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FUZZY SUBORDINATION RESULTS FOR FRACTIONAL INTEGRAL ASSOCIATED WITH GENERALIZED MITTAG-LEFFLER FUNCTION

ABBAS KAREEM WANAS[1,*] AND ABDULRAHMAN H. MAJEED[2]

[1]Department of Mathematics, College of Science, University of Al-Qadisiyah, Iraq

[2] Department of Mathematics, College of Science, University of Baghdad, Iraq

Abstract: By making use of fractional integral, we study fuzzy subordination methods to obtain some interesting results of operator defined by generalized Mittag-Leffler function in the open unit disk.

Keywords: fuzzy differential subordination; fuzzy best dominant; generalized Mittag-Leffler function; fractional integral.

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1. Introduction

Let $\mathcal{H}(U)$ denote the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. For a positive integer number n and $a \in \mathbb{C}$, we denote by

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \ , \ z \in U\}$$

and

$$\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \cdots \ , \ z \in U\},$$

[*]Corresponding author
E-mail address: abbas.kareem.w@qu.edu.iq
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with $\mathcal{A}_1 = \mathcal{A}$.

Definition 1.1 (Zadeh, [15]) Let X be a non-empty set. An application $F : X \to [0,1]$ is called fuzzy subset. An alternate definition, more precise, would be the following:

A pair (A, F_A), where $F_A : X \to [0,1]$ and $A = \{x \in X : 0 < F_A(x) \le 1\} = supp(A, F_A)$ is called fuzzy subset. The function F_A is called membership function of the fuzzy subset (A, F_A).

Definition 1.2 (Oros and Gh Oros, [10]) Let two fuzzy subsets of X, (M, F_M) and (N, F_N). We say that the fuzzy subsets M and N are equal if and only if $F_M(x) = F_N(x), x \in X$ and we denote this by $(M, F_M) = (N, F_N)$. The fuzzy subset (M, F_M) is contained in the fuzzy subset (N, F_N) if and only if $F_M(x) \le F_N(x), x \in X$ and we denote the inclusion relation by $(M, F_M) \subseteq (N, F_N)$.

Let $D \subseteq \mathbb{C}$ and f, g analytic functions. We denote by

$$f(D) = supp\big(f(D), F_{f(D)}\big) = \big\{f(z) :\ 0 < F_{f(D)}\big(f(z)\big) \le 1, z \in D\big\}$$

and

$$g(D) = supp\big(g(D), F_{g(D)}\big) = \big\{g(z) :\ 0 < F_{g(D)}\big(g(z)\big) \le 1, z \in D\big\}.$$

Definition 1.3 (Oros and Gh Oros, [10]) Let $D \subseteq \mathbb{C}$, $z_0 \in D$ be a fixed point, and let the functions $f, g \in \mathcal{H}(D)$. The function f is said to be fuzzy subordinate to g and write $f \prec_F g$ or $f(z) \prec_F g(z)$ if the following conditions are satisfied:

1) $f(z_0) = g(z_0)$,
2) $F_{f(D)}\big(f(z)\big) \le F_{g(D)}\big(g(z)\big), z \in D$.

Definition 1.4 (Oros and Gh Oros, [11]) Let $\psi : \mathbb{C}^3 \times U \longrightarrow \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the (second-order) fuzzy differential subordination

$$F_{\psi(\mathbb{C}^3\times U)}\big(\psi(p(z), zp'(z), z^2p''(z); z)\big) \le F_{h(U)}\big(h(z)\big), \tag{1.1}$$

i.e.

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec_F h(z)\ , z \in U,$$

then p is called a fuzzy solution of the fuzzy differential subordination. The univalent function q is called a fuzzy dominant of the fuzzy solutions of the fuzzy differential subordination, or more simple a fuzzy dominant, if $p(z) \prec_F q(z), z \in U$ for all p satisfying (1.1). A fuzzy dominant $\tilde{q}$ that satisfies $\tilde{q}(z) \prec_F q(z), z \in U$ for all fuzzy dominant q of (1.1) is said to be the fuzzy best dominant of (1.1).

The Mittag-Leffler function $E_\alpha(z), (z \in \mathbb{C})$ (see [4,5]) is defined by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad (\alpha \in \mathbb{C}, Re(\alpha) > 0).$$

Several researchers have investigated properties of Mittag-Leffer function and generalized Mittag-Leffler function, see for example [2,3,6,7]. Moreover, Srivastava and Tomovski [9] introduced the function $E_{\alpha,\beta}^{\gamma,k}(z), (z \in \mathbb{C})$ in the form:

$$E_{\alpha,\beta}^{\gamma,k}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nk} z^n}{\Gamma(\alpha n + \beta) n!},$$

where $\alpha, \beta, \gamma \in \mathbb{C}, Re(\alpha) > \max\{0, Re(k) - 1\}, Re(k) > 0$ and $(x)_n$ is the Pochhammer symbol defined by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1 & (n = 0), \\ x(x+1) \dots (x+n-1) & (n \in \mathbb{N}). \end{cases}$$

Let $f_i \in \mathcal{A}$ $(i = 1,2)$ defined by

$$f_i(z) = z + \sum_{n=2}^{\infty} a_{n,i} z^n \ (i = 1,2),$$

the Hadamard product of f_1 and f_2 is defined by

$$(f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n = (f_2 * f_1)(z).$$

Definition 1.5 (Attiya, [1]) For $f \in \mathcal{A}$ the operator $\mathcal{H}_{\alpha,\beta}^{\gamma,k} : \mathcal{A} \longrightarrow \mathcal{A}$ is defined by

$$\mathcal{H}_{\alpha,\beta}^{\gamma,k} f(z) = Q_{\alpha,\beta}^{\gamma,k}(z) * f(z) \ (z \in U),$$

where

$$Q_{\alpha,\beta}^{\gamma,k}(z) = \frac{\Gamma(\alpha+\beta)}{(\gamma)_k} \left(E_{\alpha,\beta}^{\gamma,k}(z) - \frac{1}{\Gamma(\beta)} \right),$$

$\beta, \gamma \in \mathbb{C}, Re(\alpha) > \max\{0, Re(k) - 1\}, Re(k) > 0.$

By some easy calculations, we have

$$\mathcal{H}_{\alpha,\beta}^{\gamma,k} f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\gamma + nk)\Gamma(\alpha + \beta)}{\Gamma(\gamma + k)\Gamma(\beta + \alpha n) n!} a_n z^n .$$

Definition 1.6 (Srivastava and Owa, [8]) The fractional integral of order $\lambda, (\lambda > 0)$ is defined for a function f by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\epsilon)}{(z - \epsilon)^{1-\lambda}} d\epsilon,$$

where f is analytic function in a simply-connected region of the z-plane containing the origin and the multiplicity of $(z - \epsilon)^{\lambda - 1}$ is removed by requiring $\log(z - \epsilon)$ to be real, when $(z - \epsilon) > 0$.

We now, by making use of Definition 1.5 and Definition 1.6, we have

$$D_z^{-\lambda} \mathcal{H}_{\alpha,\beta}^{\gamma,k} f(z) = \frac{1}{\Gamma(2 + \lambda)} z^{1+\lambda} + \sum_{n=2}^{\infty} \frac{\Gamma(\gamma + nk)\Gamma(\alpha + \beta)}{\Gamma(n + 1 + \lambda)\Gamma(\gamma + k)\Gamma(\beta + \alpha n)} a_n z^{n+\lambda}. \quad (1.2)$$

It is easily verified from (1.2) that

$$z\left(D_z^{-\lambda} \mathcal{H}_{\alpha,\beta}^{\gamma,k} f(z)\right)' = \left(\frac{\gamma + k}{k}\right) D_z^{-\lambda} \mathcal{H}_{\alpha,\beta}^{\gamma+1,k} f(z) - \left(\frac{\gamma - \lambda k}{k}\right) D_z^{-\lambda} \mathcal{H}_{\alpha,\beta}^{\gamma,k} f(z), \quad Re(\gamma - \lambda k) \neq 0. \quad (1.3)$$

In our investigations we shall need the following lemmas.

Lemma 1.1 (Oros and Gh Oros, [12]) Let h be a convex function with $h(0) = a$, and let $\mu \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ be a complex number with $\mathrm{Re}(\mu) \geq 0$. If $p \in \mathcal{H}[a, n]$ with $p(0) = a$ and $\psi: \mathbb{C}^2 \times U \to \mathbb{C}$, $\psi(p(z), zp'(z)) = p(z) + \frac{1}{\mu} zp'(z)$ is analytic in U, then

$$F_{\psi(\mathbb{C}^2 \times U)}\left[p(z) + \frac{1}{\mu} zp'(z)\right] \leq F_{h(U)} h(z),$$

implies

$$F_{p(U)} p(z) \leq F_{q(U)} q(z) \leq F_{h(U)} h(z), \;\; z \in U,$$

i.e.

$$p(z) \prec_F q(z) \prec_F h(z),$$

where

$$q(z) = \frac{\mu}{n z^{\frac{\mu}{n}}} \int_0^z h(t) \, t^{\frac{\mu}{n} - 1} dt.$$

The function q is convex and is the fuzzy best dominant.

Lemma 1.2 (Oros and Gh Oros, [12]) Let q be a convex function in U and let the function $h(z) = q(z) + n\nu z q'(z)$, where $\nu > 0$ and $n \in \mathbb{N}$. If the function $p \in \mathcal{H}[q(0), n]$ and $\psi: \mathbb{C}^2 \times U \to \mathbb{C}$, $\psi(p(z), zp'(z)) = p(z) + \nu z p'(z)$ is analytic in U, then

$$F_{\psi(\mathbb{C}^2 \times U)}[p(z) + \nu z p'(z)] \leq F_{h(U)} h(z),$$

implies

$$F_{p(U)} p(z) \leq F_{q(U)} q(z), \;\; z \in U,$$

i.e.

$$p(z) \prec_F q(z)$$

and q is the fuzzy best dominant.

Recently, Oros and Oros [11,12] and Wanas and Majeed [13,14] have obtained fuzzy differential subordination results for certain classes of analytic functions.

2. MAIN RESULT

Theorem 2.1. Let h be a convex function such that $h(0) = 1$. Let $f \in \mathcal{A}$ and $G(\gamma, k, \alpha, \beta, \lambda; z)$ is analytic in U, where

$$\begin{aligned} G(\gamma, k, \alpha, \beta, \lambda; z) = {} & \frac{(1-\lambda)\lambda!}{k z^{1+\lambda}} \left((\gamma + k) D_z^{-\lambda} \mathcal{H}_{\alpha,\beta}^{\gamma+1,k} f(z) - (\gamma - \lambda k) D_z^{-\lambda} \mathcal{H}_{\alpha,\beta}^{\gamma,k} f(z) \right) \\ & + \frac{\lambda!}{z^{-1+\lambda}} \left(D_z^{-\lambda} \mathcal{H}_{\alpha,\beta}^{\gamma,k} f(z) \right)'' . \end{aligned} \tag{2.1}$$

If

$$F_{\psi(\mathbb{C}^2 \times U)}[G(\gamma, k, \alpha, \beta, \lambda; z)] \leq F_{h(U)} h(z), \tag{2.2}$$

then

$$F_{\left(D_z^{-\lambda} \mathcal{H}_{\alpha,\beta}^{\gamma,k} f\right)'(U)} \left(\frac{\lambda! \left(D_z^{-\lambda} \mathcal{H}_{\alpha,\beta}^{\gamma,k} f(z) \right)'}{z^{\lambda}} \right) \leq F_{q(U)} q(z) \leq F_{h(U)} h(z),$$

i.e.

$$\frac{\lambda!\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f(z)\right)'}{z^{\lambda}} \prec_F q(z) \prec_F h(z),$$

where $q(z)=\frac{1}{z}\int_0^z h(t)\,dt$ is convex and is the fuzzy best dominant.

Proof. Suppose that

$$p(z)=\frac{\lambda!\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f(z)\right)'}{z^{\lambda}}\,. \tag{2.3}$$

Then $p\in\mathcal{H}[1,1]$ and $p(0)=1$. Therefore, by making use of (1.3) and (2.3), we have

$$p(z)+zp'(z)=1+\sum_{n=2}^{\infty}\frac{n(n+\lambda)\Gamma(\gamma+nk)\Gamma(\alpha+\beta)\lambda!}{\Gamma(n+1+\lambda)\Gamma(\gamma+k)\Gamma(\beta+\alpha n)}a_n z^{n-1}$$

$$=\frac{(1-\lambda)(\gamma+k)\lambda!}{kz^{1+\lambda}}\left[\frac{1}{\Gamma(2+\lambda)}z^{1+\lambda}+\sum_{n=2}^{\infty}\frac{\Gamma(\gamma+1+nk)\Gamma(\alpha+\beta)}{\Gamma(n+1+\lambda)\Gamma(\gamma+1+k)\Gamma(\beta+\alpha n)}a_n z^{n+\lambda}\right]$$

$$+\frac{\gamma(\lambda-1)+2\lambda k}{k(1+\lambda)}-\sum_{n=2}^{\infty}\frac{\Gamma(\gamma+nk)\Gamma(\alpha+\beta)(1-\lambda)(\gamma-\lambda k)\lambda!}{\Gamma(n+1+\lambda)\Gamma(\gamma+k)\Gamma(\beta+\alpha n)k}a_n z^{n-1}$$

$$+\sum_{n=2}^{\infty}\frac{\Gamma(\gamma+nk)\Gamma(\alpha+\beta)(n+\lambda)(n+\lambda-1)\lambda!}{\Gamma(n+1+\lambda)\Gamma(\gamma+k)\Gamma(\beta+\alpha n)k}a_n z^{n-1}=G(\gamma,k,\alpha,\beta,\lambda;z), \tag{2.4}$$

where $G(\gamma,k,\alpha,\beta,\lambda;z)$ is given by (2.1).

From (2.2) and (2.4), we get

$$F_{\psi(\mathbb{C}^2\times U)}[p(z)+zp'(z)]\le F_{h(U)}h(z).$$

Thus, by applying Lemma 1.1 with $\mu=1$, we obtain

$$F_{p(U)}p(z)\le F_{q(U)}q(z)\le F_{h(U)}h(z).$$

By (2.3), we have

$$F_{\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f\right)'(U)}\left(\frac{\lambda!\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f(z)\right)'}{z^{\lambda}}\right)\le F_{q(U)}q(z)\le F_{h(U)}h(z),$$

i.e.

$$\frac{\lambda!\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f(z)\right)'}{z^{\lambda}} \prec_F q(z) \prec_F h(z),$$

where $q(z) = \frac{1}{z}\int_0^z h(t)\,dt$ is convex and is the fuzzy best dominant.

If we take $\gamma = k = 1, \alpha = 0$ and $h(z) = \frac{1+(2\rho-1)z}{1+z}$ $(0 \le \rho < 1)$ in Theorem 2.1, we obtain the following corollary:

Corollary 2.1. Let $f \in \mathcal{A}$ and

$$\frac{(1-\lambda)\lambda!}{z^{1+\lambda}}\left(\lambda D_z^{-\lambda} f(z) + D_z^{-\lambda} z f'(z)\right) + \frac{\lambda!}{z^{-1+\lambda}}\left(D_z^{-\lambda} f(z)\right)''$$

is analytic in U. If

$$\frac{(1-\lambda)\lambda!}{z^{1+\lambda}}\left(\lambda D_z^{-\lambda} f(z) + D_z^{-\lambda} z f'(z)\right) + \frac{\lambda!}{z^{-1+\lambda}}\left(D_z^{-\lambda} f(z)\right)'' \prec_F \frac{1+(2\rho-1)z}{1+z},$$

then

$$\frac{\lambda!\left(D_z^{-\lambda} f(z)\right)'}{z^{\lambda}} \prec_F q(z) \prec_F \frac{1+(2\rho-1)z}{1+z},$$

where $q(z) = 2\rho - 1 + \frac{2(1-\rho)}{z}\ln(1+z)$ is convex and is the fuzzy best dominant.

Theorem 2.2. Let h be a convex function such that $h(0) = 1$. Let $f \in \mathcal{A}$ and $\frac{\lambda!\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k} f(z)\right)'}{z^{\lambda}}$ is analytic in U. If

$$F_{\psi(\mathbb{C}^2\times U)}\left[\frac{\lambda!\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k} f(z)\right)'}{z^{\lambda}}\right] \le F_{h(U)}h(z), \qquad (2.5)$$

then

$$F_{\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k} f\right)(U)}\left(\frac{(1+\lambda)!\,D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k} f(z)}{z^{1+\lambda}}\right) \le F_{q(U)}q(z) \le F_{h(U)}h(z),$$

i.e.

$$\frac{(1+\lambda)!\,D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k} f(z)}{z^{1+\lambda}} \prec_F q(z) \prec_F h(z),$$

where $q(z) = \frac{1}{z}\int_0^z h(t)\,dt$ is convex and is the fuzzy best dominant.

Proof. Suppose that

$$p(z)=\frac{(1+\lambda)!\,D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f(z)}{z^{1+\lambda}}. \tag{2.6}$$

Then $p\in\mathcal{H}[1,1]$ and $p(0)=1$.

We have

$$p(z)+\frac{1}{\lambda+1}zp'(z)=\frac{\lambda!\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f(z)\right)'}{z^{\lambda}}. \tag{2.7}$$

From (2.7), the fuzzy differential subordination (2.5) becomes

$$F_{\psi(\mathbb{C}^2\times U)}\left[p(z)+\frac{1}{\lambda+1}zp'(z)\right]\leq F_{h(U)}h(z).$$

Thus, by applying Lemma 1.1 with $\mu=\lambda+1$, we obtain

$$F_{p(U)}p(z)\leq F_{q(U)}q(z)\leq F_{h(U)}h(z).$$

By (2.6), we get

$$F_{\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f\right)(U)}\left(\frac{(1+\lambda)!\,D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f(z)}{z^{1+\lambda}}\right)\leq F_{q(U)}q(z)\leq F_{h(U)}h(z),$$

i.e.

$$\frac{(1+\lambda)!\,D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f(z)}{z^{1+\lambda}}\prec_F q(z)\prec_F h(z),$$

where $q(z)=\frac{1}{z}\int_0^z h(t)\,dt$ is convex and is the fuzzy best dominant.

If we take $\gamma=k=1,\alpha=0$ and $h(z)=e^{bz},|b|\leq 1$ in Theorem 2.2, we obtain the following corollary:

Corollary 2.2. Let $f\in\mathcal{A}$ and $\frac{\lambda!\left(D_z^{-\lambda}f(z)\right)'}{z^{\lambda}}$ is analytic in U. If

$$\frac{\lambda!\left(D_z^{-\lambda}f(z)\right)'}{z^{\lambda}}\prec_F e^{bz},$$

then

$$\frac{(1+\lambda)!\,D_z^{-\lambda}f(z)}{z^{1+\lambda}}\prec_F q(z)\prec_F e^{bz},$$

where $q(z)=\frac{e^{bz}-1}{bz}$ is convex and is the fuzzy best dominant.

Theorem 2.3. Let q be a convex function in U such that $q(0)=1$ and let h be the function

$h(z) = q(z) + \frac{k}{\gamma+k} zq'(z)$. Let $f \in \mathcal{A}$ and $\frac{\lambda!\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma+1,k} f(z)\right)'}{z^\lambda}$ is analytic in U. If

$$F_{\psi(\mathbb{C}^2\times U)}\left[\frac{\lambda!\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma+1,k} f(z)\right)'}{z^\lambda}\right] \leq F_{h(U)} h(z), \tag{2.8}$$

then

$$F_{\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k} f\right)'(U)}\left(\frac{\lambda!\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k} f(z)\right)'}{z^\lambda}\right) \leq F_{q(U)} q(z),$$

i.e.

$$\frac{\lambda!\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k} f(z)\right)'}{z^\lambda} \prec_F q(z)$$

and q is fuzzy best dominant.

Proof. Suppose that

$$p(z) = \frac{\lambda!\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k} f(z)\right)'}{z^\lambda}. \tag{2.9}$$

Then $p \in \mathcal{H}[1,1]$.

Differentiating both sides of (2.9) with respect to z, we have

$$p(z) + \frac{k}{\gamma+k} zp'(z) = \frac{\lambda!\, k\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k} f(z)\right)''}{(\gamma+k)z^{-1+\lambda}} + \frac{\lambda!\left(\gamma + k(1-\lambda)\right)\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k} f(z)\right)'}{(\gamma+k)z^\lambda}. \tag{2.10}$$

By using (1.3) and differentiating with respect to z, we obtain

$$\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma+1,k} f(z)\right)' = \frac{kz\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k} f(z)\right)''}{\gamma+k} + \frac{\left(\gamma + k(1-\lambda)\right)\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k} f(z)\right)'}{\gamma+k}.$$

So

$$\frac{\lambda!\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma+1,k} f(z)\right)'}{z^\lambda} = \frac{\lambda!\, k\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k} f(z)\right)''}{(\gamma+k)z^{-1+\lambda}} + \frac{\lambda!\left(\gamma + k(1-\lambda)\right)\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k} f(z)\right)'}{(\gamma+k)z^\lambda}. \tag{2.11}$$

From (2.10) and (2.11), the fuzzy differential subordination (2.8) becomes

$$F_{\psi(\mathbb{C}^2\times U)}\left[p(z) + \frac{k}{\gamma+k} zp'(z)\right] \leq F_{h(U)} h(z).$$

Thus, by applying Lemma 1.2 with $\nu = \frac{k}{\gamma+k}$, we obtain

$$F_{\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f\right)'(U)}\left(\frac{\lambda!\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f(z)\right)'}{z^{\lambda}}\right) \leq F_{q(U)}q(z),$$

i.e.

$$\frac{\lambda!\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f(z)\right)'}{z^{\lambda}} \prec_F q(z)$$

and q is fuzzy best dominant.

If we take $\gamma = k = 1, \alpha = 0$ and $q(z) = \frac{1+z}{1-z}$ in Theorem 2.3, we obtain the following corollary:

Corollary 2.3. Let $f \in \mathcal{A}$ and $\frac{\lambda!\left(D_z^{-\lambda}f(z)+D_z^{-\lambda}zf'(z)\right)'}{2z^{\lambda}}$ is analytic in U. If

$$\frac{\lambda!\left(D_z^{-\lambda}f(z) + D_z^{-\lambda}zf'(z)\right)'}{2z^{\lambda}} \prec_F \frac{1+z-z^2}{(1-z)^2},$$

then

$$\frac{\lambda!\left(D_z^{-\lambda}f(z)\right)'}{z^{\lambda}} \prec_F \frac{1+z}{1-z}$$

and $q(z) = \frac{1+z}{1-z}$ is fuzzy best dominant.

Theorem 2.4. Let q be a convex function in U such that $q(0) = 1$ and let h be the function $h(z) = q(z) + zq'(z)$. Let $f \in \mathcal{A}$ and $\left(\frac{z\,D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma+1,k}f(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f(z)}\right)'$ is analytic in U. If

$$F_{\psi(\mathbb{C}^2\times U)}\left[\left(\frac{z\,D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma+1,k}f(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f(z)}\right)'\right] \leq F_{h(U)}h(z), \tag{2.12}$$

then

$$F_{\left(\frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma+1,k}f}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f}\right)(U)}\left(\frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma+1,k}f(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f(z)}\right) \leq F_{q(U)}q(z),$$

i.e.

$$\frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma+1,k}f(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f(z)} \prec_F q(z)$$

and q is fuzzy best dominant.

Proof. Suppose that

$$p(z) = \frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma+1,k}f(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f(z)}. \tag{2.13}$$

Then $p \in \mathcal{H}[1,1]$.

Differentiating both sides of (2.13) with respect to z, we have

$$p'(z) = \frac{\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma+1,k}f(z)\right)'}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f(z)} - p(z)\frac{\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f(z)\right)'}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f(z)}.$$

Then

$p(z) + zp'(z)$

$$= \frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f(z)\left(z\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma+1,k}f(z)\right)' + D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma+1,k}f(z)\right) - zD_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma+1,k}f(z)\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f(z)\right)'}{\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f(z)\right)^2}.$$

$$= \left(\frac{z\, D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma+1,k}f(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f(z)}\right)'. \tag{2.14}$$

By using (2.14) in (2.12), we have

$$F_{\psi(\mathbb{C}^2\times U)}[p(z) + zp'(z)] \leq F_{h(U)}h(z).$$

Thus, by applying Lemma 1.2 with $\nu = 1$, we obtain

$$F_{\left(\frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma+1,k}f}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f}\right)(U)}\left(\frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma+1,k}f(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f(z)}\right) \leq F_{q(U)}q(z),$$

i.e.

$$\frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma+1,k}f(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f(z)} \prec_F q(z)$$

and q is fuzzy best dominant.

If we take $\gamma = k = 1, \alpha = 0$ and $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 2.4, we obtain the following corollary:

Corollary 2.4. Let $f \in \mathcal{A}$ and $\left(\frac{z\left(D_z^{-\lambda} f(z) + D_z^{-\lambda} z f'(z)\right)}{2D_z^{-\lambda} f(z)}\right)'$ is analytic in U. If

$$\left(\frac{z\left(D_z^{-\lambda} f(z) + D_z^{-\lambda} z f'(z)\right)}{2D_z^{-\lambda} f(z)}\right)' \prec_F \frac{1 + 2z - z^2}{(1 + Bz)^2},$$

then

$$\frac{1}{2}\left(1 + \frac{D_z^{-\lambda} z f'(z)}{D_z^{-\lambda} f(z)}\right) \prec_F \frac{1 + Az}{1 + Bz}$$

and $q(z) = \frac{1+Az}{1+Bz}$ is fuzzy best dominant.

Conflict of Interests

The authors declare that there is no conflict of interests.

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