q-\alpha]\,(a_n+b_n)r^{p+1}$$

$$\geq (1-b_p)r^p - \frac{(p-q)[(p-q-1)(\beta(1-\gamma)-\alpha\gamma)+p-q-\alpha](1-b_p)}{(p+1)[(p-q)(\beta(1-\gamma)+1-\alpha\gamma)+1-\alpha]}\ .$$

Relation (2.6) can be proved by using the similar statements. So the proof is completes.

The following covering result follows from the inequality (2.5) of theorem 2.3.

Corollary 2.1. Let $f \in AW_{\mathcal{H}}(p,q,\gamma,\alpha,\beta)$. Then for $|z| = r < 1$, we have

$$\left\{w:|w|<\left(1-b_p\right)-\frac{(p-q)[(p-q-1)(\beta(1-\gamma)-\alpha\gamma)+p-q-\alpha]\left(1-b_p\right)}{(p+1)[(p-q)(\beta(1-\gamma)+1-\alpha\gamma)+1-\alpha]}\right\}\subset f(U)$$

In the next result, we discuss extreme points of $AW_{\mathcal{H}}(p,q,\gamma,\alpha,\beta)$.

Theorem2.4. Let f be given by (1.2). Then $f\in AW_{\mathcal{H}}(p,q,\gamma,\alpha,\beta)$ if and only if f can be expressed as

$$f(z)=\sum_{n=p}^{\infty}(\sigma_n h_n(z)+\xi_n g_n(z)),\qquad (z\in U),$$

where $h_p(z)=z^p$,

$$h_n(z)=z^p-\frac{p!\,(n-q-1)!\,[(p-q-1)(\beta(1-\gamma)-\alpha\gamma)+p-q-\alpha]}{n!\,(p-q-1)!\,[(n-q-1)(\beta(1-\gamma)+1-\alpha\gamma)+n-q-\alpha]}z^n,\ \ n=p+1,p+2,\ldots$$

and

$$g_n(z)=z^p-\frac{p!\,(n-q-1)!\,[(p-q-1)(\beta(1-\gamma)-\alpha\gamma)+p-q-\alpha]}{n!\,(p-q-1)!\,[(n-q-1)(\beta(1-\gamma)+1-\alpha\gamma)+n-q-\alpha]}(\bar{z})^n,\ \ n=p,p+1,\ldots,$$

$$\sum_{n=p}^{\infty}(\sigma_n+\xi_n)=1,\qquad (\sigma_n\geq 0,\ \ \xi_n\geq 0).$$

In particular , the extreme points of $AW_{\mathcal{H}}(p,q,\gamma,\alpha,\beta)$ are $\{h_n\}$ and $\{g_n\}$.

Proof: Assume that f can be expressed by (2.7). Then, we have

$$f(z)=\sum_{n=p}^{\infty}(\sigma_n h_n(z)+\xi_n g_n(z))$$

$$=\sum_{n=p}^{\infty}(\sigma_n+\xi_n)z^p-\sum_{n=p+1}^{\infty}\frac{p!\,(n-q-1)!\,[(p-q-1)(\beta(1-\gamma)-\alpha\gamma)+p-q-\alpha]}{n!\,(p-q-1)!\,[(n-q-1)(\beta(1-\gamma)+1-\alpha\gamma)+n-q-\alpha]}\sigma_n z^n$$

$$-\sum_{n=p}^{\infty}\frac{p!\,(n-q-1)!\,[(p-q-1)(\beta(1-\gamma)-\alpha\gamma)+p-q-\alpha]}{n!\,(p-q-1)!\,[(n-q-1)(\beta(1-\gamma)+1-\alpha\gamma)+n-q-\alpha]}\ \xi_n(\bar{z})^n$$

$$=z^p-\sum_{n=p+1}^{\infty}\frac{p!\,(n-q-1)!\,[(p-q-1)(\beta(1-\gamma)-\alpha\gamma)+p-q-\alpha]}{n!\,(p-q-1)!\,[(n-q-1)(\beta(1-\gamma)+1-\alpha\gamma)+n-q-\alpha]}\sigma_n z^n$$

$$-\sum_{n=p}^{\infty}\frac{p!\,(n-q-1)!\,[(p-q-1)(\beta(1-\gamma)-\alpha\gamma)+p-q-\alpha]}{n!\,(p-q-1)!\,[(n-q-1)(\beta(1-\gamma)+1-\alpha\gamma)+n-q-\alpha]}\ \xi_n(\bar{z})^n.$$

Hence

$$\sum_{n=p+1}^{\infty} \frac{n!}{(n-q-1)!}[(n-q-1)(\beta(1-\gamma)-\alpha\gamma)+n-q-\alpha] \times$$
$$\times \frac{p!\,(n-q-1)!\,[(p-q-1)(\beta(1-\gamma)-\alpha\gamma)+p-q-\alpha]}{n!\,(p-q-1)!\,[(n-q-1)(\beta(1-\gamma)+1-\alpha\gamma)+n-q-\alpha]}\sigma_n$$
$$+\sum_{n=p}^{\infty} \frac{n!}{(n-q-1)!}[(n-q-1)(\beta(1-\gamma)-\alpha\gamma)+n-q-\alpha] \times$$
$$\times \frac{p!\,(n-q-1)!\,[(p-q-1)(\beta(1-\gamma)-\alpha\gamma)+p-q-\alpha]}{n!\,(p-q-1)!\,[(n-q-1)(\beta(1-\gamma)+1-\alpha\gamma)+n-q-\alpha]}\xi_n$$
$$= \frac{p!\,[(p-q-1)(\beta(1-\gamma)-\alpha\gamma)+p-q-\alpha]}{(p-q-1)!}\left(\sum_{n=p}^{\infty}(\sigma_n+\ \xi_n)-\sigma_p\right)$$
$$= \frac{p!\,[(p-q-1)(\beta(1-\gamma)-\alpha\gamma)+p-q-\alpha]}{(p-q-1)!}\left(1-\sigma_p\right)$$
$$\leq \frac{p!\,[(p-q-1)(\beta(1-\gamma)-\alpha\gamma)+p-q-\alpha]}{(p-q-1)!},$$

and so $f \in AW_{\mathcal{H}}(p,q,\gamma,\alpha,\beta)$.

Conversely, let $f \in AW_{\mathcal{H}}(p,q,\gamma,\alpha,\beta)$. putting

$$\sigma_n = \frac{n!\,(p-q-1)!\,[(n-q-1)(\beta(1-\gamma)+1-\alpha\gamma)+n-q-\alpha]}{p!\,(n-q-1)!\,[(p-q-1)(\beta(1-\gamma)-\alpha\gamma)+p-q-\alpha]}a_n, \qquad (n=p+1,p+2,\dots)$$

and

$$\xi_n = \frac{n!\,(p-q-1)!\,[(n-q-1)(\beta(1-\gamma)+1-\alpha\gamma)+n-q-\alpha]}{p!\,(n-q-1)!\,[(p-q-1)(\beta(1-\gamma)-\alpha\gamma)+p-q-\alpha]}b_n, \qquad (n=p,p+1,\dots).$$

We define $\sigma_p = 1-\sum_{n=p+1}^{\infty}\sigma_n-\sum_{n=p}^{\infty}\xi_n$. There fore

$$f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n - \sum_{n=p}^{\infty} b_n\,(\bar{z})^n$$

$$= z^p - \sum_{n=p+1}^{\infty} \frac{p!\,(n-q-1)!\,[(p-q-1)(\beta(1-\gamma)-\alpha\gamma)+p-q-\alpha]}{n!\,(p-q-1)!\,[(n-q-1)(\beta(1-\gamma)+1-\alpha\gamma)+n-q-\alpha]}\sigma_n z^n$$

$$-\sum_{n=p}^{\infty} \frac{p!\,(n-q-1)!\,[(p-q-1)(\beta(1-\gamma)-\alpha\gamma)+p-q-\alpha]}{n!\,(p-q-1)!\,[(n-q-1)(\beta(1-\gamma)+1-\alpha\gamma)+n-q-\alpha]}\xi_n(\bar{z})^n$$

$$= z^p - \sum_{n=p+1}^{\infty}(z^p-h_n(z))\sigma_n - \sum_{n=p}^{\infty}(z^p-g_n(z))\xi_n$$

$$= \left(1 - \sum_{n=p+1}^{\infty} \sigma_n - \sum_{n=p}^{\infty} \xi_n\right) z^p + \sum_{n=p+1}^{\infty} \sigma_n h_n(z) - \sum_{n=p}^{\infty} \xi_n \, g_n(z)$$

$$= \sigma_p h_p(z) + \sum_{n=p+1}^{\infty} \sigma_n h_n(z) - \sum_{n=p}^{\infty} \xi_n \, g_n(z)$$

$$= \sum_{n=p}^{\infty} \big(\sigma_n h_n(z) + \xi_n \, g_n(z)\big),$$

and this completes the proof of theorem 2.4.

Theorem 2.5. The class $AW_{\mathcal{H}}(p, q, \gamma, \alpha, \beta)$ is closed under convex combinations.

Proof. For $j = 1,2,3, \dots,$ let $f_j \in AW_{\mathcal{H}}(p, q, \gamma, \alpha, \beta)$ where f_j is given by

$$f_j(z) = z^p - \sum_{n=p+1}^{\infty} a_{n,j} z^n - \sum_{n=p}^{\infty} b_{n,j} \, (\bar{z})^n.$$

Then by (2.3), we have

$$\sum_{n=p+1}^{\infty} \frac{n!}{(n-q-1)!} [(n-q-1)(\beta(1-\gamma) - \alpha\gamma) + n - q - \alpha] \, a_{n,j}$$
$$+ \sum_{n=p}^{\infty} \frac{n!}{(n-q-1)!} [(n-q-1)(\beta(1-\gamma) - \alpha\gamma) + n - q - \alpha] \, b_{n,j}$$
$$\leq \frac{p! \, [(p-q-1)(\beta(1-\gamma) - \alpha\gamma) + p - q - \alpha]}{(p-q-1)!} \tag{2.8}$$

For $\sum_{j=1}^{\infty} \lambda_j = 1$, $0 \leq \lambda_j \leq 1$, the convex combination of f_j may be written as

$$\sum_{j=1}^{\infty} \lambda_j \; f_j(z) = z^p - \sum_{n=p+1}^{\infty} \left(\sum_{j=1}^{\infty} \lambda_j \, a_{n,j}\right) Z^n - \sum_{n=p}^{\infty} \left(\sum_{j=1}^{\infty} \lambda_j \; b_{n,j}\right) (\bar{z})^n.$$

Then by (2.8), we have

$$\sum_{n=p+1}^{\infty} \frac{n!}{(n-q-1)!} [(n-q-1)(\beta(1-\gamma) - \alpha\gamma) + n - q - \alpha] \left(\sum_{j=1}^{\infty} \lambda_j \, a_{n,j}\right)$$
$$+ \sum_{n=p}^{\infty} \frac{n!}{(n-q-1)!} [(n-q-1)(\beta(1-\gamma) - \alpha\gamma) + n - q - \alpha] \left(\sum_{j=1}^{\infty} \lambda_j \; b_{n,j}\right)$$
$$= \sum_{j=1}^{\infty} \lambda_j \left\{ \sum_{n=p+1}^{\infty} \frac{n!}{(n-q-1)!} [(n-q-1)(\beta(1-\gamma) - \alpha\gamma) + n - q - \alpha] \, a_{n,j} \right.$$

$$\left. + \sum_{n=p}^{\infty} \frac{n!}{(n-q-1)!} [(n-q-1)(\beta(1-\gamma)-\alpha\gamma)+n-q-\alpha] \ b_{n,j} \right\}$$

$$\leq \sum_{j=1}^{\infty} \lambda_j \frac{p!\,[(p-q-1)(\beta(1-\gamma)-\alpha\gamma)+p-q-\alpha]}{(p-q-1)!}$$

$$= \frac{p!\,[(p-q-1)(\beta(1-\gamma)-\alpha\gamma)+p-q-\alpha]}{(p-q-1)!}.$$

Therefore

$$\sum_{j=1}^{\infty} \lambda_j \, f_j(z) \in \ AW_{\mathcal{H}}(p,q,\gamma,\alpha,\beta).$$

This completes the proof.

Conflict of Interests

The authors declare that there is no conflict of interests.

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