Available online at http://scik.org
Eng. Math. Lett. 2019, 2019:12
https://doi.org/10.28919/eml/4118
ISSN: 2049-9337

SOME RESULTS FOR STRONG DIFFERENTIAL SUBORDINATION OF ANALYTIC FUNCTIONS DEFINED BY DIFFERENTIAL OPERATOR

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Abstract: The purpose of this paper is to introduce some results associated with the strong differential subordinations of analytic functions defined in the open unit disk and closed unit disk of the complex plane.

Keywords: analytic functions; convex functions; strong differential subordinations; differential operator.

2010 AMS Subject Classification: 30C45.

1. INTRODUCTION

Denote by $\mathcal{H}(U \times \overline{U})$ the family of all analytic functions in $U \times \overline{U}$. Let $U = \{z \in \mathbb{C} : |z| < 1\}$ and $\overline{U} = \{z \in \mathbb{C} : |z| \leq 1\}$ indicate the open unit disk and the closed unit disk of the complex plane, respectively. For $n \in \mathbb{N} = \{1,2,\dots\}$ and $a \in \mathbb{C}$, let $\mathcal{H}^*[a,n,\zeta] = \{f \in \mathcal{H}(U \times \overline{U}): f(z,\zeta) = a + a_n(\zeta)z^n + a_{n+1}(\zeta)z^{n+1} + \cdots, z \in U, \zeta \in \overline{U}\}$, where $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \geq n$.

Also, let $\mathcal{A}^*_{n\,\zeta} = \{f \in \mathcal{H}(U \times \overline{U}): f(z,\zeta) = z + a_{n+1}(\zeta)z^{n+1} + \cdots, z \in U, \zeta \in \overline{U}\}$, where $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \geq n+1$.

A function $f \in \mathcal{H}^*[a,n,\zeta]$ is said to be starlike in $U \times \overline{U}$ if

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Received May 12, 2019

$$Re\left\{\frac{zf_z'(z,\zeta)}{f(z,\zeta)}\right\} > 0\,, \qquad (\,z \in U, \zeta \in \overline{U}).$$

Denote the class of all starlike functions in $U \times \overline{U}$ by S_ζ^*.

Similar, $f \in \mathcal{H}^*[a,n,\zeta]$ is said to be convex in $U \times \overline{U}$ if

$$Re\left\{\frac{zf_{z^2}''(z,\zeta)}{f_z'(z,\zeta)} + 1\right\} > 0\ , \quad (\,z \in U, \zeta \in \overline{U}).$$

Denote the class of all convex functions in $U \times \overline{U}$ by K_ζ^*.

Definition 1.1 (Oros and Gh Oros, [9]) Let $f(z,\zeta), g(z,\zeta)$ analytic in $U \times \overline{U}$. The function $f(z,\zeta)$ is said to be strongly subordinate to $g(z,\zeta)$, written $f(z,\zeta) \prec\prec F(z,\zeta)$, $z \in U, \zeta \in \overline{U}$, if there exists an analytic function w in U with $w(0) = 0$ and $|w(z)| < 1, z \in U$ such that $f(z,\zeta) = g(w(z),\zeta)$ for all $\zeta \in \overline{U}$.

Remark 1.1 (Oros and Gh Oros, [9])

1) Since $f(z,\zeta)$ is analytic in $U \times \overline{U}$, for all $\zeta \in \overline{U}$ and univalent in U, for all $\zeta \in \overline{U}$, Definition1.1 is equivalent to $f(0,\zeta) = g(0,\zeta)$ for all $\zeta \in \overline{U}$ and $f(U \times \overline{U}) \subset g(U \times \overline{U})$.
2) If $f(z,\zeta) = f(z)$ and $g(z,\zeta) = g(z)$, the strong subordination becomes the usual notion of subordination.

Let $\mathcal{A}_\zeta^*$ denote the subclass of the functions $f(z,\zeta) \in \mathcal{H}(U \times \overline{U})$ of the form:

$$f(z,\zeta) = z + \sum_{k=2}^{\infty} a_k(\zeta) z^k \ \ ,\ \ z \in U, \zeta \in \overline{U} \tag{1.1}$$

which are analytic and univalent in $U \times \overline{U}$.

For $\eta \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ $\alpha, \tau \geq 0, \mu, \lambda, \beta > 0$ and $\alpha \neq \lambda$, we consider the differential operator $A_{\mu,\lambda,\tau}^{\eta}(\alpha,\beta) : \mathcal{A}_\zeta^* \longrightarrow \mathcal{A}_\zeta^*$, introduced by Amourah and Darus [2], where

$$A_{\mu,\lambda,\tau}^{\eta}(\alpha,\beta) f(z) = z + \sum_{k=2}^{\infty}\left[1 + \frac{(n-1)[(\lambda-\alpha)\beta + n\tau]}{\mu+\lambda}\right]^{\eta} a_k(\zeta) z^k\ . \tag{1.2}$$

It is readily verified from (1.2) that

$$z\left(A_{\mu,\lambda,\tau}^{\eta}(\alpha,\beta) f(z,\zeta)\right)_z'$$

$$=\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\tau}A_{\mu,\lambda,\tau}^{\eta+1}(\alpha,\beta)f(z,\zeta)-\left(1-\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\tau}\right)A_{\mu,\lambda,\tau}^{\eta}(\alpha,\beta)f(z,\zeta). \qquad (1.3)$$

Some of the special cases of the operator defined by (1.2) can be found in [1,4,10,11].

In recent years, many authors obtained various interesting results associated with strong differential subordination and superordination for example (see [3,5,6,12,13,14]).

In order to derive our main results, we need the following Lemmas.

Lemma 1.1 (Miller and Mocanu, [8]) Let $h(z,\zeta)$ be a convex function with $h(0,\zeta)=a$, for every $\zeta\in\overline{U}$ and let $\gamma\in\mathbb{C}^*=\mathbb{C}\setminus\{0\}$ with $Re\,(\gamma)\geq 0$. If $p\in\mathcal{H}^*[a,n,\zeta]$ and

$$p(z,\zeta)+\frac{1}{\gamma}zp'_z(z,\zeta)\prec\prec h(z,\zeta),\quad (z\in U,\zeta\in\overline{U}), \qquad (1.4)$$

then

$$p(z,\zeta)\prec\prec q(z,\zeta)\prec\prec h(z,\zeta),\ (z\in U,\zeta\in\overline{U}),$$

where $q(z,\zeta)=\frac{\gamma}{nz^{\frac{\gamma}{n}}}\int_0^z t^{\frac{\gamma}{n}-1}h(t,\zeta)dt$ is convex and it is the best dominant of (1.4).

Lemma 1.2 (Miller and Mocanu, [7]) Let $q(z,\zeta)$ be a convex function in $U\times\overline{U}$ for all $\zeta\in\overline{U}$ and let $h(z,\zeta)=q(z,\zeta)+n\delta zq'_z(z,\zeta)$, $z\in U,\zeta\in\overline{U}$, where $\delta>0$ and n is a positive integer. If

$$p(z,\zeta)=q(0,\zeta)+p_n(\zeta)z^n+p_{n+1}(\zeta)z^{n+1}+\cdots,$$

is analytic in $U\times\overline{U}$ and

$$p(z,\zeta)+\delta zp'_z(z,\zeta)\prec\prec h(z,\zeta),\quad (z\in U,\zeta\in\overline{U}),$$

then

$$p(z,\zeta)\prec\prec q(z,\zeta),\ (z\in U,\zeta\in\overline{U}),$$

and this result is sharp.

2. MAIN RESULT

Theorem 2.1. Let $h(z,\zeta)$ be a convex function such that $h(0,\zeta)=1$. If $f\in\mathcal{A}^*_\zeta$ satisfies the strong differential subordination:

$$\left(A_{\mu,\lambda,\tau}^{\eta}(\alpha,\beta)f(z,\zeta)\right)'_z\prec\prec h(z,\zeta), \qquad (2.1)$$

then

$$\frac{A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)}{z} \prec\prec q(z,\zeta) \prec\prec h(z,\zeta),$$

where $q(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)dt$ is convex and it is the best dominant

Proof. Suppose that

$$p(z,\zeta)=\frac{A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)}{z}\ , \ z\in U,\zeta\in \overline{U}\,. \tag{2.2}$$

Then the function $p(z,\zeta)$ is analytic in $U\times\overline{U}$ and $p(0,\zeta)=1$.

Simple computations from (2.2), we get

$$p(z,\zeta)+zp'_z(z,\zeta)=\left(A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)\right)'_z. \tag{2.3}$$

Using (2.3), (2.1) becomes

$$p(z,\zeta)+zp'_z(z,\zeta)\prec\prec h(z,\zeta).$$

An application of Lemma 1.1 with $n=1,\gamma=1$ yields

$$\frac{A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)}{z} \prec\prec q(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)dt \prec\prec h(z,\zeta).$$

By taking $h(z,\zeta)=\frac{\zeta+(2\rho-\zeta)z}{1+z}, 0\le\rho<1$ in Theorem 2.1, we obtain the following corollary:

Corollary 2.1. If $f\in\mathcal{A}^*_\zeta$ satisfies the strong differential subordination:

$$\left(A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)\right)'_z \prec\prec \frac{\zeta+(2\rho-\zeta)z}{1+z},$$

then

$$\frac{A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)}{z} \prec\prec \frac{1}{z}\int_0^z \frac{\zeta+(2\rho-\zeta)t}{1+t}dt=2\rho-\zeta+\frac{2(\zeta-\rho)}{z}\ln(1+z).$$

Theorem 2.2. Let $q(z,\zeta)$ be a convex function such that $q(0,\zeta)=1$ and let h be the function $h(z,\zeta)=q(z,\zeta)+zq'_z(z,\zeta)$. If $f\in\mathcal{A}^*_\zeta$ satisfies the strong differential subordination:

$$\left(\frac{z\,A^{\eta+1}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)}{A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)}\right)'_z \prec\prec h(z,\zeta), \tag{2.4}$$

then

$$\frac{A^{\eta+1}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)}{A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)} \prec\prec q(z,\zeta).$$

Proof. Suppose that

$$p(z,\zeta) = \frac{A^{\eta+1}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)}{A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)}, \quad z \in U, \zeta \in \overline{U}. \tag{2.5}$$

Then the function $p(z,\zeta)$ is analytic in $U \times \overline{U}$ and $p(0,\zeta) = 1$.

Differentiating both sides of (2.5) with respect to z and using (2.4), we have

$$\begin{aligned}
&p(z,\zeta) + zp'_z(z,\zeta)\\
&= \frac{A^{\eta+1}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)}{A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)}\\
&+ \frac{A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)\left(A^{\eta+1}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)\right)'_z - A^{\eta+1}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)\left(A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)\right)'_z}{\left[A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)\right]^2}\\
&= \frac{A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)\left(z\,A^{\eta+1}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)\right)'_z - zA^{\eta+1}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)\left(A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)\right)'_z}{\left[A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)\right]^2}\\
&= \left(\frac{z\,A^{\eta+1}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)}{A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)}\right)'_z \prec\prec h(z,\zeta).
\end{aligned}$$

An application of Lemma 1.2, we obtain

$$\frac{A^{\eta+1}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)}{A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)} \prec\prec q(z,\zeta).$$

Theorem 2.3. Let $h(z,\zeta)$ be a convex function such that $h(0,\zeta) = 1$. If $0 \le \sigma < p, \theta \in \mathbb{C}$ and $f \in \mathcal{A}^*_\zeta$ satisfies the strong differential subordination:

$$\frac{1-\theta}{1-\sigma}\left(\frac{A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)}{z} - \sigma\right) + \frac{\theta}{1-\sigma}\left(\left(A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)\right)'_z - \sigma\right) \prec\prec h(z,\zeta), \tag{2.6}$$

then

$$\frac{1}{1-\sigma}\left(\frac{A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)}{z} - \sigma\right) \prec\prec q(z,\zeta) \prec\prec h(z,\zeta),$$

where $q(z,\zeta)=\frac{1}{\theta}z^{-\frac{1}{\theta}}\int_0^z t^{\frac{1}{\theta}-1}h(t,\zeta)dt$ is convex and it is the best dominant.

Proof. Suppose that

$$p(z,\zeta)=\frac{1}{1-\sigma}\left(\frac{A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)}{z}-\sigma\right), z\in U,\zeta\in\overline{U}\,. \tag{2.7}$$

Then the function $p(z,\zeta)$ is analytic in $U\times\overline{U}$ and $p(0,\zeta)=1$.

Differentiating both sides of (2.7) with respect to z, we have

$$p(z,\zeta)+\theta zp'_z(z,\zeta)$$

$$=\frac{1-\theta}{1-\sigma}\left(\frac{A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)}{z}-\sigma\right)+\frac{\theta}{1-\sigma}\left(\left(A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)\right)'_z-\sigma\right). \tag{2.8}$$

From (2.6) and (2.8), we get

$$p(z,\zeta)+\theta zp'_z(z,\zeta)\prec\prec h(z,\zeta).$$

An application of Lemma 1.1 with $n=1,\gamma=\frac{1}{\theta}$ yields

$$\frac{1}{1-\sigma}\left(\frac{A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)}{z}-\sigma\right)\prec\prec q(z,\zeta)=\frac{1}{\theta}z^{-\frac{1}{\theta}}\int_0^z t^{\frac{1}{\theta}-1}h(t,\zeta)dt\prec\prec h(z,\zeta).$$

Theorem 2.4. Let $q(z,\zeta)$ be a convex function such that $q(0,\zeta)=1$ and let h be the function $h(z,\zeta)=q(z,\zeta)+\frac{1}{1+\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\tau}}zq'_z(z,\zeta)$, where $\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\tau}>0$. Suppose that

$$F(z,\zeta)=\frac{1+\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\tau}}{z^{\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\tau}}}\int_0^z t^{\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\tau}-1}f(t,\zeta)dt,\;\; z\in U,\zeta\in\overline{U}. \tag{2.9}$$

If $f\in\mathcal{A}^*_{\zeta}(p)$ satisfies the strong differential subordination

$$\left(A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)\right)'_z\prec\prec h(z,\zeta), \tag{2.10}$$

then

$$\left(A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta)F(z,\zeta)\right)'_z\prec\prec q(z,\zeta).$$

Proof. Suppose that

$$p(z,\zeta)=\left(A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta)F(z,\zeta)\right)'_z,\;\; z\in U,\zeta\in\overline{U}\,. \tag{2.11}$$

Then the function $p(z,\zeta)$ is analytic in $U \times \overline{U}$ and $p(0,\zeta) = 1$.

From (2.9), we have

$$z^{\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\tau}} F(z,\zeta) = \left(1 + \frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\tau}\right)\int_0^z t^{\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\tau}-1} f(t,\zeta)dt \,. \tag{2.12}$$

Differentiating both sides of (2.12) with respect to z, we get

$$\left(1 + \frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\tau}\right) f(z,\zeta) = \frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\tau} F(z,\zeta) + zF_z'(z,\zeta)$$

and

$$\left(1 + \frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\tau}\right) A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta) f(z,\zeta) \\ = \frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\tau} A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta)F(z,\zeta) + z\left(A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta)F(z,\zeta)\right)_z'.$$

So

$$\left(A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)\right)_z' = \left(A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta)F(z,\zeta)\right)_z' + \frac{z\left(A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta)F(z,\zeta)\right)_{z^2}''}{1 + \frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\tau}} \,. \tag{2.13}$$

From (2.11) and (2.13), we obtain

$$p(z,\zeta) + \frac{1}{1 + \frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\tau}} zp_z'(z,\zeta) = \left(A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta)f(z,\zeta)\right)_z'. \tag{2.14}$$

Using (2.14), (2.10) becomes

$$p(z,\zeta) + \frac{1}{1 + \frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\tau}} zp_z'(z,\zeta) \prec\prec q(z,\zeta) + \frac{1}{1 + \frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\tau}} zq_z'(z,\zeta).$$

An application of Lemma 1.2 yields $p(z,\zeta) \prec\prec q(z,\zeta)$. By using (2.10), we obtain

$$\left(A^{\eta}_{\mu,\lambda,\tau}(\alpha,\beta)F(z,\zeta)\right)_z' \prec\prec q(z,\zeta).$$

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] F. M. Al-Oboudi, On univalent functions defined by a generalized Salagean operator, Int. J. Math. Math. Sci., 27(2004), 1429–1436.

[2] A. Amourah and M. Darus, Some properties of a new class of univalent functions involving a new generalized differential operator with negative coefficients, Indian J. Sci. Tech., 9(36)(2016), 1–7.

[3] N. E. Cho, O. S. Kwon and H. M. Srivastava, Strong differential subordination and superordination for multivalently meromorphic functions involving the Liu- Srivastava operator, Integral transforms Spec. Funct., 21(8)(2010), 589-601.

[4] M. Darus and R. W. Ibrahim, On subclasses for generalized operators of complex order, Far East J. Math. Sci., 33(3)(2009), 299–308.

[5] M. P. Jeyaraman and T. K. Suresh, Strong differential subordination and superordination of analytic functions, J. Math. Anal. Appl., 385(2)(2012), 854-864.

[6] A. A. Lupas, A note on strong differential superordinations using a generalized Salagean operator and Ruscheweyh operator, Stud. Univ. Babes-Bolyai Math., 57(2) (2012), 135-165.

[7] S. S. Miller and P. T. Mocanu, On some classes of first-order differential subordinations, Michigan Math. J., 32(2)(1985), 185-195.

[8] S. S. Miller and P. T. Mocanu, Differential Subordinations: Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics Vol. 225, Marcel Dekker Inc., New York and Basel, 2000.

[9] G. I. Oros and Gh. Oros, Strong differential subordination, Turk. J. Math., 33(2009), 249-257.

[10] G. S. Salagean, Subclasses of univalent functions, Lecture Notes in Math., Springer Verlag, Berlin, 1013(1983), 362-372.

[11] A. K. Wanas and A. Alb Lupas, On a new strong differential subordinations and superordinations of analytic functions involving the generalized differential operator, Int. J. Pure Appl. Math., 116(3)(2017), 571-579.

[12] A. K. Wanas and A. H. Majeed, New strong differential subordinations and superordinations of symmetric analytic functions, Int. J. Math. Anal., 11(11)(2017), 543-549.

[13] A. K. Wanas and A. H. Majeed, Strong differential subordinations for higher-order derivatives of multivalent analytic functions defined by linear operator, Khayyam J. Math., 3(2)(2017), 160-171.

[14] A. K. Wanas and A. H. Majeed, Strong differential subordination and superordination of meromorphic multivalent quasi-convex functions, Kragujevac J. Math., 44(1)(2020), 27-39.