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EXPONENTIAL STABILITY OF RANDOM IMPULSIVE SEMILINEAR INTEGRO-DIFFERENTIAL SYSTEMS

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Abstract: In this article, we study the existence, uniqueness and exponential stability of random impulsive semilinear integro-differential systems. The results are obtained by using the contraction mapping principle. finally, an example is given to illustrate the applications of the abstract results.

Keywords: Semilinear integro-differential equation; random impulses; exponential stability; contraction principle.

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1. INTRODUCTION

Impulsive differential equations have become more important in recent years in some mathematical models of processes and phenomena studied in physics, optimal control, chemotherapy, biotechnology, population dynamics and ecology. There have been much research activity concerning the theory of impulsive differential equations see [2-6]. The impulses may exist at deterministic or random points. There are a lot of papers which investigate the properties of deterministic impulses see [9] and the references therein.

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Thus the random impulsive equations gives more realistic than deterministic impulsive equations. There are few publications in this field, Wu and Duan brought forword random impulsive ordinary differential equations and investigated boundedness of soloutions to these models by Liapunov's direct function in [11] . Wu et al, studied some qualitative properties of random impulses in [7,8,10]. In [12-14] the author studied the existence results for the random impulsive neutral functional differential equations and differential inclutions with delays. In [13], the authors generalized the distribution of random impulses with the Erlang distribution.

The stabilities like continuous dependence, Hyers- Ulam stability, Hyers- Ulam-Rassias stability, exponential stability and asymptotic stability have attracted the attention of many mathematicians see [15-18]. Motivated by the above mentioned works, the main purpose of this paper is to study of random impulsive semilinear integrodifferential systems. We relaxed the Lipchitz condition on the impulsive term and under our assumption it is enough to be bounded.

This article is organized as follows: In section 2, we recall some notations, definitions, concepts of random impulsive semilinear integrodifferential systems, In section 3, the assumptions, existence and uniqueness of solutions of random impulsive semilinear integrodifferential systems, In section 4, we study the exponential stability of random impulsive semilinear integrodifferential systems, In section 5, we provide an example to illustrate the applications of the obtained result.

2. PRELIMINARIES

Let ⦀ • ⦀ denote the Euclidean norm in $\mathcal{R}^n$. If B is a vector or a matrix, its transpose is denoted by B^T; if b is a matrix, its Frobenius norm is represented by ⦀ B ⦀ = $\{trace(B^TB)\}^{\frac{1}{2}}$. Let $\mathcal{R}^n$ be the n-dimensional Euclidean space and Ω a nonempty set. Assume that δ_k is a random variable defined from Ω to $D_k \overset{\text{def}}{=} (0\ ,d_k)$ for all k=1,2,…where $0< d_k < +\infty$. Furthermore, assume that δ_i and δ_j are independent with each other as i ≠ j for i,j = 1,2,…. Let δ,T ∈$\mathcal{R}$ be two constants satisfying δ< T. For the sake of simplicity, we denote $\mathcal{R}_\delta = [\delta, T]$.

We consider semilinear integro- differential systems with random impulses of the form

x' (t) = Bx(t) + f (t,x(t)) + $\int_0^t g(s,x_s)$, t ≠ η_k, t ≥ δ , (2.1

x(η_k) = $b_k(\delta_k)$ x(η_k^-) , k = 1,2,… , (2.2)

x_{t_0} = x_0 (2.3)

Where B is a matrix of dimension $n \times n$: the functions f,g : $\mathcal{R}^n \times \mathcal{R}^n \to \mathcal{R}^n$; $b_k : D_k \to \mathcal{R}^{n\times n}$ is a matrix valued function for each k = 1,2,…; $\eta_0 = t_0$ and $\eta_k = \eta_{k-1} + \delta_k$ for k = 1,2,…, here $t_0 \in \mathcal{R}_\delta$ is arbitrary real number. Obviously, $t_0 = \eta_0 < \eta_1 < \ldots < \eta_k < \ldots$; x($\eta_k^-$) = $\lim_{t\uparrow\eta_k} x(t)$ according to their paths with the norm $|||x||| = sup_{\delta\le s\le t} \| x(s) \|$ for each t satisfying $\delta \le s \le T$.

Let us denote $\{A_t, t \ge 0\}$ by the simple counting process generated by $\{\eta_n\}$, that is , $\{A_t \ge n\} = \{\eta_n \le t\}$, and denote $\mathcal{F}_t$ the $\sigma - algebra$ generated by $\{A_t, t \ge 0\}$. Then $(\Omega, P, \{\mathcal{F}_t\})$ is a probability space. Let $\mathfrak{B}$ be the Banach space with the norm defined for any $\psi \in \mathfrak{B}$, $\|\psi\|^2 = (sup_{t\in[\delta,T]} E\|\psi(t)\|^2)$, where $\psi(t)$, for any given t ∈ $[\delta, T]$.

Definition 2.1: For a given T ∈ $(\delta, +\infty)$, a stochastic process $\{x(t), \delta \le t \le T\}$ is called a solution to equations (2.1)-(2.3) in $(\Omega, P, \{\mathcal{F}_t\})$, if

(i) x(t) is $\mathcal{F}_t$ - adapted;

(ii) x(t) = $\sum_{k=0}^{+\infty}[\prod_{i=1}^{k} b_i(\delta_i)\, \Phi(t-t_0)x_0 + \sum_{i=1}^{k}\prod_{j=1}^{k} b_j(\delta_j)\int_{\eta_{i-1}}^{\eta_i} \Phi(t-s)[f(s,x(s)) + \int_0^s g(\delta,x_\delta)]ds + \int_{\eta_k}^{t} \Phi(t-s)[f(s,x(s)) + \int_0^s g(\delta,x_\delta)]ds\;]I_{[\eta_k,\eta_{k+1})}(t)$, t ∈ $[\delta, T]$,

(2.4)

where $\prod_{j=m}^{n}(.) = 1$ as m > n, $\prod_{j=1}^{k} b_j(\delta_i) = b_k(\delta_k)\, b_{k-1}(\delta_{k-1}) \ldots b_i(\delta_i)$, $I_B(.)$ is the index function, ie., $I_B(t) = \begin{cases} 1, if\ t \in B \\ 0, if\ t \notin B \end{cases}$.

3. MAIN RESULTS

Existence and uniqueness

In this section we give the existence and uniqueness of the system (2.1) – (2.3). We start with the following assumptions,

(I) The function f satisfies the Lipchitz condition. ie; for $\alpha, \gamma \in \mathcal{R}^n$ and $\delta \leq t \leq T$ there exists a constant L > 0 such that

$E\|f(t,\alpha) - f(t,\gamma)\|^2 \leq LE \| \alpha - \gamma\|^2$,

$E\|f(t,0)\|^2 \leq \frac{k}{2}$, *where* $k > 0$ *is a constant.*

(II) The condition $\max_{i,k}\{\prod_{j=i}^{k} ||b_j\left(\delta_j\right)||\}$ is uniformly bounded if, there is a constant C > 0 such that $\max_{i,k}\{\prod_{j=i}^{k} ||b_j\left(\delta_j\right)||\} \leq C \; for\; all\; \delta_j \in D_j$,j = 1,2,…

Theorem 3.1: Let the hypotheses (I), (II) hold. If the following inequality $\Delta = M^2 \max\{1, C^2\} L(T-\delta)^2 [1+ (T-\delta)] < 1$, is satisfied , then the (2.1)-(2.3) has a unique solution in $\mathfrak{B}$.

Proof: Let T be an arbitrary number $\delta \leq T < +\infty$. First we define the nonlinear operator $\oplus$: $\mathfrak{B} \to \mathfrak{B}$ as follows $(\oplus x)(t) = \sum_{k=0}^{+\infty}[\prod_{i=1}^{k} b_i(\delta_i)\, \Phi(t-t_0)x_0 + \sum_{i=1}^{k}\prod_{j=1}^{k} b_j(\delta_j) \int_{\eta_{i-1}}^{\eta_i} \Phi(t-s)[f(s,x(s)) + \int_0^s g(\delta, x_\delta)]ds + \int_{\eta_k}^{t} \Phi(t-s)[f(s,x(s)) + \int_0^s g(\delta,x_\delta)]ds\;]I_{[\eta_k,\eta_{k+1})}(t)$, $t \in [\delta, T]$,

It is easy to prove the continuity of $\oplus$. Now, we have to show that $\oplus$ *maps* $\mathfrak{B}$ into itself.

$$\|(\oplus x)(t)\|^2 \leq [\sum_{k=0}^{+\infty}[\|\prod_{i=1}^{k} b_i(\delta_i)\|\|\Phi(t-t_0)\|\|x_0\|$$
$$+ \sum_{i=1}^{k}\|\prod_{j=1}^{k} b_j(\delta_j)\|\int_{\eta_{i-1}}^{\eta_i}\|\Phi(t-s)[f(s,x(s)) + \int_0^s g(\delta,x_\delta)]\|ds$$
$$+ \int_{\eta_k}^{t}\|\Phi(t-s)[f(s,x(s)) + \int_0^s g(\delta,x_\delta)]\|ds\;]I_{[\eta_k,\eta_{k+1})}(t)\;]^2$$

$\leq 2M^2 \max_k\{\prod_{i=1}^{k}\|b_i(\delta_i)\|^2\}\|x_0\|^2 + 2M^2[\max_{i,k}\{1, \prod_{j=1}^{k}\|b_j(\delta_j)\|\}]^2[\int_{t_0}^{t}\|[f(s,x(s)) + \int_0^s g(\delta,x_\delta)]\|ds\, I_{[\eta_k,\eta_{k+1})}(t)]^2$

$\leq 2M^2C^2\|x_0\|^2 + 2M^2\max\{1,C^2\}\,[\int_{t_0}^{t}\|[f(s,x(s)) + \int_0^s g(\delta,x_\delta)]\|ds\,]^2$

$\leq 2M^2C^2\|x_0\|^2 + 2M^2\max\{1,C^2\}(t-t_0)\int_{t_0}^{t}\|[f(s,x(s)) + \int_0^s g(\delta,x_\delta)]\|^2 ds$

$E\ \|\ (\oplus\ x)(\mathrm{t})\ \|^2 \le\ 2M^2C^2\|x_0\|^2 + 2M^2 \max\ \{1, C^2\}(T-\ \delta)\ \int_{t_0}^{t} \mathrm{E}\|\ [\mathrm{f}\ (\mathrm{s}, \mathrm{x}(\mathrm{s})\ ) + \int_0^s g(\delta, x_\delta)]\|^2 ds$

$\le\ 2M^2C^2\|x_0\|^2 + 4M^2 \max\ \{1, C^2\}(\mathrm{T}-\ \delta)^2(\frac{k}{2}+\frac{k}{2}) + 4M^2 \max\ \{1, C^2\}(T-\delta)\ L\{\int_{t_0}^{t} \mathrm{E}\ \|\mathrm{x}(\mathrm{s})\|^2 \mathrm{ds} + \int_0^s E\|x_\delta\ \|^2\} ds.$

Thus,

$sup_{\mathrm{t} \in [\delta, T]}\ E\ \|\ (\oplus\ x)(\mathrm{t})\ \|^2\ \le\ 2M^2C^2\|x_0\|^2 + 4M^2 \max\ \{1, C^2\}(\mathrm{T}-\ \delta)^2 \mathrm{k} + 4M^2 \max\ \{1, C^2\}(T-\ \delta)\ ^2 \mathrm{L}\ sup_{\mathrm{t} \in [\delta, T]} \mathrm{E}\|\mathrm{x}(\mathrm{s})\|^2 + 4M^2 \max\{1, C^2\}(T-\ \delta)\ ^3 sup_{\mathrm{t} \in [\delta, T]} \mathrm{E}\|x_\delta\|^2$ for all $\mathrm{t} \in [\delta, T]$.

Therefore $\oplus$ maps $\mathfrak{B}$ into itself.

Now, we have to show $\oplus$ is a contraction mapping

$$\|\ (\oplus\ x)(\mathrm{t}) - (\oplus\ y)(\mathrm{t})\ \|^2$$

$$\le \sum_{\mathrm{k}=0}^{+\infty}[\sum_{i=1}^{k} \prod_{j=1}^{k} \|\ b_j\ (\ \delta_j)\ \left\|\int_{\eta_{i-1}}^{\eta_i} \|\Phi(t-s)\|\right\| \left[\mathrm{f}\left(\mathrm{s}, \mathrm{x}(\mathrm{s})\right) + \int_0^s g(\delta, x_\delta\right)\right]$$

$$- [\mathrm{f}\ (\mathrm{s}, \mathrm{y}(\mathrm{s})\ )\ +\ \int_0^s g(\delta, y_\delta)]\|\mathrm{ds}\ +\ \int_{\eta_k}^{t} \|\Phi(t-s)\|\|\ [\mathrm{f}\ (\mathrm{s}, \mathrm{x}(\mathrm{s})\ )$$

$$+\ \int_0^s g(\delta, x_\delta)] - [\mathrm{f}\ (\mathrm{s}, \mathrm{y}(\mathrm{s})\ )\ +\ \int_0^s g(\delta, y_\delta)]\|\mathrm{ds}\ \ ]I_{[\eta_k, \eta_{k+1})}(\mathrm{t})\ ]^2$$

$$\le M^2\ [\max_{i,k}\{\ 1, \prod_{j=1}^{k} \|\ b_j\ (\delta_j)\ \|\}]^2\ (\int_{t_0}^{t} \|\ [\mathrm{f}\ (\mathrm{s}, \mathrm{x}(\mathrm{s})\ )\ +\ \int_0^s g(\delta, x_\delta)] - [\mathrm{f}\ (\mathrm{s}, \mathrm{y}(\mathrm{s})\ )$$

$$+\ \int_0^s g(\delta, y_\delta)]\|\mathrm{ds}\ \ ]I_{[\eta_k, \eta_{k+1})}(\mathrm{t}))^2$$

$\le\ M^2\ \max\{1,\ C^2\}(\mathrm{t}\ -\ t_0) \int_{t_0}^{t} \|\ [\mathrm{f}\ (\mathrm{s}, \mathrm{x}(\mathrm{s})\ )\ +\ \int_0^s g(\delta, x_\delta)] - [\mathrm{f}\ (\mathrm{s}, \mathrm{y}(\mathrm{s})\ )\ + \int_0^s g(\delta, y_\delta)]\|^2 \mathrm{ds}$

$\mathrm{E}\|\ (\oplus\ x)(\mathrm{t}) - (\oplus\ y)(\mathrm{t})\ \|^2 \le\ M^2\ \max\{1,\ C^2\}(\mathrm{t}\ -\ t_0) \int_{t_0}^{t} \mathrm{E}\|\ [\mathrm{f}\ (\mathrm{s}, \mathrm{x}(\mathrm{s})\ )\ +\ \int_0^s g(\delta, x_\delta)] - [\mathrm{f}\ (\mathrm{s}, \mathrm{y}(\mathrm{s})\ )\ +\ \int_0^s g(\delta, y_\delta)]\|^2 \mathrm{ds}$

$\leq\; M^2\max\{1, C^2\}(\mathrm{T} - \delta)\; L\{\int_{t_0}^{t} \mathrm{E}\,\|\mathrm{x(s)} - \mathrm{y(s)}\|^2\mathrm{ds} + \int_0^s E\|x_\delta - y_\delta\,\|^2\}ds$

Taking the supremum over t, we get,

$\|\,(\oplus\, x)(\mathrm{t}) - (\oplus\, y)(\mathrm{t})\,\|^2 \leq\; M^2\;\; \max\{1,\;\; C^2\}[(T -\; \delta)\,^2\mathrm{L}\|\mathrm{x(s)} - \mathrm{y(s)}\|^2\;\; +(T -\; \delta)\,^3L\|x_\delta - y_\delta\|^2]$.

Thus,

$\|\,(\oplus\, x)(\mathrm{t}) - (\oplus\, y)(\mathrm{t})\,\|^2 \leq\;\; \Delta\,\{\|\mathrm{x(s)} - \mathrm{y(s)}\|^2 +\; \|x_\delta - y_\delta\|^2\}$,

Since $0 < \Delta < 1$. This shows that the operator $\oplus$ satiafies the Contraction mapping principle and therefore, $\oplus$ has a unique fixed point which is the solution of the system (2.1)- (2.3).

4. Exponential stability

In this section, we study the exponential stability of the solution of the system (2.1)-(2.3). Let $\mathcal{P}: [0, \infty) \to \mathcal{R}$ be any $\mathcal{F}_1$ - adapted process which is almost surely continuous in t and we assume that f(t,0) ≡ 0, for any $t \geq \delta$, so that the system (2.1)-(2.3) admits a trivial solution.

Definition 4.1. Equations (2.1)-(2.3) is said to be exponentially stable in the mean square , if there exist positive constants C_1 and $\lambda > 0$ such that

$$E\|x(s)\|^2 \leq C_1 E\|x_0\|^2 e^{-\lambda(s-s_0)}\;\; , s \leq s_{0.}$$

We now make the following additional assumption:

(III) : $\|\phi(s)\| \leq Me^{-\gamma(s-s_0)},\; s \geq s_{0.}$, where $M \geq 1\;\; and\, \gamma > 0$.

Theorem 4.1. Suppose that all the conditions of theorem (3.1) and (III) hold. Then the system (2.1)-(2.3) is exponentially stable in the mean square, if the following inequality holds

$$\max\{1, \mathrm{C}^2\}\; \mathrm{M}^2\mathrm{L}/(\gamma - \alpha) < \gamma \qquad (4.1)$$

Proof. Let $\oplus$ be the map defined in theorem 3.1. In order to prove the exponential stability of the solution of (2.1)-(2.3) by contraction mapping principle, first we have to prove the continuity of $\oplus$ on $[\delta, T]$. Let $x\, \in \mathfrak{B}$ and $t_1 \geq t_0$ and |h| be sufficiently small, then by using hypotheses (I)-(III) and condition (4.1), we have

$(\oplus\, x)(t_1 + h)$ - $(\oplus\, x)(t_1)$ =$\{\, \sum_{k=0}^{+\infty}[\prod_{i=1}^{k} b_i(\,\delta_i)\; \Phi(t_1 + h - t_0)x_0 + \sum_{i=1}^{k}\prod_{j=1}^{k}\, b_j(\,\delta_j)\int_{\eta_{i-1}}^{\eta_i} \Phi(t_1 + h -$

$s)$[f (s,x(s)) + $\int_0^s g(\tau,x_\tau)$]ds + $\int_{\eta_k}^{t_1+h}\Phi(t_1+h-s)$[f (s,x(s)) + $\int_0^s g(\tau,x_\tau)$]ds]$I_{[\eta_k,\eta_{k+1})}(t_1+h)$}-{

$\sum_{k=0}^{+\infty}[\prod_{i=1}^{k} b_i(\delta_i)\ \Phi(t_1-t_0)x_0 + \sum_{i=1}^{k}\prod_{j=1}^{k} b_j(\delta_j)\int_{\eta_{i-1}}^{\eta_i}\Phi(t_1-s)$[f (s,x(s)) + $\int_0^s g(\tau,x_\tau)$]ds +

$\int_{\eta_k}^{t_1}\Phi(t_1-s)$[f (s,x(s)) + $\int_0^s g(\tau,x_\tau)$]ds]$I_{[\eta_k,\eta_{k+1})}(t_1)$}

Thus,

$(\bigoplus x)(t_1+h)$ - $(\bigoplus x)(t_1)$ = $\sum_{k=0}^{+\infty}[\prod_{i=1}^{k} b_i(\delta_i)\ \Phi(t_1+h-t_0)x_0 + \sum_{i=1}^{k}\prod_{j=1}^{k} b_j(\delta_j)\int_{\eta_{i-1}}^{\eta_i}\Phi(t_1+h-$

$s)$[f (s,x(s)) + $\int_0^s g(\tau,x_\tau)$]ds + $\int_{\eta_k}^{t_1}\Phi(t_1+h-s)$[f (s,x(s)) + $\int_0^s g(\tau,x_\tau)$]ds] [$I_{[\eta_k,\eta_{k+1})}(t_1+h)$ -

$I_{[\eta_k,\eta_{k+1})}(t_1)$] + $\sum_{k=0}^{+\infty}[\prod_{i=1}^{k} b_i(\delta_i)\ [\Phi(t_1+h-t_0)-\Phi(t_1-t_0)]x_0 + \int_{\eta_{i-1}}^{\eta_i}[\Phi(t_1+h-s)-\Phi(t_1-$

$s)]$[f(s,x(s))+$\int_0^s g(\tau,x_\tau)$]ds +$\int_{\eta_k}^{t_1}[\Phi(t_1+h-s)-\Phi(t_1-s]$[f(s,x(s))+$\int_0^s g(\tau,x_\tau)$]ds]+ $\int_{t_1}^{t_1+h}\Phi(t_1+$

$h-s)$ [f (s, x(s)) + $\int_0^s g(\tau,x_\tau)$]ds] $I_{[\eta_k,\eta_{k+1})}(t_1+h)$.

E ‖ $(\bigoplus x)(t_1+h)-(\bigoplus x)(t_1)$ ‖$^2 \le 2E$ ‖ I_1 ‖^2+ $2E$ ‖ I_2 ‖^2 (4.2)

Where

I_1 = $\sum_{k=0}^{+\infty}[\prod_{i=1}^{k} b_i(\delta_i)\ \Phi(t_1+h-t_0)x_0 + \sum_{i=1}^{k}\prod_{j=1}^{k} b_j(\delta_j)\int_{\eta_{i-1}}^{\eta_i}\Phi(t_1+h-s)$[f (s,x(s)) +

$\int_0^s g(\tau,x_\tau)$]ds + $\int_{\eta_k}^{t_1}\Phi(t_1+h-s)$[f (s,x(s)) + $\int_0^s g(\tau,x_\tau)$]ds] [$I_{[\eta_k,\eta_{k+1})}(t_1+h)$ - $I_{[\eta_k,\eta_{k+1})}(t_1)$]

I_2 = $\sum_{k=0}^{+\infty}[\prod_{i=1}^{k} b_i(\delta_i)[\Phi(t_1+h-t_0)-\Phi(t_1-t_0)]x_0+\int_{\eta_{i-1}}^{\eta_i}[\Phi(t_1+h-s)-\Phi(t_1-$

$s)]$[f(s,x(s))+$\int_0^s g(\tau,x_\tau)$]ds + $\int_{\eta_k}^{t_1}[\Phi(t_1+h-s)-\Phi(t_1-s]$[f(s,x(s)) + $\int_0^s g(\tau,x_\tau)$]ds]+ $\int_{t_1}^{t_1+h}\Phi(t_1+$

$h-s)$ [f (s, x(s)) + $\int_0^s g(\tau,x_\tau)$]ds] $I_{[\eta_k,\eta_{k+1})}(t_1+h)$.

$$E\|l_1\|^2 \leq \left(2E\left[\max_k\left\{\prod_{i=1}^{k}\|b_i(\delta_i)\|^2\right\}\right]\|\phi(t_1+h-t_0)\|^2\, E\|x_0\|^2\left(I_{[\eta_k,\eta_{k+1})}(\mathrm{t}_1+h)\right.\right.$$
$$\left.\left.-I_{[\eta_k,\eta_{k+1})}(\mathrm{t}_1)\right)^2\right)$$
$$+2\left[\max_{i,k}\left\{1,\prod_{j=1}^{k}\|b_j(\delta_j)\|\right\}\right]^2\left(\mathrm{E}\sum_{\mathrm{k}=0}^{+\infty}\int_{t_0}^{t_1}\|\phi(t_1+h-s)-\phi(t_1-s)\|\,\|f\big(s,x(s)\big)\right.$$
$$\left.+\int_0^s g(\tau,x_\tau)\|ds\times\left(I_{[\eta_k,\eta_{k+1})}(\mathrm{t}_1+h)-I_{[\eta_k,\eta_{k+1})}(\mathrm{t}_1)\right)\right)^2$$
$$\leq\left[2C^2M^2e^{-2\gamma(t_1+h-t_0)}\,E\|x_0\|^2+2max\{1,C^2\}(t_1\right.$$
$$\left.-t_0)E\int_{t_0}^{t_1}e^{-2\gamma(t_1+h-s)}\|f(s,x(s))+\int_0^s g(\tau,x_\tau)]\,\|^2ds\right]$$
$$\times E\left(I_{[\eta_k,\eta_{k+1})}(\mathrm{t}_1+h)-I_{[\eta_k,\eta_{k+1})}(\mathrm{t}_1)\right)\to 0\ \ as\ \ h\to 0.$$

And,

$$E\|l_2\|^2\leq 3E\left[\max_k\left\{\prod_{i=1}^{k}\|b_i(\tau_i)\|^2\,\|\phi(t_1+h-t_0)-\phi(t-t_0)\|^2\,E\|x_0\|^2\right\}\right]$$
$$+3E\left[\max_{i,k}\left\{1,\prod_{j=1}^{k}\|b_j(\delta_j)\|\right\}\right]^2$$
$$\times E\left[\sum_{\mathrm{k}=0}^{+\infty}\int_{t_0}^{t_1}\|\phi(t_1+h-s)-\phi(t_1-s)\|\,\|f(s,x(s))\right.$$
$$\left.+\int_0^s g(\delta,x_\delta)]\,\|dsI_{[\eta_k,\eta_{k+1})}(\mathrm{t}_1+h)\right]^2$$
$$+3E\left(\sum_{\mathrm{k}=0}^{+\infty}\int_{t_1}^{t_1+h}\|\phi(t_1+h-s)\|\,\|f(s,x(s))+\int_0^s g(\tau,x_\tau)]\,\|dsI_{[\eta_k,\eta_{k+1})}(\mathrm{t}_1+h)\right)^2$$

$\leq\ 3\mathrm{C}^2\,\|\phi(t_1+h-t_0)-\phi(t-t_0)\|^2\,E\|x_0\|^2+3\max|1.\mathrm{C}^2|\,(t_1-t_0)E\ \times\int_{t_0}^{t_1}\|\phi(t_1+h-s)-$

$\phi(t_1-s)\|^2\,\|f(s,x(s))+\int_0^s g(\tau,x_\tau)]\,\|^2ds+3(h)E\int_{t_1}^{t_1+h}\|\phi(t_1+h-s)\|^2\|f(s,x(s))+$

$\int_0^s g(\tau, x_\tau)] \|^2 ds$

$\leq\ 3C^2 \|\phi(t_1 + h - t_0) - \phi(t - t_0)\|^2 E\|x_0\|^2 + 3 \max|1. C^2| (t_1 - t_0) \times \int_{t_0}^{t_1} \|\phi(t_1 + h - s) -$

$\phi(t_1 - s)\|^2 E\|x(s)\|^2 ds + 3l\, h \int_{t_1}^{t_1+h} \|\phi(t_1 + h - s)\|^2 E\|x(s)\|^2 ds \to 0\ as\ h \to 0.$

Thus, the right - hand side of (4.2) tends to 0 as $h \to 0$. Hence $\oplus$ is continuous on $[\delta, T]$. Next, we have to show that $\oplus$ maps $\mathfrak{B}$ into itself. Let β is a positive constant such that $0 < \beta < \gamma$.

$$
\begin{aligned}
e^{\beta(t-t_0)} E\|(\oplus x)(t)\|^2 \\
&\leq 2e^{\beta(t-t_0)} \max_k \left\{ \prod_{i=1}^{k} \|b_i(\delta_i)\|^2 \right\} \|\phi(t - t_0)\|^2 E\|x_0\|^2 \\
&+\ 2e^{\beta(t-t_0)} \left[\max_{i,k} \left\{ \prod_{j=1}^{k} \|b_j(\delta_j)\| \right\} \right]^2 E \left(\int_{t_0}^{t} \|\phi(t_1 + h - s) \right. \\
&\left. - \phi(t_1 - s)\| \, \|f(s, x(s)) + \int_0^s g(\tau, x_\tau)\| ds\ I_{[\eta_k, \eta_{k+1})}(t) \right)^2
\end{aligned}
$$

$$e^{\alpha(t-t_0)} E\|(sx)(t)\|^2 \leq I_3 + I_4 \tag{4.3}$$

Where,

$$
\begin{aligned}
I_3 &= 2\ e^{\beta(t-t_0)} \max_k \left\{ \prod_{i=1}^{k} \|b_i(\delta_i)\|^2 \right\} \|\phi(t - t_0)\|^2 E\|x_0\|^2 \\
&\leq 2C^2 M^2 e^{\beta(t-t_0)}\, e^{-2\gamma(t-t_0)} E\|x_0\|^2 \to 0\ \ as\ \ t \to \infty,
\end{aligned} \tag{4.4}
$$

and

$$
\begin{aligned}
I_4 = 2\ e^{\beta(t-t_0)} \left[\max_{i,k} \left\{ \prod_{j=1}^{k} \|b_j(\delta_j)\| \right\} \right]^2 E \left(\int_{t_0}^{t} \|\phi(t - s)\| \, \|f(s, x(s)) \right. \\
\left. + \int_0^s g(\tau, x_\tau)\| ds\ I_{[\eta_k, \eta_{k+1})}(t) \right)^2
\end{aligned}
$$

$\leq 2 \max\{1, C^2\}\ \ e^{\beta(t-t_0)}\ E\left(\int_{t_0}^{t} M\ \ e^{-\gamma(t-s)} \|f(s, x(s)) + \int_0^s g(\tau, x_\tau)\| ds\ \right)^2$

$$\leq 2\max\{1, C^2\} M^2 \; e^{\beta(t-t_0)} \; E\left(\int_{t_0}^{t} \; e^{-\frac{\gamma(t-s)}{2}} \quad e^{-\frac{\gamma(t-s)}{2}} \|f(s,x(s)) + \int_0^s g(\tau,x_\tau)\| ds \;\right)^2$$

$$\leq 2\max\{1, C^2\} \; M^2 \; e^{\beta(t-t_0)} \left(\int_{t_0}^{t} \; e^{-\gamma(t-s)} \; ds \right)\left(\int_{t_0}^{t} \; e^{-\gamma(t-s)} \; E \; \|f(s,x(s)) + \int_0^s g(\tau,x_\tau)\|^2 ds\right)$$

$$\leq \frac{2\max\{1, C^2\} \; M^2 \; e^{\beta(t-t_0)}}{\gamma} \left(\int_{t_0}^{t} \; e^{-\gamma(t-s)} \; E \; \|f(s,x(s)) + \int_0^s g(\tau,x_\tau) + \int_0^s g(\tau,x_\tau)\|^2 ds\right)$$

$$\leq \frac{2\max\{1, C^2\} \; M^2 \; e^{-(\gamma-\beta)(t-t_0)}}{\gamma} \int_{t_0}^{t} \; e^{(\gamma-\beta)(s-t_0)} \quad e^{\alpha(s-t_0)} E \; \|f(s,x(s)) + \int_0^s g(\tau,x_\tau)\|^2 ds$$

$$I_4 \leq \frac{2\max\{1, C^2\} \; M^2 \; e^{-(\gamma-\beta)(t-t_0)} L}{\gamma} \int_{t_0}^{t} \; e^{(\gamma-\beta)(s-t_0)} \quad e^{\beta(s-t_0)} E \; \|x(s)\|^2 ds \tag{4.5}$$

For any x(t)∈ $\mathfrak{B}$ and $\epsilon > 0$, there exists at $t_1 > 0$, such that

$e^{\beta(s-t_0)} E \; \|x(s)\|^2 < \epsilon$ for $t \geq t_1$. Thus from (4.5) we get

$$I_4 \leq \frac{2\max\{1, C^2\} \; M^2 \; e^{-(\gamma-\beta)(t-t_0)} L}{\gamma} \int_{t_0}^{t_1} \; e^{(\gamma-\beta)(s-t_0)} \quad e^{\beta(s-t_0)} E \; \|x(s)\|^2 ds +$$

$$\frac{2\max\{1, C^2\} \; M^2 \; e^{-(\gamma-\beta)(t-t_0)} L}{\gamma} \int_{t_0}^{t} \; e^{(\gamma-\beta)(s-t_0)} \quad e^{\beta(s-t_0)} E \; \|x(s)\|^2 ds$$

$$\leq \frac{2\max\{1, C^2\} \; M^2 \; e^{-(\gamma-\alpha)(t-t_0)} L}{\gamma} \int_{t_0}^{t} \; e^{(\gamma-\beta)(s-t_0)} \quad e^{\alpha(s-t_0)} E \; \|x(s)\|^2 ds$$

$$+ \frac{2\max\{1, C^2\} M^2 L}{\gamma} \left(\frac{1}{\gamma-\beta}\right) \epsilon. \tag{4.6}$$

As $e^{-(\gamma-\beta)(t-t_0)} \to 0$ as $t \to \infty$ and the condition (4.1), there exists $t_2 \geq t_1$ such that for any $t \geq t_2$, we have

$$\frac{2\max\{1, C^2\} \; M^2 \; e^{-(\gamma-\beta)(t-t_0)} L}{\gamma} \int_{t_0}^{t_1} \; e^{(\gamma-\beta)(s-t_0)} \quad e^{\beta(s-t_0)} E \; \|x(s)\|^2 ds$$

$$\leq \epsilon - \frac{2\max\{1, C^2\} M^2 L}{\gamma} \left(\frac{1}{\gamma-\beta}\right) \epsilon \tag{4.7}$$

So from (4.6) and (4.7), we obtain $I_4 < \epsilon$, for any $t \geq t_2$. That is

$$I_4 \to 0 \;\text{ as }\; t \to \infty \,. \tag{4.8}$$

Thus from (4.3), (4.4) and (4.8) we have

$$e^{\beta(t-t_0)} E \; \|(\oplus \, x)(t)\|^2 \; \to 0 \text{ as } t \to \infty \,. \tag{4.9}$$

Thus $\oplus$ maps $\mathfrak{B}$ into itself. Now, we have to show that $\oplus$ is a contraction mapping. For any $x, y \in \mathfrak{B}$, we have

$$E\|(\oplus x)(t)-(\oplus y)(t)\|^2 \le \left[\max_{i,k}\left\{\prod_{j=1}^{k}\|b_j(\delta_j)\|\right\}\right]^2 E\left(\int_{t_0}^{t}\|\phi(t-s)\|\ \left\|f(s,x(s))+\int_0^s g(\tau,y_\tau)\right.\right.$$

$$\left.\left.-f(s,y(s))-\int_0^s g(\delta,y_\delta)\right\|ds\ I_{[\eta_k,\eta_{k+1})}(t)\right)^2$$

$$\le \max\{1,C^2\}\,M^2E\left(\int_{t_0}^{t} e^{-\gamma(t-s)}\left\|f(s,x(s))+\int_0^s g(\delta,x_\delta)-f(s,y(s))-\int_0^s g(\delta,y_\delta)\right\|ds\right)^2$$

$$\le \max\{1,C^2\}\,M^2E\left(\int_{t_0}^{t} e^{-\frac{\gamma(t-s)}{2}}\ e^{-\frac{\gamma(t-s)}{2}}\left\|f(s,x(s))+\int_0^s g(\delta,x_\delta)-f(s,y(s))-\int_0^s g(\delta,y_\delta)\right\|ds\right)^2$$

$$\sup_{\tau\le t\le T}E\|(\oplus x)(t)-(\oplus y)(t)\|^2 \le \frac{\max\{1,C^2\}\,M^2L}{\gamma}\int_{t_0}^{t} e^{-\gamma(t-s)}\sup_{\delta\le t\le T}\|x(s)-y(s)\|^2ds$$

$$\le \frac{\max\{1,C^2\}\,M^2L}{\gamma}\left(\frac{1}{\gamma-\beta}\right)\sup_{\delta\le t\le T}E\|x(t)-y(t)\|^2$$

Thus,

$$\|(\oplus x)-(\oplus y)\|^2 \le \frac{\max\{1,C^2\}\,M^2L}{\gamma}\left(\frac{1}{\gamma-\beta}\right)\|x-y\|^2$$

Thus by (4.1), this shows that $\oplus$ is contraction mapping. Hence $\oplus$ has a unique fixed point $x(t)\in\mathfrak{B}$, which is the solution of (2.1)-(2.3). This completes the proof.

5. Application

Let $\widetilde{\Omega}\subset\mathcal{R}^n$ be a bounded domain with smooth boundary $\partial\widetilde{\Omega}$.

$$\begin{cases} u_s(x,s)=u_{xx}(x,s)+\int_{-r}^{t}\mu(\theta)u(s+\theta,x)d\theta, s\ne\xi_k, s\ge\tau, \\ u(x,\xi_k)=q(k)\tau_k u(x,\xi_k^-) \qquad a.s.\ x\in\widetilde{\Omega}, \\ u(x,s)=\varphi(x,s) \qquad a.s.\ x\in\widetilde{\Omega}, -r\le s\le 0, \\ u(x,s)=0 \qquad a.s.\ x\in\partial\widetilde{\Omega} \end{cases} \tag{5.1}$$

Let X = $L^2(\widetilde{\Omega})$, and τ_k be a random variable defined on $D_k\equiv(0,d_k)$ for k = 1,2,…, where $0<d_k<+\infty$ and $\mu:[-r,0]\to\mathcal{R}$ is a positive function. Furthermore, assume that τ_k follow Erlang distribution, where k = 1,2,…; and τ_i *and* τ_j *are* independent with each other as i ≠ j for i,j = 1,2,…; q is a function of k; $\xi_0=s_0$; $\xi_k=\xi_{k-1}+\tau_k$ for k = 1,2,… and $s_0=\mathcal{R}^+$ is an arbitrarily given real number.

Define B is an operator on X by Bu = $\frac{\partial^2u}{\partial x^2}$ with the domain

D(B) = {u ∈ X| u and $\frac{\partial u}{\partial x}$ are absolutely continuous, $\frac{\partial^2 u}{\partial x^2} \in X, u = 0 \ on\ \partial\widetilde{\Omega}$}.

It is well known that B generates a strongly continuous semigroup S(s) which is compact, analytic and self adjoint. Moreover, the operator B can be expressed as

B $(u)= \sum_{n=1}^{\infty} n^2 \langle u, u_n \rangle u_n, u \in$ D(B),

Where $u_n(\omega) = (\frac{2}{\pi})^{1/2}$ sin(nω), n = 1, 2,…, is the orthonormal set of eigenvectors of B and for every $u \in X$,

S(s)u = $\sum_{n=1}^{\infty} \exp(-n^2 s) < u,$ which satisfies ‖ S(s) ‖≤ $\exp(-\pi^2(s - s_0))$, s >s_0. Hence S(s) is a contraction semigroup.

Conflict of Interests

The authors declare that there is no conflict of interests.

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