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EXISTENCE RESULTS OF HYBRID FRACTIONAL SEQUENTIAL INTEGRO-DIFFERNTIAL EQUATIONS

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Abstract. We study in this paper, the existence results for initial value problems for hybrid fractional sequential integro-differential equations. By using fixed point theorems for the sum of three operators are used for proving the main results.

Keywords: existence results; hybrid integro-differential equations; fractional sequential; fixed point theorems.

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1. Introduction

Fractional differential equations arise in the mathematical modeling of systems and processes occurring in many engineering and scientific disciplines such as physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, etc. [18]-[21].

For some recent developments on the topic, see [3]-[23], and the references therein. Hybrid fractional differential equations have also been studied by several researchers.

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This class of equations involves the fractional derivative of an unknown function hybrid with the nonlinearity depending on it. Some recent results on hybrid differential equations can be found in a series of papers [25]-[14].

In [22], Surang Sitho, Sotiris K Ntouyas, and Jessada Tariboon discussed the following Existence results for hybrid fractional integro-differential equations

$$\begin{cases} D^{\alpha}\left(\frac{x(t)-\sum_{i=1}^{m} I^{\beta_i} h_i(t,x(t))}{f(t,x(t))}\right) = g(t,x(t)) \quad a.e. \quad t \in J=[0,T], \quad 0<\alpha\leq 1 \\ x(0)=0 \end{cases}$$

where D^{α} denotes the Riemann-Liouville fractional derivative of order α, $0<\alpha\leq 1$, I^{ϕ} is the Riemann-Liouville fractional integral of order $\phi>0$, $\phi \in \{\beta_1,\beta_2,...,\beta_m\}$, $f \in C(J\times\mathfrak{R},\mathfrak{R}\setminus\{0\})$, $g \in C(J\times\mathfrak{R},\mathfrak{R})$, with $h_i \in C(J\times\mathfrak{R},\mathfrak{R})$ with $h_i(0,0)=0, i=1,2,\ldots,m$.

In [19], K Hilal and A Kajouni considered boundary value problems for hybrid differential equations with fractional order (BVPHDEF of short) involving Caputo differential operators of order $0<\alpha<1$

$$\begin{cases} D^{\alpha}\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t)) \quad a.e. \quad t \in J=[0,T] \\ a\frac{x(0)}{f(0,x(0))} + b\frac{x(T)}{f(T,x(T))} = c \end{cases}$$

where $f \in C(J\times\mathfrak{R},\mathfrak{R}\backslash\{0\})$, $g \in \mathscr{C}(J\times\mathfrak{R},\mathfrak{R})$ and a, b, c are real constants with $a+b\neq 0$.

Dhage and Lakshmikantham [12] discussed the following first order hybrid differential equation

$$\begin{cases} \frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t)) \quad a.e. \quad t \in J=[0,T] \\ x(t_0)=x_0 \in \mathfrak{R} \end{cases}$$

where $f \in C(J\times\mathfrak{R},\mathfrak{R}\backslash\{0\})$ and $g \in \mathscr{C}(J\times\mathfrak{R},\mathfrak{R})$. They established the existence, uniqueness results and some fundamental differential inequalities for hybrid differential equations initiating the study of theory of such systems and proved utilizing the theory of inequalities, its existence of extremal solutions and a comparison results.

Zhao, Sun, Han and Li [25] are discussed the following fractional hybrid differential equations involving Riemann-Liouville differential operators

$$\begin{cases} D^q\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t)) \quad a.e. \quad t \in J = [0,T] \\ \\ x(0) = 0 \end{cases}$$

where $f \in C(J \times \mathfrak{R}, \mathfrak{R}\backslash\{0\})$ and $g \in \mathscr{C}(J \times \mathfrak{R}, \mathfrak{R})$. They established the existence theorem for fractional hybrid differential equation, some fundamental differential inequalities are also established and the existence of extremal solutions.

Benchohra and al.[24] are discussed the following boundary value problems for differential equations with fractional order

$$\begin{cases} {}^cD^\alpha y(t) = f(t,y(t)), \quad for \quad each \quad t \in J = [0,T], \quad 0 < \alpha < 1 \\ ay(0) + by(T) = c \end{cases}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative, $f : [0,T] \times \mathfrak{R} \to \mathfrak{R}$, is a continuous function, a, b, c are real constants with $a + b \neq 0$.

Motivated by some recent studies on hybrid fractional differential equations see [19],[22], we consider the following value problem :

$$(1) \qquad \begin{cases} D^\alpha\left(\frac{x(t) - \sum_{i=1}^m I^{\beta_i} h_i(t,x(t))}{f(t,x(t))}\right) = g(t,x(t)) \quad a.e. \quad t \in J = [0,T], \quad 0 < \alpha < 1 \\ a\frac{x(0)}{f(0,x(0))} + b\frac{x(T)}{f(T,x(T))} = c, \end{cases}$$

where D^α denotes the Caputo fractional derivative of order α, $0 < \alpha < 1$, I^ϕ is the Riemann-Liouville fractional integral of order $\phi > 0$, $\phi \in \{\beta_1, \beta_2, \ldots, \beta_m\}$, $f \in C(J \times \mathfrak{R}, \mathfrak{R} \setminus \{0\})$, $g \in \mathscr{C}(J \times \mathfrak{R}, \mathfrak{R})$, a, b, c are real constants with $a + b \neq 0$, and $h_i \in C(J \times \mathfrak{R}, \mathfrak{R})$ with $h_i(0, x(0)) = 0, i = 1, 2, \ldots, m$. An existence result is obtained for the initial value problem (1) by using a hybrid fixed point theorem for three operators in a Banach algebra due to Dhage [15].

The problem (1) considered here is general in the sense that it includes the following three well-known classes of initial value problems of fractional differential equations.

Case I: Let $f(t,x(t)) = 1$ and $I^{\beta_i}h(t,x(t)) = 0$, $i = 1,2,\dots,m$ for all $t \in J$ and $x \in \mathfrak{R}$. Then the problem (1) reduces to standard initial value problem of fractional differential equation,

$$\begin{cases} D^{\alpha}(x(t)) = g(t,x(t)) \quad a.e. \quad t \in J = [0,T], 0 < \alpha < 1 \\ ax(0) + bx(T) = c, \end{cases}$$

Case II: If $I^{\beta_i}h(t,x(t)) = 0$, $i = 1,2,\dots,m$ for all $t \in J$ and $x \in \mathfrak{R}$ in (1). We obtain the following quadratic fractional differential equation,

$$\begin{cases} D^{\alpha}\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t)) \quad a.e. \quad t \in J = [0,T], \quad 0 < \alpha < 1 \\ a\frac{x(0)}{f(0,x(0))} + b\frac{x(T)}{f(T,x(T))} = c, \end{cases}$$

Case III: If $f(t,x(t)) = 1$ for all $t \in J$ and $x \in \mathfrak{R}$ in (1).We obtain the following interesting fractional differential equation,

$$\begin{cases} D^{\alpha}\Big(x(t) - \sum_{i=1}^{m} I^{\beta_i}h(t,x(t))\Big) = g(t,x(t)) \quad a.e. \quad t \in J = [0,T], \quad 0 < \alpha < 1 \\ ax(0) + bx(T) = c, \end{cases}$$

Therefore, the main result of this paper also includes the existence the results for the solutions of above mentioned initial value problems of fractional differential equations as special cases. As a second problem we discuss in Section 4 an initial value problem for hybrid fractional sequential integro-differential equations,

$$(2) \quad \begin{cases} D^{\alpha}\left(\frac{D^{\beta}x(t) - \sum_{i=1}^{m} I^{\beta_i}h_i(t,x(t))}{f(t,x(t))}\right) = g(t,x(t),I^{\gamma}x(t)) \quad a.e. \quad t \in J = [0,T], \quad 0 < \alpha < 1 \\ a\frac{x(0)}{f(0,x(0))} + b\frac{x(T)}{f(T,x(T))} = c, \qquad D^{\beta}x(0) = 0, \end{cases}$$

where $0 < \alpha, \beta \leq 1$, $1 < \alpha + \beta \leq 2$, functions f, h, and constants $\beta_1, \beta_2, \dots, \beta_m$ are defined as in problem (2), $g \in C(J \times \mathfrak{R}, \mathfrak{R})$, a, b, c are real constants with $a + b \neq 0$, and I^{γ} is the Riemann-Liouville fractional integral of order γ. By using a useful generalization of Krasnoselskii's fixed point theorem due to Dhage [17], we prove an existence result for the initial value problem (2). This paper is arranged as follows. In Section 2, we recall some concepts and some fractional calculation law and establish preparation results. In Section 3, we study the existence of the initial value problem (1), based on the Dhage fixed point theorem,while in Section 4 we deal with the initial value problem (2).

2. Preliminaries

In this section, we introduce notations, definitions, and preliminaries facts which are used throughout this paper. By $E = C(J,\mathfrak{R})$ we denote the Banach space of all continuous functions from $J = [0,T]$ into $\mathfrak{R}$ with the norm

$$\|y\| = \sup\{|y(t)|, t \in J\}$$

and a multiplication in E by

$\|x\| = \sup_{t\in J} |x(t)|$ and $(xy)(t) = x(t)y(t)$, $\forall t \in J$.

Clearly E is a Banach algebra with respect to above supremum norm and the multiplication in it.

Let $\mathscr{C}(J\times\mathfrak{R},\mathfrak{R})$ denote the class of functions $g : J\times\mathfrak{R} \longrightarrow \mathfrak{R}$ such that

(i) the map $t \mapsto g(t,x)$ is mesurable for each $x \in \mathfrak{R}$, and

(ii) the map $x \mapsto g(t,x)$ is continuous for each $t \in J$.

The class $\mathscr{C}(J\times\mathfrak{R},\mathfrak{R})$ is called the Carathéodory class of functions on $J\times\mathfrak{R}$ which are Lebesgue integrable when bounded by a Lebesgue integrable function on J .

By $L^1(J,\mathfrak{R})$ denote the space of Lebesgue integrable real-valued functions on J equipped with the norm $\|.\|_{L^1}$ defined by

$\|x\|_{L^1} = \int_0^T |x(s)|ds$

Definition 2.1: The fractional integral of the function $h \in L^1([a,b],\mathfrak{R}^+)$ of order $\alpha \in \mathfrak{R}^+$ is defined by

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds$$

where Γ is the gamma function.

Definition 2.2: For a function h given on the interval $[a,b]$, the The Riemann-Liouville fractional-order derivative of h, is defined by

$$({}^cD_{a^+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(\alpha)}(s)ds$$

where $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

Definition 2.3:[5] For a function h given on the interval $[a,b]$, the Caputo fractional-order derivative of h, is defined by

$$({}^{c}D_{a^+}^{\alpha}h)(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(\alpha)}h^{(n)}(s)ds$$

where $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

Lemma 2.1: [18] Let $\alpha > 0$ and $x \in C(0,T)\cap L(0,T)$. Then the fractional differential equation

$$D^{\alpha}x(t) = 0$$

has a unique solution

$$x(t) = k_1 t^{\alpha-1} + k_2 t^{\alpha-2} + \ldots + k_n t^{\alpha-n},$$

where $k_i \in \Re$, $i = 1,2,\ldots,n$, and $n-1 < \alpha < n$.

Lemma 2.2: Let $\alpha > 0$. Then for $x \in C(0,T)\cap L(0,T)$ we have

$$I^{\alpha}D^{\alpha}x(t) = x(t) + c_0 + c_1 t + \ldots + c_{n-1}t^{n-1},$$

fore some $c_i \in \Re$, $i = 1,2,\ldots,n-1$. where $n = [\alpha]+1$.

3. Hybrid Fractional Integro-Differential Equations

In this section we consider the initial value problem (1). The following hybrid fixed point theorem for three operators in a Banach algebra E, due to Dhage [15], will be used to prove the existence result for the initial value problem (1).

lemma 3.1: Let S be a nonempty, closed convex and bounded subset of a Banach algebra E and let $A,C : E \longrightarrow E$ and $B : S \longrightarrow E$ be three operators satisfying:

- (a_1) A and C are Lipschitzian with Lipschitz constants δ and ρ, respectively,
- (b_1) B is compact and continuous,
- (c_1) $x = AxBy + Cx \Longrightarrow x \in S$ for all $y \in S$,
- (d_1) $\delta M + \rho < 1$, where $M = \|B(S)\|$.

Then the operator equation $x = AxBx + Cx$ has a solution.

lemma 3.2: Suppose that $0 < \alpha < 1$ and a, b, c are real constants with $a+b \neq 0$.

Then, for any $h \in L^1(J,\Re)$, the function $x \in C(J,\Re)$ is a solution of the

$$\begin{cases} D^{\alpha}\left(\frac{x(t)-\sum_{i=1}^{m} I^{\beta_i}h_i(t,x(t))}{f(t,x(t))}\right) = h(t) \quad a.e. \quad t \in J = [0,T] \\ a\frac{x(0)}{f(0,x(0))} + b\frac{x(T)}{f(T,x(T))} = c \end{cases} \tag{3}$$

if and only if x satisfies the hybrid integral equation

$$\begin{aligned} (4)\; x(t) &= [f(t,x(t))]\Big[\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds - \frac{1}{a+b}\Big(\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)ds - c \\ &+ \frac{b\sum_{i=1}^{m} I^{\beta_i}h_i(T,x(T))}{f(T,x(T))}\Big)\Big] + \sum_{i=1}^{m} I^{\beta_i}h_i(t,x(t)), \quad t \in [0,T] \end{aligned}$$

proof Assume that x is a solution of the problem (3). By definition, $\frac{x(t)}{f(t,x(t))}$ is continuous. Applying the Caputo fractional operator of the order α, we obtain the first equation in (3). Again, substituting $t=0$ and $t=T$ in (3) we have

$$\frac{x(0)}{f(0,x(0))} = \frac{-1}{a+b}\Big(\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)ds - c + \frac{b\sum_{i=1}^{m} I^{\beta_i}h_i(T,x(T))}{f(T,x(T))}\Big) + \frac{\sum_{i=1}^{m} I^{\beta_i}h_i(0,x(0))}{f(0,x(0))}$$

$$\begin{aligned} \frac{x(T)}{f(T,x(T))} &= \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)ds \\ &- \frac{1}{a+b}\Big(\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)ds - c + \frac{b\sum_{i=1}^{m} I^{\beta_i}h_i(T,x(T))}{f(T,x(T))}\Big) \\ &+ \frac{\sum_{i=1}^{m} I^{\beta_i}h_i(T,x(T))}{f(T,x(T))} \end{aligned}$$

then

$$\begin{aligned} a\frac{x(0)}{f(0,x(0))} &+ b\frac{x(T)}{f(T,x(T))} = \frac{-ab}{(a+b)\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)ds + \frac{ac}{a+b} \\ &+ \frac{b}{(\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)ds - \frac{b^2}{(a+b)\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)ds \\ &+ \frac{bc}{a+b} + \Big(\frac{-ab}{a+b} - \frac{b^2}{a+b} + b\Big)\frac{\sum_{i=1}^{m} I^{\beta_i}h_i(T,x(T))}{f(T,x(T))} \end{aligned}$$

this implies that

$$a\frac{x(0)}{f(0,x(0))} + b\frac{x(T)}{f(T,x(T))} = c$$

Conversely, $$D^{\alpha}\left(\frac{x(t)-\sum_{i=1}^{m} I^{\beta_i}h_i(t,x(t))}{f(t,x(t))}\right)=h(t)$$
so we get

$$\frac{x(t)-\sum_{i=1}^{m} I^{\beta_i}h_i(t,x(t))}{f(t,x(t))}=I^{\alpha}h(t)+\frac{x(0)}{f(0,x(0))}$$

$$\frac{x(t)}{f(t,x(t))}=I^{\alpha}h(t)+\frac{x(0)}{f(0,x(0))}+\frac{\sum_{i=1}^{m} I^{\beta_i}h_i(t,x(t))}{f(t,x(t))}$$

Then

$$a\frac{x(0)}{f(0,x(0))}+b\frac{x(T)}{f(T,x(T))}=bI^{\alpha}h(T)+(a+b)\frac{x(0)}{f(0,x(0))}+b\frac{\sum_{i=1}^{m} I^{\beta_i}h_i(T,x(T)}{f(T,x(T))}$$

$$\frac{x(0)}{f(0,x(0))}=\frac{1}{a+b}\Big(c-bI^{\alpha}h(T)-\frac{b\sum_{i=1}^{m} I^{\beta_i}h_i(T,x(T))}{f(T,x(T))}\Big)$$
In consequence, we have
$$\begin{aligned} x(t) &= (f(t,x(t)))\Big[\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds-\frac{1}{a+b}\Big(\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)ds-c \\ &+ \frac{b\sum_{i=1}^{m} I^{\beta_i}h_i(T,x(T))}{f(T,x(T))}\Big)\Big]+\sum_{i=1}^{m} I^{\beta_i}h_i(t,x(t)), \quad t\in[0,T] \end{aligned}$$
. In the forthcoming analysis, we need the following assumptions. Assume that :

(H_1) The functions $f: J\times\Re\longrightarrow\times\Re\setminus\{0\}$ and $h_i: J\times\Re\longrightarrow\times\Re$, $h_i(0,x(0))=0, i=1,2,\dots,m$, are continuous and there exist two positive functions ϕ, ψ_i, $i=1,2,\dots,m$ with bound $\|\phi\|$ and $\|\psi\|$, $i=1,2,\dots,m$, respectively, such that

$$|f(t,x(t))-f(t,y(t))|\leq\phi(t)|x(t)-y(t)| \tag{5}$$

and

$$|h_i(t,x(t))-h_i(t,y(t))|\leq\psi_i(t)|x(t)-y(t)|,\quad i=1,2,\dots,m, \tag{6}$$

for $t \in J$ and $x, y \in \mathfrak{R}$.

(H_2) There exists a function $h \in L^1(J, \mathfrak{R})$ such that .

$$|g(t,x)| \leq h(t) \quad a.e \quad t \in J \tag{7}$$

for all $x \in \mathfrak{R}$.

(H_3) There exists a number $r > 1$ such that

$$r \geq \frac{F_0\Big[\Big(1+\frac{|b|}{|a+b|}\Big)\Big(\|h\|_{L^1}\frac{T^\alpha}{\Gamma(\alpha+1)}\Big)+\frac{|c|}{|a+b|}+\Big|\frac{b\sum_{i=1}^m I^{\beta_i}h_i(T,x(T))}{(a+b)f(T,x(T))}\Big|\Big]+K_0\sum_{i=1}^m\frac{T^{\beta_i}}{\Gamma(\beta_i+1)}}{1-\|\phi\|\Big[\Big(1+\frac{|b|}{|a+b|}\Big)\Big(\|h\|_{L^1}\frac{T^\alpha}{\Gamma(\alpha+1)}\Big)+\frac{|c|}{|a+b|}+\Big|\frac{b\sum_{i=1}^m I^{\beta_i}h_i(T,x(T))}{(a+b)f(T,x(T))}\Big|\Big]-\sum_{i=1}^m\frac{\|\psi_i\|T^{\beta_i}}{\Gamma(\beta_i+1)}} \tag{8}$$

where $F_0 = \sup_{t\in J}|f(t,0)|$ and $K_0 = \sup_{t\in J}|h_i(t,0)|$, $i = 1,2,\ldots,m$.

Theorem 3.1 Assume that the conditions $(H_1)-(H_3)$ hold. Then the initial value problem (1) has at least one solution on J provided that

$$\begin{aligned} \|\phi\|\Big[\Big(1+\frac{|b|}{|a+b|}\Big)\Big(\|h\|_{L^1}\frac{T^\alpha}{\Gamma(\alpha+1)}\Big) &+ \frac{|c|}{|a+b|}+\Big|\frac{b\sum_{i=1}^m I^{\beta_i}h_i(T,x(T))}{(a+b)f(T,x(T))}\Big|\Big] \\ &+ \sum_{i=1}^m\frac{\|\phi\|T^{\beta_i}}{\Gamma(\beta_i+1)} < 1. \end{aligned} \tag{9}$$

proof:. Set $E = C(J,\mathfrak{R})$ and define a subset S of E as

$$S = \{x \in E : \|x\| \leq r\}$$

where r satisfies inequality (8).

Clearly S is closed, convex, and bounded subset of the Banach space E. By Lemma 3.2, problem (1) is equivalent to the integral equation (3). Now we define three operators;

$\mathscr{A} : E \longrightarrow E$ by

$$\mathscr{A}x(t) = f(t,x(t)), \quad t \in J, \tag{10}$$

$\mathscr{B} : S \longrightarrow E$ by

$$\begin{aligned} \mathscr{B}x(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s,x(s))ds - \frac{1}{a+b}\Big(\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}g(s,x(s))ds - c \\ &+ \frac{b\sum_{i=1}^m I^{\beta_i}h_i(T,x(T))}{f(T,x(T))}\Big), \quad t \in J, \end{aligned} \tag{11}$$

and $\mathscr{C}: E \longrightarrow E$ by

$$\mathscr{C}x(t) = \sum_{i=1}^{m} I^{\beta_i} h_i(t,x(t)) = \sum_{i=1}^{m} \int_0^t \frac{(t-s)^{\beta_i-1}}{\Gamma(\beta_i)} h_i(s,x(s))ds, \quad t \in J \tag{12}$$

We shall show that the operators $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ satisfy all the conditions of Lemma 3.1. This will be achieved in the following series of steps.

Step 1. We first show that $\mathscr{A}$ and $\mathscr{C}$ are Lipschitzian on E.

Let $x,y \in E$. Then by (H_1), for $t \in J$ we have

$$\begin{aligned} |\mathscr{A}x(t) - \mathscr{A}y(t)| &= |f(t,x(t)) - f(t,y(t))| \\ &\leq \phi(t)|x(t)-y(t)| \leq \|\phi\|\|x-y\| \end{aligned}$$

which implies $\|\mathscr{A}x - \mathscr{A}y\| \leq \|\phi\|\|x-y\|$ for all $x,y \in E$. Therefore, $\mathscr{A}$ is a Lipschitzian on E with Lipschitz constant $\|\phi\|$.

Analogously, for any $x,y \in E$, we have

$$\begin{aligned} |\mathscr{C}x(t) - \mathscr{C}y(t)| &= \Big| \sum_{i=1}^{m} I^{\beta_i} h_i(t,x(t)) - \sum_{i=1}^{m} I^{\beta_i} h_i(t,y(t)) \Big| \\ &\leq \sum_{i=1}^{m} \int_0^t \frac{(t-s)^{\beta_i-1}}{\Gamma(\beta_i)} \psi_i(s)|x(s)-y(s)|ds \\ &\leq \|x-y\| \sum_{i=1}^{m} \frac{\|\psi_i\| T^{\beta_i}}{\Gamma(\beta_i+1)} \end{aligned}$$

This means that

$$\|\mathscr{C}x - \mathscr{C}y\| \leq \sum_{i=1}^{m} \frac{\|\psi_i\| T^{\beta_i}}{\Gamma(\beta_i+1)} \|x-y\|$$

Thus, $\mathscr{C}$ is a Lipschitzian on E with Lipschitz constant $\sum_{i=1}^{m} \frac{\|\psi_i\| T^{\beta_i}}{\Gamma(\beta_i+1)}$.

Step 2. The operator $\mathscr{B}$ is completely continuous on S.

We first show that the operator $\mathscr{B}$ is continuous on E. Let $\{x_n\}$ be a sequence in S converging to a point $x \in S$. Then by the Lebesgue dominated convergence theorem, for all $t \in J$, we obtain

$$\begin{aligned} \lim_{n\to\infty} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s,x_n(s))ds &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \lim_{n\to\infty} g(s,x_n(s))ds \\ &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s,x(s))ds \end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty}\frac{b}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}g(s,x_n(s))ds &= \frac{b}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}\lim_{n\to\infty}g(s,x_n(s))ds\\ &= \frac{b}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}g(s,x(s))ds\end{aligned}$$

In consequence, we have

$$\lim_{n\to\infty}\mathscr{B}x_n=\mathscr{B}x$$

This implies that $\mathscr{B}$ is continuous on S.

Next we will prove that the set $\mathscr{B}(S)$ is a uniformly bounded in S. For any $x\in S$, we have

$$\begin{aligned}|\mathscr{B}x(t)| &= \Big|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x(s))ds-\frac{1}{a+b}\Big(\frac{b}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}g(s,x(s))ds-c\\ &+ \frac{b\sum_{i=1}^m I^{\beta_i}h_i(T,x(T))}{f(T,x(T))}\Big)\Big|\\ &\leq \|h\|_{L^1}\frac{T^\alpha}{\Gamma(\alpha+1)}+\frac{|b|}{|a+b|}\frac{T^\alpha}{\Gamma(\alpha+1)}\|h\|_{L^1}+\frac{|c|}{|a+b|}+\Big|\frac{b\sum_{i=1}^m I^{\beta_i}h_i(T,x(T))}{(a+b)f(T,x(T))}\Big|\\ &\leq \Big(1+\frac{|b|}{|a+b|}\Big)(\|h\|_{L^1}\frac{T^\alpha}{\Gamma(\alpha+1)})+\frac{|c|}{|a+b|}+\Big|\frac{b\sum_{i=1}^m I^{\beta_i}h_i(T,x(T))}{(a+b)f(T,x(T))}\Big|=K_1\end{aligned}$$

for all $t\in J$. Therefore, $\|\mathscr{B}\|\leq K_1$, which shows that $\mathscr{B}$ is uniformly bounded on S.

Now, we will show that $\mathscr{B}(S)$ is an equicontinuous set in E. Let $\tau_1,\tau_2\in J$ with $\tau_1<\tau_2$ and $x\in S$. Then we have

$$\begin{aligned}|\mathscr{B}x(\tau_2)-\mathscr{B}x(\tau_1)| &= \Big|\int_0^{\tau_2}\frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x(s))ds-\int_0^{\tau_1}\frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x(s))ds\Big|\\ &\leq \int_0^{\tau_1}\frac{|(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}|}{\Gamma(\alpha)}|g(s,x(s))|ds+\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)}|g(s,x(s))|ds\\ &\leq \int_0^{\tau_1}\frac{|(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}|}{\Gamma(\alpha)}\|h\|_{L^1}ds+\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)}\|h\|_{L^1}ds\end{aligned}$$

which is independent of $x\in S$. As $\tau_1\longrightarrow\tau_2$, the right-hand side of the above inequality tends to zero. Therefore, it follows from the Arzelá-Ascoli theorem that $\mathscr{B}$ is a completely continuous operator on S.

Step 3. The hypothesis (c_1) of Lemma 3.1 is satisfied.

Let $x \in E$ and $y \in S$ be arbitrary elements such that $x = \mathscr{A}x\mathscr{B}y + \mathscr{C}x$. Then we have

$$\begin{aligned}
|x(t)| &\leq |\mathscr{A}x(t)||\mathscr{B}y(t)| + |\mathscr{C}x(t)| \\
&\leq |f(t,x(t))|\Big|\Big[\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s,x(s))ds + \frac{1}{a+b}\Big(\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} g(s,x(s))ds + c \\
&+ \frac{b\sum_{i=1}^m I^{\beta_i} h_i(T,x(T))}{f(T,x(T))}\Big)\Big]\Big| + \Big|\sum_{i=1}^m I^{\beta_i} h_i(s,s(t))\Big| \\
&\leq (|f(t,x(t) - f(t,0)| + |f(t,0)|)\Big[\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|h\|_{L^1} ds \\
&+ \frac{1}{|a+b|}\Big(\frac{|b|}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\|h\|_{L^1} ds + |c| \\
&+ \Big|\frac{b\sum_{i=1}^m I^{\beta_i} h_i(T,x(T))}{f(T,x(T))}\Big|\Big)\Big] + \sum_{i=1}^m I^{\beta_i}(|h_i(s,x(s)) - h_i(s,0) + |h_i(s,0)|) \\
&\leq (r\|\phi\| + F_0)\Big[\Big(1 + \frac{|b|}{|a+b|}\Big)\Big(\|h\|_{L^1}\frac{T^\alpha}{\Gamma(\alpha+1)}\Big) + \frac{|c|}{|a+b|} + \Big|\frac{b\sum_{i=1}^m I^{\beta_i} h_i(T,x(T))}{(a+b)f(T,x(T))}\Big|\Big] \\
&+ \sum_{i=1}^m \frac{(r\|\psi_i\| + k_0)T^{\beta_i}}{\Gamma(\beta_i+1)}
\end{aligned}$$

which leads to

$$\begin{aligned}
\|x\| &\leq (r\|\phi\| + F_0)\Big[\Big(1 + \frac{|b|}{|a+b|}\Big)\Big(\|h\|_{L^1}\frac{T^\alpha}{\Gamma(\alpha+1)}\Big) + \frac{|c|}{|a+b|} \\
&+ \Big|\frac{b\sum_{i=1}^m I^{\beta_i} h_i(T,x(T))}{(a+b)f(T,x(T))}\Big|\Big] + \sum_{i=1}^m \frac{(r\|\psi_i\| + k_0)T^{\beta_i}}{\Gamma(\beta_i+1)} \leq r
\end{aligned} \tag{13}$$

Therefore, $x \in S$.

Step 4. Finally we show that $\delta M + \rho < 1$, that is, (d_1) of Lemma 3.1 holds.

Since

$$\begin{aligned}
M = \|\mathscr{B}(S)\| &= \sup_{x\in S}\{\sup_{t\in J}|\mathscr{B}x(t)|\} \\
&\leq \Big(1 + \frac{|b|}{|a+b|}\Big)\Big(\|h\|_{L^1}\frac{T^\alpha}{\Gamma(\alpha+1)}\Big) \\
&+ \frac{|c|}{|a+b|} + \Big|\frac{b\sum_{i=1}^m \frac{T^{\beta_i}}{\Gamma(\beta_i+1)} I^{\beta_i} h_i(T,x(T))}{(a+b)f(T,x(T))}\Big|
\end{aligned} \tag{14}$$

and by (H_3) we have

$$\|\phi\|M + \sum_{i=1}^m \frac{T^{\beta_i}}{\Gamma(\beta_i+1)}\|\psi_i\| < 1.$$

with $\delta = \|\phi\|$ and $\rho = \sum_{i=1}^{m} \frac{T^{\beta_i}}{\Gamma(\beta_i+1)} \|\psi_i\|$.

Thus all the conditions of Lemma 3.1 are satisfied and hence the operator equation $x = \mathscr{A}x\mathscr{B}x + \mathscr{C}x$ has a solution in S. In consequence, problem (3) has a solution on J. This completes the proof.

4. Hybrid Fractional Sequential Integro-Differential Equations

In this section we consider the initial value problem (2). An existence result will be proved by using the following fixed point theorem due to Dhage.

Lemma 4.1: Let M be a nonempty, closed, convex and bounded subset of the Banach space X and let $A : X \longrightarrow X$ and $B : M \longrightarrow X$ be two operators such that

(i) A is a contraction,

(ii) B is completely continuous, and

(iii) $x = Ax + By$ for all $y \in M \Longrightarrow x \in M$.

Then the operator equation $Ax + Bx = x$ has a solution.

Lemma 4.2: Suppose that $0 < \alpha, \beta \leq 1$, $0 < \alpha + \beta \leq 1$, $\gamma > 0$, $a + b \neq 0$ and the functions f , g, $h_i, i = 1, 2, \ldots, m$ satisfy problem (2). Then the unique solution of the hybrid fractional sequential integro-differential problem (2) is given by

$$\begin{aligned} x(t) &= f(t,x(t))\Big[\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\Big(\frac{1}{\Gamma(\alpha)}\int_0^s (s-u)^{\alpha-1} g(u,x(u),I^\gamma x(u))du \\ &- \frac{1}{a+b}\Big(\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} g(u,x(u),I^\gamma x(u))du\Big) - c \\ &+ \frac{b\sum_{i=1}^m I^{\beta+\beta_i} h_i(T,x(T))}{f(T,x(T))}\Big)ds\Big] + \sum_{i=1}^m I^{\beta+\beta_i} h_i(t,x(t)) \quad ,t\in[0,T] \end{aligned} \tag{15}$$

Proof: By lemma 2.2 we have

$$\frac{D^\beta x(t) - \sum_{i=1}^m I^{\beta_i} h_i(t,x(t))}{f(t,x(t))} = I^\alpha g(t,x(t),I^\gamma x(t)) + c_0$$

by condition $D^\beta x(0) = 0$, implies that $c_0 = 0$

applying the semigroup property, i.e., $I^\beta I^{\beta_i} h_i = I^{\beta_i+\beta} h_i$, $i = 1, 2, \ldots, m$, we obtain the,

$$\frac{x(t)}{f(t,x(t))} = I^{\alpha+\beta} g(t,x(t),I^\gamma x(t)) + \frac{\sum_{i=1}^m I^{\beta+\beta_i} h_i(T,x(T))}{f(t,x(t))} + \frac{x(0)}{f(0,x(0))}$$

Then

$$a\frac{x(0)}{f(0,x(0))}+b\frac{x(T)}{f(T,x(T))}=bI^{\alpha+\beta}gT,x(T),I^{\gamma}x(T))+\frac{b\sum_{i=1}^{m}I^{\beta+\beta_i}h_i(T,x(T))}{f(T,x(T))}+(a+b)\frac{x(0)}{f(0,x(0))}$$

Thus,

$$\frac{x(0)}{f(0,x(0))}=\frac{1}{a+b}\Big(c-bI^{\alpha+\beta}g(T,x(T),I^{\gamma}x(T))-\frac{b\sum_{i=1}^{m}I^{\beta+\beta_i}h_i(T,x(T))}{f(T,x(T))}\Big)$$

Consequently,

$$\begin{aligned} x(t) &= f(t,x(t))\Big[\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}\Big(\frac{1}{\Gamma(\alpha)}\int_0^s(s-u)^{\alpha-1}g(u,x(u),I^{\gamma}x(u))du \\ &- \frac{b}{(a+b)\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}g(u,x(u),I^{\gamma}x(u))du+\frac{c}{a+b} \\ &- \frac{b\sum_{i=1}^{m}I^{\beta+\beta_i}h_i(T,x(T))}{(a+b)f(T,x(T))}\Big)ds\Big]+\sum_{i=1}^{m}I^{\beta+\beta_i}h_i(t,x(t)), \quad t\in[0,T] \end{aligned}$$

In the forthcoming analysis, we need the following assumptions. Assume that :

(A_1) The functions $f:J\times\mathfrak{R}\longrightarrow\mathfrak{R}\setminus\{0\}$ and $g:J\times\mathfrak{R}^2\longrightarrow\mathfrak{R}$, are continuous and there exist two positive functions ϕ, χ with bound $\|\phi\|$ and $\|\chi\|$, respectively, such that

$$|f(t,x(t))-f(t,y(t))|\leq\phi(t)|x(t)-y(t)| \tag{16}$$

and

$$|g(t,x(t),y(t))-g(t,\bar{x}(t),\bar{y}(t))|\leq\chi(t)|x(t)-y(t)|+|\bar{x}(t)-\bar{y}(t)| \tag{17}$$

for $t\in J$ and $x,y,\bar{x},\bar{y}\in\mathfrak{R}$.

(A_2) $|f(t,x)|\leq\mu(t)$, $\forall(t,x)\in J\times\mathfrak{R}$, $\mu\in C(J,\mathfrak{R}^+)$, $|g(t,x,y)|\leq\nu(t)$, $\forall(t,x,y)\in J\times\mathfrak{R}\times\mathfrak{R}$, $\nu\in C(J,\mathfrak{R}^+)$, and $|h_i(t,x)|\leq\theta_i(t)$, $\forall(t,x)\in J\times\mathfrak{R}$, $\theta_i\in C(J,\mathfrak{R}^+)$, $i=1,2,\ldots,m$.

Theorem 4.1: Assume that the conditions $(A_1)-(A_2)$ hold. Then the initial value problem (2) has at least one solution on J provided that

$$\begin{aligned}\frac{T^{\beta}}{\Gamma(\beta+1)}\Big[\frac{T^{\alpha}}{\Gamma(\alpha+1)}\Big(1 &+ \frac{|b|}{|a+b|}\Big)\|\nu\|+\frac{|c|}{|a+b|}+\frac{b\sum_{i=1}^{m}I^{\beta_i}h_i(T,x(T)}{f(T,x(T))}\Big]\|\phi\| \\ &+ \|\mu\|\|\chi\|\Big(1+\frac{|b|}{|a+b|}\Big)\Big(\frac{T^{\beta}}{\Gamma(\beta+1)}\frac{T^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)}\Big)<1 \end{aligned}\tag{18}$$

then problem (2) has at least one solution on J.

Proof: Setting $\sup\limits_{t\in J}|\mu(t)|=\|\mu\|$, $\sup\limits_{t\in J}|\nu(t)|=\|\nu\|$, $\sup\limits_{t\in J}|\theta_i(t)|=\|\theta_i\|$, $i=1,2,\dots,m$, and choosing

$$\begin{aligned} R &\geq \sum_{i=1}^{m}\frac{T^{\beta_i+\beta}}{\Gamma(\beta_i+\beta+1)}\|\theta_i\|+\|\mu\|\Big[\Big(1+\frac{|a|}{|a+b|}\Big)\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\|\nu\| \\ &+ \frac{|c|}{|a+b|}+\Big|\frac{b\sum_{i=1}^{m}I^{\beta_i+\beta}h_i(T,x(T))}{(a+b)f(T,x(T))}\Big|\Big]\end{aligned}\tag{19}$$

we consider $B_R=\{x\in C(J,\mathbb{R}):\|x\|\leq R\}$. We define the operators $\mathscr{A}:E\longrightarrow E$ by

$$\mathscr{A}x(t)=f(t,x(t)),\quad t\in J,\tag{20}$$

$\mathscr{D}:B_R\longrightarrow E$ by

$$\mathscr{D}x(t) = \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x(s),I^{\gamma}x(s))ds-\frac{b}{(a+b)\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}g(u,x(u),I^{\gamma}x(u))du,\quad t\in J$$

and

$$\mathscr{Q}x(t)=\sum_{i=1}^{m}\frac{1}{\Gamma(\beta_i+\beta)}\int_0^t(t-s)^{\beta_i+\beta-1}h_i(s,x(s)),\quad t\in J\tag{21}$$

and

$$\begin{aligned}\mathscr{T}x(t) &= \mathscr{A}x(t)\Big(\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}\mathscr{D}x(s)ds+\frac{c}{a+b} \\ &+ \frac{b\sum_{i=1}^{m}I^{\beta_i}h_i(T,x(T))}{(a+b)f(T,x(T))}\Big),t\in J\end{aligned}\tag{22}$$

For any $y \in B_R$, we have

$$
\begin{aligned}
|x(t)| &= |\mathscr{Q}x(t)+\mathscr{T}y(t)| \\
&\leq \sum_{i=1}^{m}\frac{1}{\Gamma(\beta_i+\beta)}\int_0^t (t-s)^{\beta_i+\beta-1}|h_i(s,x(s))|ds + |\mathscr{A}y(t)|\Big(\frac{1}{\Gamma(\beta)}\int_0^t |\mathscr{D}y(s)|ds+\frac{|c|}{|a+b|} \\
&+ \Big|\frac{b\sum_{i=1}^{m} I^{\beta_i+\beta}h_i(T,x(T))}{(a+b)f(T,x(T))}\Big|\Big) \\
&\leq \sum_{i=1}^{m}\frac{1}{\Gamma(\beta_i+\beta)}\int_0^t (t-s)^{\beta_i+\beta-1}|\theta_i(s)||ds \\
&+ |\mu(t)|\Big[\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\Big(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}|\nu(\tau)|d\tau+\frac{1}{|a+b|}\frac{|b|}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}|\nu(s)|\Big)ds \\
&+ \frac{|c|}{|a+b|}+\Big|\frac{b\sum_{i=1}^{m} I^{\beta_i+\beta}h_i(T,x(T))}{(a+b)f(T,x(T))}\Big|\Big] \\
&\leq \sum_{i=1}^{m}\frac{T^{\beta_i+\beta}}{\Gamma(\beta_i+\beta+1)}\|\theta_i\|+\|\mu\|\Big[\Big(1+\frac{|a|}{|a+b|}\Big)\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\|\nu\|+\frac{|c|}{|a+b|} \\
&+ \Big|\frac{b\sum_{i=1}^{m} I^{\beta_i+\beta}h_i(T,x(T))}{(a+b)f(T,x(T))}\Big|\Big]
\end{aligned}
$$

and therefore $\|x\| \leq R$, which means that $x \in B_R$. Hence, the condition (iii) of Lemma 4.1 holds.

Next we will show that $\mathscr{Q}$ satisfy the condition (ii) of Lemma 4.1. The operator $\mathscr{Q}$ is obviously continuous. Also, $\mathscr{Q}$ is uniformly bounded on B_R as

$$\|Qx\| \leq \sum_{i=1}^{m}\frac{T^{\beta_i+\beta}}{\Gamma(\beta_i+\beta+1)}\|\theta_i\|$$

Let $\tau_1,\tau_2 \in J$ with $\tau_1<\tau_2$ and $x\in B_R$. We define $\sup\limits_{(t,x)\in J\times B_R}|h_i(t,x)|=\bar{h}_i<\infty$, $i=1,2,\ldots,m$. Then we have

$$
\begin{aligned}
|\mathscr{Q}x(\tau_2)-\mathscr{Q}x(\tau_1)| &= \sum_{i=1}^{m}\frac{1}{\Gamma(\beta_i+\beta)}\Big|\int_0^{\tau_2}(\tau_2-s)^{\beta_i+\beta-1}h_i(s,x(s))ds \\
&- \sum_{i=1}^{m}\frac{1}{\Gamma(\beta_i+\beta)}\int_0^{\tau_1}(\tau_1-s)^{\beta_i+\beta-1}h_i(s,x(s))ds\Big| \\
&\leq \sum_{i=1}^{m}\frac{\bar{h}_i}{\Gamma(\beta_i+\beta)}\Big|\int_0^{\tau_1}[(\tau_2-s)^{\beta_i+\beta-1}-(\tau_1-s)^{\beta_i+\beta-1}]+\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\beta_i+\beta-1}ds\Big| \\
&\leq \sum_{i=1}^{m}\frac{\bar{h}_i}{\Gamma(\beta_i+\beta+1)}|\tau_2^{\beta_i+\beta}-\tau_1^{\beta_i+\beta}|
\end{aligned}
$$

which is independent of x and tends to zero as $\tau_2-\tau_1 \longrightarrow 0$. Thus, $\mathscr{Q}$ is equicontinuous. So $\mathscr{Q}$ is relatively compact on B_R. Hence, by the Arzelá-Ascoli theorem, $\mathscr{Q}$ is compact on B_R.

Now we show that $\mathscr{T}$ is a contraction mapping. Let $x, y \in B_R$. Then for $t \in J$ we have

$$
\begin{aligned}
& |\mathscr{T}x(t) - \mathscr{T}y(t)| \\
= & \left|\mathscr{A}x(t)\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\mathscr{D}x(s)ds - \mathscr{A}y(t)\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\mathscr{D}y(s)ds\right| \\
= & \left|\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}[\mathscr{A}x(s)\mathscr{D}x(s) - \mathscr{A}y(s)\mathscr{D}x(s) + \mathscr{A}y(s)\mathscr{D}y(s) - \mathscr{A}x(s)\mathscr{D}x(s)]ds\right| \\
\leq & \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\Big\{|\mathscr{D}x(s)||\mathscr{A}x(s) - \mathscr{A}y(s)| + |\mathscr{A}y(s)||\mathscr{D}x(s) - \mathscr{D}y(s)]\Big\}ds \\
\leq & \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\Big\{\Big(1+\frac{|a|}{|a+b|}\Big)\frac{T^\alpha}{\Gamma(\alpha+1)}\|\nu\|\|\phi\|\|x-y\| \\
+ & \|\mu\|\|\chi\|\Big(1+\frac{|a|}{|a+b|}\Big)\frac{T^\alpha}{\Gamma(\alpha+1)}\Big(\|x-y\| + \|x-y\|\int_0^s (s-u)^{\gamma-1}\frac{1}{\Gamma(\gamma)}du\Big)\Big\}ds \\
\leq & \Big\{\frac{T^\alpha}{\Gamma(\alpha+1)}\frac{T^\beta}{\Gamma(\beta+1)}\|\nu\|\|\phi\| + \Big(1+\frac{|a|}{|a+b|}\Big)\|\mu\|\|\chi\|\Big(\frac{T^\alpha}{\Gamma(\alpha+1)}\frac{T^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)}\Big)\Big\}\|x-y\|
\end{aligned}
$$

Hence, by (18), $\mathscr{T}$ is a contraction mapping, and thus the condition (i) of Lemma 4.1 is satisfied.

Thus all the assumptions of Lemma 4.1 are satisfied. Therefore, the conclusion of Lemma 4.1 implies that problem (2) has at least one solution on J.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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