Available online at http://scik.org
Eng. Math. Lett. 2019, 2019:9
https://doi.org/10.28919/eml/4003
ISSN: 2049-9337

AN APPLICATION OF FRACTIONAL CALCULUS ON A CERTAIN CLASS OF MULTIVALENT ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract: The object of this paper to study an application of the fractional calculus techniques for a certain class $MR(p,m,\varepsilon,\sigma,\mu)$ of multivalent analytic functions with negative coefficients in the open unit disk. Distortion theorems for the fractional derivative and fractional integration are obtained. Also, we gain results about coefficient inequality, neighborhood property and radii of starlikeness and convexity.

Keywords: multivalent functions; fractional calculus; neighborhood; radii of starlikeness and convexity.

2010 AMS Subject Classification: 30C45, 30C50.

1. INTRODUCTION

Let $M(p,m)$ denote the class of functions of the form:

$$f(z)=z^p+\sum_{n=p+m}^{\infty}a_n z^n \qquad (p,m\in\mathbb{N}=\{1,2,\dots\}), \tag{1.1}$$

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Received January 18, 2019

which are analytic and multivalent in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

Let $R(p, m)$denote the subclass of $M(p, m)$ consisting of functions of the form:

$$f(z) = z^p - \sum_{n=p+m}^{\infty} a_n z^n \qquad (a_n \geq 0\ p, m \in \mathbb{N} = \{1,2, \dots\}), \qquad (1.2)$$

A function $f \in M(p,m)$ is said to be multivalent starlike of order $\alpha(0 \leq \alpha < p)$ if it satisfies the condition:

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in U),$$

and is said to be multivalent convex of order $\alpha(0 \leq \alpha < p)$ if it satisfies the condition:

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \qquad (z \in U).$$

Denote by $S_m^*\ (p,\alpha)$ and $C_m\ (p,\alpha)$ the classes of multivalent starlike and multivalent convex functions of order α, respectively, which were introduced and studied by Owa [8]. It is known that (see [6] and [8])

$$f \in C_m\ (p,\alpha)\ if\ and\ only\ if\ \frac{zf'(z)}{p} \in S_m^*\ (p,\alpha).$$

The classes $S_m^* = S^*(p,\alpha)\ and\ C_1\ (p,\alpha) = C(p,\alpha)$. were studied by Own [7].

Definition 1.1. (Srivastava and Owa, [10]) The fractional integral of order $\lambda\ (\lambda > 0)$ is defined for a function f by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt,$$

where f is analytic function in a simple connected region of z- plane containing the origin and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $log(z-t)$ to be real when $(z-t) > 0$.

Definition 1.2. (Srivastava and Owa, [10]) The fractional derivative of order $\lambda\ (0 \leq \lambda < 1)$ is defined for a function f by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\lambda}}\, dt,$$

where f is as in Definition 1.1 and the multiplicity of $(z-t)^{-\lambda}$ is removed like Definition 1.1.

Definition 1.3. (Srivastava and Owa, [10]) Under the hypothesis of Definition 1.2, the fractional derivative of order $\lambda+k$ is defined for a function f by

$$D_z^{\lambda+k} f(z) = \frac{d^k}{dz^k} D_z^{\lambda} f(z)\,, \quad (0 \le \lambda < 1, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

For $f \in R(p,m)$, from Definitions 1.1 and 1.2 by applying a simple calculation, we get

$$D_z^{-\lambda} f(z) = \frac{\Gamma(p+1)}{\Gamma(p+\lambda+1)} z^{p+\lambda} - \sum_{n=p+m}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+\lambda+1)} a_n\, z^{n+\lambda} \tag{1.3}$$

and

$$D_z^{\lambda} f(z) = \frac{\Gamma(p+1)}{\Gamma(p-\lambda+1)} z^{p-\lambda} - \sum_{n=p+m}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n-\lambda+1)} a_n\, z^{n-\lambda} \tag{1.4}$$

Definition 1.4. A function $f \in R(p,m)$ is said to be in the class $MR(p,m,\varepsilon,\sigma,\mu)$ if and only if satisfies the inequality:

$$\left| \frac{\frac{zf''(z)}{f'(z)} + 1 - p}{\varepsilon \frac{zf''(z)}{f'(z)} + p + \varepsilon - \sigma(\varepsilon+1)} \right| < \mu\,, \tag{1.5}$$

where $0 \le \varepsilon < 1\,, 0 \le \sigma < p\,, 0 < \mu \le 1$ and $z \in U$.

Such type of study was carried out by various authors for another classes, like, Ghanim and Darus [4], Aouf [1], Aouf and Mostafa [2] Atshan and Wanas [3] and Wanas [11].

2. MAIN RESULT

The first theorem gives a necessary and sufficient condition for a function $f$to be in the class $MR(p,m,\varepsilon,\sigma,\mu)$.

Theorem 2.1. A function $f \in R(p,m)$ is in the class $MR(p,m,\varepsilon,\sigma,\mu)$ if and only if

$$\sum_{n=p+m}^{\infty} n[n-p+\mu(\varepsilon(n-\sigma)+p-\sigma)]a_n \le \mu p(p-\sigma)(\varepsilon+1), \tag{2.1}$$

where $0 \le \varepsilon < 1\,, 0 \le \sigma < p\,, 0 < \mu \le 1$ and $z \in U$.

The result is sharp for the function f given by.

$$f(z) = z^p - \frac{\mu p(p-\sigma)(\varepsilon+1)}{n[n-p+\mu(\varepsilon(n-\sigma)+p-\sigma)]} z^n\ , \ (n \ge p+m,\ \ p,m \in \mathbb{N}). \tag{2.2}$$

Proof. Assume that the inequality (2.1) holds true $|z| = 1$. Then, we obtain

$$|zf''(z) + (1-p)f'(z)| - \mu\left|\varepsilon zf''(z) + \big(p + \varepsilon - \sigma(\varepsilon+1)\big)f'(z)\right|$$

$$= \left|-\sum_{n=p+m}^{\infty} n(n-p)a_n z^{n-1}\right| - \left|\mu p(p-\sigma)(\varepsilon+1)z^{p-1} - \sum_{n=p+m}^{\infty} \mu n(\varepsilon(n-\sigma)+p-\sigma)a_n z^{n-1}\right|$$

$$\leq \sum_{n=p+m}^{\infty} n(n-p)a_n|z|^{n-1} - \mu p(p-\sigma)(\varepsilon+1)|z|^{p-1} + \sum_{n=p+m}^{\infty} \mu n(\varepsilon(n-\sigma)+p-\sigma)a_n|z|^{n-1}$$

$$= \sum_{n=p+m}^{\infty} n[n-p+\mu(\varepsilon(n-\sigma)+p-\sigma)]a_n - \mu p(p-\sigma)(\varepsilon+1) \leq 0\,,$$

by hypothesis. Hence, by maximum modulus principle, we have $f \in MR(p,m,\varepsilon,\sigma,\mu)$.

Conversely, let $f \in MR(p,m,\varepsilon,\sigma,\mu)$. Then from (1.5), we obtain

$$\left|\frac{\frac{zf''(z)}{f'(z)} + 1 - p}{\varepsilon\frac{zf''(z)}{f'(z)} + p + \varepsilon - \sigma(\varepsilon+1)}\right| = \left|\frac{\sum_{n=p+m}^{\infty} n(n-p)a_n z^{n-1}}{p(p-\sigma)(\varepsilon+1)z^{p-1} - \sum_{n=p+m}^{\infty} n(\varepsilon(n-\sigma)+p-\sigma)a_n z^{n-1}}\right| < \mu.$$

Since $Re(z) \leq |z|$ for all z ($z \in U$), we get

$$Re\left\{\frac{\sum_{n=p+m}^{\infty} n(n-p)a_n z^{n-1}}{p(p-\sigma)(\varepsilon+1)z^{p-1} - \sum_{n=p+m}^{\infty} n(\varepsilon(n-\sigma)+p-\sigma)a_n z^{n-1}}\right\} < \mu. \tag{2.3}$$

We choose the value of z on the real axis so that $\frac{zf''(z)}{f'(z)}$ is real. Upon clearing the denominator of (2.3) and letting $z \to 1^-$, through real values, so we can write (2.3) as

$$\sum_{n=p+m}^{\infty} n[n-p+\mu(\varepsilon(n-\sigma)+p-\sigma)]a_n \leq \mu p(p-\sigma)(\varepsilon+1).$$

Corollary 2.1. If $f \in MR(p,m,\varepsilon,\sigma,\mu)$. Then

$$a_n \leq \frac{\mu p(p-\sigma)(\varepsilon+1)}{n[n-p+\mu(\varepsilon(n-\sigma)+p-\sigma)]}\ , \quad (n \geq p+m, \ \ p,m \in \mathbb{N}).$$

Theorem 2.2. If $f \in MR(p,m,\varepsilon,\sigma,\mu)$, then

$$\left|D_z^{-\lambda} f(z)\right| \leq \frac{\Gamma(p+1)|z|^{p+\lambda}}{\Gamma(p+\lambda+1)} \times$$

$$\times\left[1+\frac{\mu p(p-\sigma)(\varepsilon+1)\Gamma(p+m+1)\Gamma(p+\lambda+1)}{(p+m)[m+\mu(\varepsilon(p+m-\sigma)+p-\sigma)]\Gamma(p+1)\Gamma(p+m+\lambda+1)}|z|^{m}\right] \qquad (2.4)$$

and

$$\left|D_z^{-\lambda} f(z)\right| \geq \frac{\Gamma(p+1)|z|^{p+\lambda}}{\Gamma(p+\lambda+1)} \times$$

$$\times\left[1-\frac{\mu p(p-\sigma)(\varepsilon+1)\Gamma(p+m+1)\Gamma(p+\lambda+1)}{(p+m)[m+\mu(\varepsilon(p+m-\sigma)+p-\sigma)]\Gamma(p+1)\Gamma(p+m+\lambda+1)}|z|^{m}\right] \qquad (2.5)$$

The result is sharp for the function f given by

$$f(z)=z^{p}-\frac{\mu p(p-\sigma)(\varepsilon+1)}{(p+m)[m+\mu(\varepsilon(p+m-\sigma)+p-\sigma)]}\, z^{p+m}, (p,m\in\mathbb{N}). \qquad (2.6)$$

Proof. Let $f\in MR(p,m,\varepsilon,\sigma,\mu)$. By (1.3), we have

$$\frac{\Gamma(p+\lambda+1)}{\Gamma(p+1)} z^{-\lambda} D_z^{-\lambda} f(z)=z^{p}-\sum_{n=p+m}^{\infty}\frac{\Gamma(n+1)\Gamma(p+\lambda+1)}{\Gamma(p+1)\Gamma(n+\lambda+1)} a_n z^{n}.$$

Setting

$$\psi(n,\lambda)=\frac{\Gamma(n+1)\Gamma(p+\lambda+1)}{\Gamma(p+1)\Gamma(n+\lambda+1)} \qquad (n\geq p+m, p,m\in\mathbb{N}),$$

we get

$$\frac{\Gamma(p+\lambda+1)}{\Gamma(p+1)} z^{-\lambda} D_z^{-\lambda} f(z)=z^{p}-\sum_{n=p+m}^{\infty}\psi(n,\lambda) a_n z^{n}.$$

Since for $n\geq p+m$, ψ is a decreasing function of n, then we have

$$0<\psi(n,\lambda)\leq\psi(p+m,\lambda)=\frac{\Gamma(p+m+1)\Gamma(p+\lambda+1)}{\Gamma(p+1)\Gamma(p+m+\lambda+1)}. \qquad (2.7)$$

Now, by the application of Theorem 2.1 and (2.7), we obtain

$$\left|\frac{\Gamma(p+\lambda+1)}{\Gamma(p+1)} z^{-\lambda} D_z^{-\lambda} f(z)\right| \leq |z|^{p}+\psi(p+m,\lambda)|z|^{p+m}\sum_{n=p+m}^{\infty} a_n$$

$$\leq |z|^{p}+\frac{\mu p(p-\sigma)(\varepsilon+1)\Gamma(p+m+1)\Gamma(p+\lambda+1)}{(p+m)[m+\mu(\varepsilon(p+m-\sigma)+p-\sigma)]\Gamma(p+1)\Gamma(p+m+\lambda+1)}|z|^{p+m},$$

which gives (2.4), we also have

$$\left|\frac{\Gamma(p+\lambda+1)}{\Gamma(p+1)}\, z^{-\lambda} D_z^{-\lambda} f(z)\right| \geq |z|^p - \psi(p+m,\lambda)|z|^{p+m} \sum_{n=p+m}^{\infty} a_n$$

$$\geq |z|^p - \frac{\mu p(p-\sigma)(\varepsilon+1)\Gamma(p+m+1)\Gamma(p+\lambda+1)}{(p+m)[m+\mu(\varepsilon(p+m-\sigma)+p-\sigma)]\Gamma(p+1)\Gamma(p+m+\lambda+1)} |z|^{p+m},$$

which gives (2.5).

By taking $\lambda = 1$ in Theorem 2.2, we obtain the following Corollary:

Corollary 2.2. If $f \in MR(p,m,\varepsilon,\sigma,\mu)$, then

$$\left|\int_0^z f(t)dt\right| \leq \frac{|z|^{p+1}}{p+1}\left[1+\frac{\mu p(p+1)(p-\sigma)(\varepsilon+1)}{(p+m)(p+m+1)[m+\mu(\varepsilon(p+m-\sigma)+p-\sigma)]}|z|^m\right]$$

and

$$\left|\int_0^z f(t)dt\right| \geq \frac{|z|^{p+1}}{p+1}\left[1-\frac{\mu p(p+1)(p-\sigma)(\varepsilon+1)}{(p+m)(p+m+1)[m+\mu(\varepsilon(p+m-\sigma)+p-\sigma)]}|z|^m\right].$$

Proof. By Definition 1.1 and Theorem 2.2 for $\lambda = 1$, we have $D_z^{-1} f(z) = \int_0^z f(t)dt$, the result is true.

Theorem 2.3. If $f \in MR(p,m,\varepsilon,\sigma,\mu)$, then

$$\left|D_z^{\lambda} f(z)\right| \leq \frac{\Gamma(p+1)|z|^{p-\lambda}}{\Gamma(p-\lambda+1)} \times$$

$$\times\left[1+\frac{\mu p(p-\sigma)(\varepsilon+1)\Gamma(p+m+1)\Gamma(p-\lambda+1)}{(p+m)[m+\mu(\varepsilon(p+m-\sigma)+p-\sigma)]\Gamma(p+1)\Gamma(p+m-\lambda+1)}|z|^m\right] \quad (2.8)$$

and

$$\left|D_z^{\lambda} f(z)\right| \geq \frac{\Gamma(p+1)|z|^{p-\lambda}}{\Gamma(p-\lambda+1)} \times$$

$$\times\left[1-\frac{\mu p(p-\sigma)(\varepsilon+1)\Gamma(p+m+1)\Gamma(p-\lambda+1)}{(p+m)[m+\mu(\varepsilon(p+m-\sigma)+p-\sigma)]\Gamma(p+1)\Gamma(p+m-\lambda+1)}|z|^m\right]. \quad (2.9)$$

The result is sharp for the function f given by (2.6).

Proof. Let $f \in MR(p,m,\varepsilon,\sigma,\mu)$. By (1.4), we have

$$\frac{\Gamma(p-\lambda+1)}{\Gamma(p+1)} z^{\lambda} D_z^{\lambda} f(z) = z^p - \sum_{n=p+m}^{\infty} \frac{\Gamma(n+1)\Gamma(p-\lambda+1)}{\Gamma(p+1)\Gamma(n-\lambda+1)} a_n z^n = z^p - \sum_{n=p+m}^{\infty} \phi(n,\lambda) a_n z^n,$$

where

$$\phi(n,\lambda) = \frac{\Gamma(n+1)\Gamma(p-\lambda+1)}{\Gamma(p+1)\Gamma(n-\lambda+1)} \quad (n \geq p+m, p, m \in \mathbb{N}).$$

Since for $n \geq p+m$, ϕ is a decreasing function of n, thus we have

$$0 < \phi(n,\lambda) \leq \phi\,(p+m\,,\lambda) = \frac{\Gamma(p+m+1)\Gamma(p-\lambda+1)}{\Gamma(p+1)\Gamma(p+m-\lambda+1)}.$$

Also, by using Theorem 2.1, we get

$$\sum_{n=p+m}^{\infty} a_n \leq \frac{\mu p(p-\sigma)(\varepsilon+1)}{(p+m)[m+\mu(\varepsilon(p+m-\sigma)+p-\sigma)]}.$$

Thus

$$\left|\frac{\Gamma(p-\lambda+1)}{\Gamma(p+1)} z^{\lambda}\, D_z^{\lambda} f(z)\right| \leq |z|^p + \phi(p+m,\lambda)|z|^{p+m} \sum_{n=p+m}^{\infty} a_n$$

$$\leq |z|^p + \frac{\mu p(p-\sigma)(\varepsilon+1)\Gamma(p+m+1)\Gamma(p-\lambda+1)}{(p+m)[m+\mu(\varepsilon(p+m-\sigma)+p-\sigma)]\Gamma(p+1)\Gamma(p+m-\lambda+1)} |z|^{p+m}.$$

Then

$$\left|D_z^{\lambda} f(z)\right| \leq \frac{\Gamma(p+1)|z|^{p-\lambda}}{\Gamma(p-\lambda+1)} \times$$

$$\times \left[1 + \frac{\mu p(p-\sigma)(\varepsilon+1)\Gamma(p+m+1)\Gamma(p-\lambda+1)}{(p+m)[m+\mu(\varepsilon(p+m-\sigma)+p-\sigma)]\Gamma(p+1)\Gamma(p+m-\lambda+1)} |z|^m\right],$$

and by the same way, we obtain

$$\left|D_z^{\lambda} f(z)\right| \geq \frac{\Gamma(p+1)|z|^{p-\lambda}}{\Gamma(p-\lambda+1)} \times$$

$$\times \left[1 - \frac{\mu p(p-\sigma)(\varepsilon+1)\Gamma(p+m+1)\Gamma(p-\lambda+1)}{(p+m)[m+\mu(\varepsilon(p+m-\sigma)+p-\sigma)]\Gamma(p+1)\Gamma(p+m-\lambda+1)} |z|^m\right].$$

By taking $\lambda = 0$ in Theorem 2.3, we obtain the following Corollary:

Corollary 2.3. If $f \in MR(p,m,\varepsilon,\sigma,\mu)$, then

$$|f(z)| \leq |z|^p \left[1 + \frac{\mu p(p-\sigma)(\varepsilon+1)}{(p+m)[m+\mu(\varepsilon(p+m-\sigma)+p-\sigma)]} |z|^m\right]$$

and

$$|f(z)| \geq |z|^p \left[1 - \frac{\mu p(p-\sigma)(\varepsilon+1)}{(p+m)[m+\mu(\varepsilon(p+m-\sigma)+p-\sigma)]}|z|^m\right].$$

Proof. By Definition 1.2 and Theorem 2.3 for $\lambda = 0$, we have $D_z^0\, f(z) = \frac{d}{dz}\int_0^z f(t)dt = f(z)$, the result is true.

Corollary 2.4. $D_z^{-\lambda}\, f(z)$ and $D_z^{\lambda}\, f(z)$ are included in the disk with center at the origin and radii

$$\frac{\Gamma(p+1)}{\Gamma(p+\lambda+1)}\left[1 + \frac{\mu p(p-\sigma)(\varepsilon+1)\Gamma(p+m+1)\Gamma(p+\lambda+1)}{(p+m)[m+\mu(\varepsilon(p+m-\sigma)+p-\sigma)]\Gamma(p+1)\Gamma(p+m+\lambda+1)}\right],$$

$$\frac{\Gamma(p+1)}{\Gamma(p-\lambda+1)}\left[1 + \frac{\mu p(p-\sigma)(\varepsilon+1)\Gamma(p+m+1)\Gamma(p-\lambda+1)}{(p+m)[m+\mu(\varepsilon(p+m-\sigma)+p-\sigma)]\Gamma(p+1)\Gamma(p+m-\lambda+1)}\right],$$

respectively.

3. NEIGHBORHOOD PROPERTY

Following the work of Goodman [5] and Ruscheweyh [9], we define the $(m-\delta)$ −neighborhood of a function $f \in R(p,m)$ by means of the definition below:

$$N_{m,\delta}(f) = \left\{ g \in R(p,m) : g(z) = z^p - \sum_{n=p+m}^{\infty} b_n z^n \ \ and \ \ \sum_{n=p+m}^{\infty} n|a_n - b_n| \leq \delta\,, 0 \leq \delta < 1 \right\}. \tag{3.1}$$

In Particular, for the identity function $e(z) = z^p$, we have

$$N_{m,\delta}(e) = \left\{ g \in R(p,m) : g(z) = z^p - \sum_{n=p+m}^{\infty} b_n z^n \ \ and \ \ \sum_{n=p+m}^{\infty} n|b_n| \leq \delta \right\}. \tag{3.2}$$

Definition 3.1. A function $f \in R(p,m)$ is said to be in the class $MR_y(p,m,\varepsilon,\sigma,\mu)$ if there exists a function $g \in MR(p,m,\varepsilon,\sigma,\mu)$, such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < p - y \quad (z \in U, 0 \leq y < p)\,.$$

Theorem 3.1. If $g \in MR(p,m,\varepsilon,\sigma,\mu)$ and

$$y = p - \frac{\delta[m+\mu(\varepsilon(p+m-\sigma)+p-\sigma)]}{(p+m)[m+\mu(\varepsilon(p+m-\sigma)+p-\sigma)] - \mu p(p-\sigma)(\varepsilon+1)}\,, \tag{3.3}$$

then $N_{m,\delta}(g) \subset MR_y(p,m,\varepsilon,\sigma,\mu)$.

Proof. Let $f \in N_{m,\delta}(g)$. Then we find from (3.1) that

$$\sum_{n=p+m}^{\infty} n|a_n - b_n| \leq \delta,$$

which implies the coefficient inequality

$$\sum_{n=p+m}^{\infty} |a_n - b_n| \leq \frac{\delta}{p+m}, \quad (n, p \in \mathbb{N}).$$

Since $g \in MR(p, m, \varepsilon, \sigma, \mu)$, then by using Theorem 2.1, we have

$$\sum_{n=p+m}^{\infty} b_n \leq \frac{\mu p(p-\sigma)(\varepsilon+1)}{(p+m)[m+\mu(\varepsilon(p+m-\sigma)+p-\sigma)]},$$

so that

$$\left|\frac{f(z)}{g(z)} - 1\right| \leq \frac{\sum_{n=p+m}^{\infty} |a_n - b_n||z|^{n-p}}{1 - \sum_{n=p+m}^{\infty} b_n|z|^{n-p}} < \frac{\sum_{n=p+m}^{\infty} |a_n - b_n|}{1 - \sum_{n=p+m}^{\infty} b_n}$$

$$\leq \frac{\delta[m+\mu(\varepsilon(p+m-\sigma)+p-\sigma)]}{(p+m)[m+\mu(\varepsilon(p+m-\sigma)+p-\sigma)] - \mu p(p-\sigma)(\varepsilon+1)} = p - y.$$

Hence, by Definition 3.1, equivalently to $f \in MR_y(p, m, \varepsilon, \sigma, \mu)$ for y given by (3.3).

4. Radii of Starlikeness and Convexity

Theorem 4.1. If $f \in MR(p, m, \varepsilon, \sigma, \mu)$, then f will be $p-$valently starlike of order α $(0 \leq \alpha < p)$ in the disk $|z| < r_1$, where

$$r_1 = \inf_n \left\{\frac{n(p-\alpha)[m+\mu(\varepsilon(p+m-\sigma)+p-\sigma)]}{\mu p(n-\alpha)(p-\sigma)(\varepsilon+1)}\right\}^{\frac{1}{n-p}}, \quad (n \geq p+m).$$

The result is sharp for the function f given by (2.2).

Proof. It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq p - \alpha \quad \text{for } |z| < r_1. \tag{4.1}$$

We have

$$\left|\frac{zf'(z)}{f(z)}-p\right| \leq \frac{\sum_{n=p+m}^{\infty}(n-p)a_n|z|^{n-p}}{1-\sum_{n=p+m}^{\infty}a_n|z|^{n-p}}.$$

Thus (4.1) will be satisfied if

$$\sum_{n=p+m}^{\infty}\left(\frac{n-\alpha}{p-\alpha}\right)a_n|z|^{n-p} \leq 1. \tag{4.2}$$

Also from Theorem 2.1, if $f \in MR(p,m,\varepsilon,\sigma,\mu)$, then

$$\sum_{n=p+m}^{\infty}\frac{n[m+\mu(\varepsilon(p+m-\sigma)+p-\sigma)]}{\mu p(p-\sigma)(\varepsilon+1)}a_n \leq 1. \tag{4.3}$$

In view of (4.3), we notice that (4.2) holds true if

$$\frac{n-\alpha}{p-\alpha}|z|^{n-p} \leq \frac{n[m+\mu(\varepsilon(p+m-\sigma)+p-\sigma)]}{\mu p(p-\sigma)(\varepsilon+1)},$$

or equivalently

$$|z| \leq \left\{\frac{n(p-\alpha)[m+\mu(\varepsilon(p+m-\sigma)+p-\sigma)]}{\mu p(n-\alpha)(p-\sigma)(\varepsilon+1)}\right\}^{\frac{1}{n-p}},$$

setting $|z| = r_1$, we get the desired result.

Theorem 4.2. If $f \in MR(p,m,\varepsilon,\sigma,\mu)$, then f will be $p-$valently convex of order α $(0 \leq \alpha < p)$ in the disk $|z| < r_2$, where

$$r_2 = \inf_n\left\{\frac{(p-\alpha)[m+\mu(\varepsilon(p+m-\sigma)+p-\sigma)]}{\mu p(n-\alpha)(p-\sigma)(\varepsilon+1)}\right\}^{\frac{1}{n-p}}, \quad (n \geq p+m).$$

The result is sharp for the function f given by (2.2).

Proof. It is sufficient to show that

$$\left|\frac{zf''(z)}{f'(z)}+1-p\right| \leq p-\alpha \quad \text{for } |z| < r_2.$$

The result follows by application of arguments similar to the proof of Theorem 4.1.

Conflict of Interests

The authors declare that there is no conflict of interests.

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