Available online at http://scik.org
Eng. Math. Lett. 2019, 2019:11
https://doi.org/10.28919/eml/4159
ISSN: 2049-9337

NIMBLE RANDOMIZATION DEVICE FOR ESTIMATING A RARE SENSITIVE ATTRIBUTE USING POISSON DISTRIBUTION

TANVEER A. TARRAY

Department of Mathematical Sciences, Islamic University of Science and Technology, Kashmir, India, 192122

Abstract. This paper addresses the problem of estimating the mean of the number of persons possessing a rare sensitive attribute utilizing the Poisson distribution in survey sampling. Properties of the proposed randomized response model have been studied along with recommendations. It is also shown that the proposed model is more efficient than Land et al. (2011) when the proportion of persons possessing a rare unrelated attribute is known. Numerical illustration is also given in support of the present study.

Keywords: randomized response technique; estimation of proportion; rare sensitive characteristics.

2010 AMS Subject Classification: 62D05.

1. INTRODUCTION

Warner (1965) suggested an ingenious method of collecting information on sensitive characters. According to the method, for estimating the population proportion π possessing the sensitive character "A", a simple random with replacement sample of n persons is drawn from the population. Each interviewee in the sample is furnished an identical randomization device where the outcome "I possess character A" occurs with probability P_1 while its complement "I do not possess character A" occurs with probability $(1-P_1)$. The respondent answers "Yes" if the outcome of the randomization device tallies with his actual status otherwise he answers "No". Some modifications in the model has been suggested by Chaudhuri and Mukerjee (1988, 2011), Ryu et

E-mail address: tanveerstat@gmail.com

Received December 4, 2016

al. (1993), Tracy and Mangat (1996), Singh (2003), Sidhu et al. (2008), Perri (2008), Tarray ans Singh (2015, 2016, 2017, 2018) and Singh and Tarray (2012, 2013, 2014, 2015) .

Greenberg et al. (1969) provided theoretical framework for a modification to the Warner's model proposed by Horvitiz et al. (1969). The method consisted in modifying the randomization device where the second outcome "I do not possess the character A" was replaced by the outcome "I possess the character Y" where "Y" was unrelated to character "A". This modified model is now known as 'unrelated question model, or U- model'. This model has been further investigated by Moors (1971), Lanke (1975) and Land et al. (2011). The randomization model used in the Mangat et al. (1992) model differs from U-model in the SRSWR sample of size n is instructed to say "Yes" if he belongs to group "A" and to report "Yes" or "No" according to the statement and the actual status he possesses by using outcome of randomized device, as in the case of usual U- model, if he is not in group "A". The respondent is supposed not to disclose the mode he had used to give the reply. Then, the probability of "Yes" answer is given by

$$\theta_1 = \pi_1 + (1-\pi_1)(1-P_1)\,\pi_2 \tag{1}$$

where π_1 and π_2 are the true proportion of the rare sensitive attribute A_1 and the rare unrelated attribute A_2 in the population respectively.

Solving (1) for π_1, we get estimator of π_1 as

$$\hat{\pi}_1 = \frac{\left[\hat{\theta}_1 - (1-P_1)\pi_2\right]}{\left[1-\pi_2(1-P_1)\right]},$$

where $\hat{\theta}_1$ is the proportion of "Yes" answers obtained from the n sampled respondents and the variance of the estimator $\hat{\pi}_1$ is given by

$$V(\hat{\pi}_1) = \frac{\pi_1(1-\pi_1)}{n} + \frac{\pi_2\ (1-\pi_1)(1-P_1)}{\left[n(1-\pi_2(1-P_1))\right]}\ . \tag{2}$$

In this paper we consider the problem where the number of persons possessing a rare sensitive attribute is very small and huge sample size is required to estimate this number. The study is carried out when the proportion of persons possessing a rare unrelated attributes is known in sections 2. Properties of the proposed randomized response model have been studied alongwith recommendations. Efficiency comparison is worked out to investigate the performance of the suggested procedures. Numerical studies are also worked out to demonstrate the superiority of the suggested model.

2. Estimation of a Rare Sensitive Attribute in Sampling – Known Rare Unrelated Attributes

Let π_1 is the true proportion of the rare sensitive attribute A_1 in the population Ω. For example, the proportion of AIDS patients who continue having affairs with strangers; the proportion of persons who have witnessed a murder; the proportion of persons who are told by the doctors that they will not survive long due to a ghastly disease, for more examples see Land et al. (2011). Consider selecting a large sample of n persons from the population such that as $n \to \infty$ and $\pi_1 \to 0$ then $n\pi_1 = \delta_1$ (finite). Let π_2 be the true proportion of the population having the rare unrelated attribute A_2 such that as $n \to \infty$ and $\pi_2 \to 0$ then $n\pi_2 = \delta_2$ (finite and known). For example, π_2 might be the proportion of persons who are born between 12:00 and 12:01 or 12:05 O'clock; the proportion of babies born blind; see Land et al. (2011). Each respondent selected in the sample is instructed to say "Yes" if he belongs to the rare sensitive attribute A_1 and if he is not in group A_1 then he / she is requested to rotate a spinner bearing two types of statements:

(a) Do you possess the rare sensitive attribute A_1?

and

(b) Do you possess the rare unrelated attribute A_2?

with probabilities P_1 and $(1-P_1)$ respectively; and to report "Yes" or "No" according to the statement and the actual status he / she possesses by using outcome of the randomization device (i.e. of spinner) as in Land et al. (2011). The respondent is supposed not to disclose the mode he had used to give the reply. The privacy of the respondents possessing the sensitive attribute is protected in the proposed procedure.

The probability of "Yes" answer is given by

$$\theta_0 = \pi_1 + (1-\pi_1)(1-P_1)\,\pi_2 \tag{3}$$

Note that both attributes A_1 and A_2 are very rare in population. As before, assuming that, as $n \to \infty$ and $\theta_0 \to 0$ such that $n\theta_0 = \delta_0$ (finite),

where

$$\delta_0 = \delta_1 + (1-P_1)\delta_2$$

Note that $\lim\limits_{n\to\infty} n\theta_0 = \lim\limits_{n\to\infty} n\pi_1 + (1-P_1)\lim\limits_{n\to\infty} n\pi_2 - (1-P_1)\lim\limits_{n\to\infty} n\pi_1\pi_2$

or $$\delta_0 = \delta_1 + (1-P_1)\delta_2 - (1-P_1)\lim_{n\to\infty} n\pi_1 \lim_{n\to\infty} \frac{n\pi_2}{n}$$

or $$\delta_0 = \delta_1 + (1-P_1)\delta_2 - (1-P_1)\delta_1 \lim_{n\to\infty} n\pi_2 \lim_{n\to\infty} \frac{1}{n}$$

or $$\delta_0 = \delta_1 + (1-P_1)\delta_2 - (1-P_1)\delta_1\delta_2 \times 0$$

i.e. $$\delta_0 = \delta_1 + (1-P_1)\delta_2$$

Let $y_{1,} y_2, \ldots, y_n$ be a random sample of n observations from the Poisson distribution with parameter δ_0. The likelihood function of the random sample of n observations is given by

$$L = \prod_{i=1}^{n} \frac{e^{-\delta_0}\delta_0^{y_i}}{y_i!}. \tag{4}$$

$$= \left(e^{-n\,\delta_0}\right)\prod_{i=1}^{n}\delta_0{}^{y_i}\prod_{i=1}^{n}\frac{1}{y_i!} = \left(e^{-n\,\delta_0}\right)\delta_0^{\sum_{i=1}^{n} y_i}\prod_{i=1}^{n}\frac{1}{y_i!}$$

Taking natural logarithm on both sides of (4) we have

$$\text{Log } L = \left(-n\delta_0\right) + \left(\sum_{i=1}^{n} y_i\right)\log\delta_o + \sum_{i=1}^{n}\log\frac{1}{y_i!}$$

i.e. $$\text{Log } L = -n[\delta_1 + (1-P_1)\delta_2] + \left(\sum_{i=1}^{n} y_i\right)\log[\delta_1 + (1-P_1)\delta_2] - \sum_{i=1}^{n}\log y_i!$$

On putting $\dfrac{\partial L}{\partial \delta_1} = 0.$

The maximum – likelihood estimator of δ_1 is given by

$$\hat{\delta}_1 = \left[\frac{1}{n}\sum_{i=1}^{n} y_i - (1-P_1)\delta_2\right]$$

Thus, we have the following theorem.

Theorem 2.1 The estimator $\hat{\delta}_1$ is an unbiased estimator of the parameter δ_1.

Proof. Since $y_i \sim P(\delta_0)$, that is, y_i follows a Poisson distribution with parameter $\delta_0 = \delta_1 + (1-P_1)\delta_2$, we have

$$E(\hat{\delta}_1) = \left[\frac{1}{n}\sum_{i=1}^{n} E(y_i) - (1-P_1)\delta_2\right] = \left[\frac{1}{n}\sum_{i=1}^{n}\delta_0 - (1-P_1)\delta_2\right] = \left[\delta_0 - (1-P_1)\delta_2\right] = \delta_1$$

which proves the theorem.

Theorem 2.2 The variance of the estimator $\hat{\delta}_1$ is given by

$$V(\hat{\delta}_1)=\frac{\delta_1}{n}+\frac{(1-P_1)\delta_2}{n}.$$

Proof. Since $y_i \sim P(\delta_0)$, that is, y_i follows a Poisson distribution with parameter $\delta_0=\delta_1+(1-P_1)\delta_2$, we have

$$V(\hat{\delta}_1)=V\left\{\left[\frac{1}{n}\sum_{i=1}^{n}(y_i)-(1-P_1)\delta_2\right]\right\}=\left[\frac{1}{n^2}\sum_{i=1}^{n}V(y_i)\right]$$

$$=\left[\frac{1}{n^2}\sum_{i=1}^{n}\delta_0\right] \qquad =\frac{\delta_1}{n}+\frac{(1-P_1)\delta_2}{n}=\left(\frac{1}{n}\right)\{\delta_1+(1-P_1)\delta_2\}.$$

Theorem 2.3 An unbiased estimator of the variance of the estimator $\hat{\delta}_1$ is

$$\hat{v}(\hat{\delta}_1)=\frac{1}{n^2}\sum_{i=1}^{n}(y_i) \tag{5}$$

Proof. Taking expectation of both sides of (5), we have

$$E[\hat{v}(\hat{\delta}_1)]=\frac{1}{n^2}E\left[\sum_{i=1}^{n}(y_i)\right]=\frac{1}{n^2}\left[\sum_{i=1}^{n}E(y_i)\right]=\frac{1}{n^2}\left[\sum_{i=1}^{n}\delta_0\right]$$

$$=\frac{\delta_1+(1-P_1)\delta_2}{n}=\frac{\delta_1}{n}+\frac{(1-P_1)\delta_2}{n}.$$

3. RELATIVE EFFICIENCY

The percent relative efficiency of the proposed estimator $\hat{\delta}_1$ with respect to the Land et al. (2011) estimator $\hat{\delta}_L$ is given by

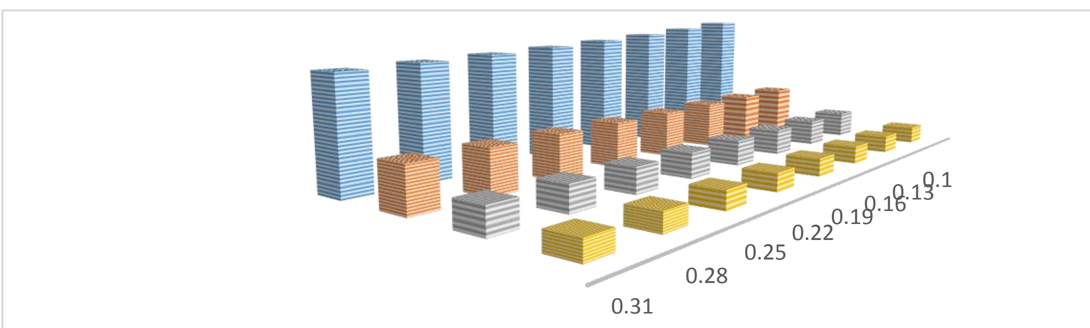
$$PRE(\hat{\delta}_1,\hat{\delta}_L)=\frac{V(\hat{\delta}_L)}{V(\hat{\delta}_1)}=\frac{P_1\delta_1+(1-P_1)\delta_2}{\{\delta_1+(1-P_1)\delta_2\}P_1^2}\times 100, \tag{6}$$

From Equation (6), it is clear that the percent relative efficiency of the proposed estimator is free from the sample size n. To look at the magnitude of the percent relative efficiency, we chose P_1 from 0.1 to 0.9. The percent relative is greater than 100 which follows that the proposed procedure is better than that of Land et al. (2011). Substantial gain in efficiency is observed when P_1 is very small.

Table 1 - The percent relative efficiency of the proposed estimator $\hat{\delta}_1$ with respect to Land et al.'s (2011) estimator $\hat{\delta}_L$.

δ_1	δ_2	P_1				
		0.1	0.3	0.5	0.7	0.9
0.10	0.40	8043.48	906.43	333.33	176.25	114.64
0.13	0.40	7612.24	864.50	321.21	172.24	114.02
0.16	0.40	7230.77	828.28	311.11	169.10	113.58
0.19	0.40	6890.91	796.69	302.56	166.56	113.26
0.22	0.40	6586.21	768.89	295.24	164.47	113.01
0.25	0.40	6311.48	744.23	288.89	162.71	112.81
0.28	0.40	6062.50	722.22	283.33	161.22	112.65
0.31	0.40	5835.82	702.45	278.43	159.94	112.52
0.10	0.60	8593.75	961.54	350.00	182.22	115.74
0.13	0.60	8253.73	927.27	339.53	178.41	115.01
0.16	0.60	7942.86	896.55	330.43	175.27	114.48
0.19	0.60	7657.53	868.85	322.45	172.64	114.07
0.22	0.60	7394.74	843.75	315.38	170.41	113.76
0.25	0.60	7151.90	820.90	309.09	168.49	113.50
0.28	0.60	6926.83	800.00	303.45	166.81	113.29
0.31	0.60	6717.65	780.82	298.36	165.35	113.11

Fig Relative efficiency of the proposed estimator $\hat{\delta}_1$ with respect to Land et al.'s (2011) estimator $\hat{\delta}_L$.



4. CONCLUSIVE REMARKS

This paper addresses the problem where the number of persons possessing a rare sensitive attribute is very small and huge sample size is required to estimate. We have developed a method to estimate the mean of the number of persons possessing a rare sensitive attribute utilizing the Poisson distribution in survey sampling. We have discussed the situation when the proportion of persons possessing a rare unrelated attributes is known. Properties of the proposed randomized response model have been studied along with recommendations. Efficiency comparison is worked out to investigate the performance of the suggested procedures.

REFERENCE

[1] Greenberg B, Abul- Ela A, Simmons W and Horvitz D. The unrelated question randomized response: Theoretical framework. *J. Amer. Statist. Assoc.,* 64(1969), 529-539.

[2] Horvitz, D.G., Shah, B.V., Simmons, W.R.: The unrelated question randomized response model. In: Proceedings of the Section on Survey Research Methods, American Statistical Association, pp. 65–72 (1967).

[3] Land M, Singh S and Sedory S. Estimation of a rare attribute using Poisson distribution. Statistics, Statistics 46(3)(2012): 351-360.

[4] Lanke J. On the choice of the unrelated question in simmons version of randomized response. *J. Amer. Stat. Assoc.,* 70(1975), 80-83.

[5] Mangat N, Singh R and Singh S. An improved unrelated question randomized response strategy. *Cal. Statist. Assoc. Bull.* 42(1992): 277-282.

[6] Moors J. Optimization of the unrelated question randomized response model. *J. Amer. Stat. Assoc.* 66(1971), 627-629.

[7] Perri P. Modified randomized devices for Simmons' model. *Model Assist. Statist. Appl.* 3(3) (2008), 233-239.

[8] Ryu J, Hong K and Lee G. Randomized response model, Freedom Academy, Seoul, Korea (1993).

[9] Sidhu S and Bansal M. Estimator of population total using Rao, Hartley and Cochran's scheme using optional randomized response technique in multi-character surveys. *Model Assist. Statist. Appl.* 3(3) (2008), 259-267.

[10] Singh S. Advanced sampling theory with applications. Kluwer Academic Publishers, Dordrecht (2003).

[11] Singh H.P. and Tarray T. A.: An alternative to stratified Kim and Warde's randomized response model using optimal (Neyman) allocation. *Model Assist. Statist. Appl.* 9(2014): 37-62.

[12] Singh H.P. and Tarray T.A.: A revisit to the Singh, Horn, Singh and Mangat's randomization device for estimating a rare sensitive attribute using Poisson distribution. Model Assist. Statist. Appl., 10 (2015),129-138.

[13] Tarray T.A. and Singh H.P.: Some improved additive randomized response models utilizing higher order moments ratios of scrambling variable. Model Assist. Statist. Appl., 10 (2015), 361-383.

[14] Tarray T.A. and Singh H.P.: An adroit randomized response new additive scrambling model. Gazi University Journal of Sciences. 29(1)(2016), 159-165.

[15] Singh H and Tarray T.A.: An improved mixed randomized response model. Model Assist. Statist. Appl. 9 (2014): 73-87.

[16] Tarray T.A. and Singh H.P.: A stratified randomized response model for sensitive characteristics using the negative hyper geometric distribution. Commun. Stat. Theory Methods. 46 (6) (2016), 2607-2629.

[17] Tarray T.A. and Singh H.P.: An optional randomized response model for estimating a rare sensitive attribute using Poisson distribution. Commun. Stat. Theory Methods. 46 (6) (2017), 2638-2654.

[18] Tarray T.A. and Singh H.P.: A randomization device for estimating a rare sensitive attribute in stratified sampling using Poisson distribution. Afr. Mat. 29 (2018): 407-423.

[19] Tarray T.A. and Singh H.P.: Missing data in clinical trials: stratified Singh and Grewal's randomized response model using geometric distribution. Trends Bioinf. 11(1) (2018),44-55.

[20] Tarray T.A., Peer Bilal and Bhat M.R.: A new approach to randomized response model using fuzzy numbers. Stat. Appl. 16(2) (2018), 49-63.

[21] Tracy D and Mangat N. Some developments in randomized response sampling during the last decade – A follow up of review by Chaudhuri and Mukherjee. J. Appl. Stat. Sci. 4 (2/3) (1996): 147 – 158.

[22] Warner S. Randomized response: A survey technique for eliminating evasive answer bias. J. Amer. Stat. Assoc. 60 (1965): 63-69.