Available online at http://scik.org
Eng. Math. Lett. 2021, 2021:1
https://doi.org/10.28919/eml/4877
ISSN: 2049-9337

A HYBRID TWO-STEP METHOD FOR DIRECT SOLUTIONS OF GENERAL SECOND ORDER INITIAL VALUE PROBLEM

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Abstract: The study is concerned with the development of a two-step hybrid linear multi-step method for direct solution of general second order initial value problem. Power series is adopted as the basis function, and the differential system arising from it are collocated at all grid and off-grid points, while the approximate solution is interpolated at the selected points. The scheme developed is tested for consistency and zero stability, and it was found to be convergent.The scheme is expanded term by term by Taylor series approach, also the efficiency of the methods is tested on some test problems, and the accuracy is compared with some existing result in literature.

Key Words: linear multi-step; approximate solution; second order; initial value problem; consistency; zero stability; grid and off-grid points; power series; basis function.

2010 AMS Subject Classification: 34A12.

1. INTRODUCTION

The use of numerical methods in obtaining approximate solutions to differential equations cannot be overemphasized. This is because, the exaxt solution to some differential equations utilized in real life situations are difficut to arrive at. And when their exact solutions cannot be established,

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Received July 24, 2020

the researcher is left with the option of obtaining an approximate via numerical approaches to be able to breakeven in hs/her studies. One of such researchers is [1] who utilised numerical approximation technique to examine an Unsteady 2-D Compressible Inviscid Flow with Heat Transfer with Slip Boundary Conditions Using MacCormack Technique.

Hence, in this work, we considered the numerical solution of initial value problem of the form:

$$y'' = f(x, y, y'); y(a) = y(0); y'(a) = \gamma \tag{1}$$

In practice, higher order ordinary differential equations of the form

$y^{n} = f(x, y', y'',..., y^{n-1})$ are solved by reducing them to systems of first order differential equation of the form:

$$y' = f(x, y), y(a) = 0, f \in [a,b], x, y \in \mathbb{R}^{n}$$

before an approximate method is applied to solve the resulting equations as widely discussed by [2] and [3]. This method of reduction is very popular, however it is treated to be insignificant due to the increased dimension of the resulting first order after reduction; time wastage; computational burden and cost of implementation. Furthermore, the approach does not utilize additional information associated with the specific ordinary differential equation, and consequently, the oscillatory nature of the solution of the differential equation is always neglected. Thus, it would be more efficient to improve on the numerical method so that higher order ordinary differential equations could be solved without having to reduce to systems of first order.The principle was suggested by [4], [5], [6]. Actually, considerable attention has been devoted to solving higher order ordinary differential equation directly without reduction for instance: methods of linear multistep method (LMM) was considered by [7], [8], [9] and [10]. Subsequently, LMM was independently proposed by [11] and [12] in the predictor-corrector mode, based on collocation method.These authors proposed LMM with continuous coefficients where they adopted Taylor series algorithm to supply the starting values. Also,some notable scholars improved on the predictor-corrector method for solving higher order ordinary differential equations for instance: [13], and [14] proposed five-step and four-step methods respectively.They adopt a continuous LMM to obtain finite difference method, applied as a block for the direct solution of the form. [15] adopted a method of collocation and interpolation to develop a continuous LMM which is evaluated at different grid points to give discrete methods to generate independent solution. Others that adopted block methods include [16], [14]. One of the advantages is that it provides direct solution of implicit LMM without developing separate predictors.

Although some of the aforementioned authors have made use of Taylor series, but little has been said with the use of Taylor series as the major method of implementation. So, Our idea is to use Taylor series algorithm to evaluate $y_{n+j}, y'_{n+j}, j = 1,2$ and $y_{n+v}, y'_{n+v}, v = \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{3}{2}, ...$ and calculate f', f'' by the use partial derivative technique. Thus, two-step hybrid methods in the Taylor series mode are developed to solve second order ordinary differential equations directly.

Oganisation of the paper:

The preliminary section captures the research topic, abstract and keywords. But section 1 of the study discuses the general background to the study, justification and a few fndings from the existing literature. Meanwhile, section 2 addreses the basic materials and methods like the derivation of the research numerical method and the general analysis of the Properties of the Scheme (i.e., Order of Accuracy of the Method, Consistency of the Scheme, Zero Stability and Region of absolute stability of the method). Section 3 handled Results/ Numerical Experiments. And finally, section 4 concludes the study followed by the references.

2. MATERIALS AND METHODS

2.1 Derivation of the method

In this section, power series is considered as an approximate solution to the general second order problems:

$$y'' = f(x, y, y'); y(a) = y(0); y'(a) = \gamma \tag{2}$$

And the series is of the form:

$$y(x) = \sum_{j=0}^{2k+1} a_j x^j \tag{3}$$

The first and second derivative of (3) are respectively given as:

$$y'(x) = \sum_{j=1}^{2k+1} j a_j x^{j-1} \tag{4}$$

$$y''(x) = \sum_{j=2}^{2k+1} j(j-1) a_j x^{j-2} \tag{5}$$

Combining (2) and (5), we generate the differential system

$$\sum_{j=2}^{2k+1} j(j-1) a_j x^{j-2} = f(x, y, y'), \tag{6}$$

we develop the hybrid scheme using (3) and (5) as interpolation and collocation equations in this work.

Collocating (6) at selected grid and off-grid points, $x = x_{n+1}, 0 \le i \le 2$ and interpolating (3) at selected grid and off-grid points, it results into a system of equations:

$$\sum_{j=2}^{2k+1} j(j-1) a_j x^{j-2} = f_{n+i}, 0 \le i \le 2 \tag{7}$$

$$\sum_{j=2}^{2k+1} a_j x^j = y_{n+i}, 0 \le i \le 2 \tag{8}$$

It implies

$$a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + a_4 x_n^4 + a_5 x_n^5 + a_6 x_n^6 = y_n \tag{9}$$

$$a_0 + a_1 x_{n+1} + a_2 x_{n+1}^2 + a_3 x_{n+1}^3 + a_4 x_{n+1}^4 + a_5 x_{n+1}^5 + a_6 x_{n+1}^6 = y_{n+1} \tag{10}$$

$$2a_2 + 6a_3 x_n + 12a_4 x_n^2 + 20a_5 x_n^3 + 30a_6 x_n^4 = f_n \tag{11}$$

$$2a_2 + 6a_3 x_{n+\frac{1}{2}} + 12a_4 x^2_{n+\frac{1}{2}} + 20a_5 x^3_{n+\frac{1}{2}} + 30a_6 x^4_{n+\frac{1}{2}} = f_{n+\frac{1}{2}} \tag{12}$$

$$2a_2 + 6a_3x_{n+1} + 12a_4x_{n+1}^2 + 20a_5x_{n+1}^3 + 30a_6x_{n+1}^4 = f_{n+1} \tag{13}$$

$$2a_2 + 6a_3x_{n+\frac{3}{2}} + 12a_4x_{n+\frac{3}{2}}^2 + 20a_5x_{n+\frac{3}{2}}^3 + 30a_6x_{n+\frac{3}{2}}^4 = f_{n+\frac{3}{2}} \tag{14}$$

$$2a_2 + 6a_3x_{n+2} + 12a_4x_{n+2}^2 + 20a_5x_{n+2}^3 + 30a_6x_{n+2}^4 = f_{n+2} \tag{15}$$

Writing these system of equations in matrix form:

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 \\ 0 & 0 & 2 & 6x_{n+1/2} & 12x_{n+1/2}^2 & 20x_{n+1/2}^3 & 30x_{n+1/2}^4 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 \\ 0 & 0 & 2 & 6x_{n+3/2} & 12x_{n+3/2}^2 & 20x_{n+3/2}^3 & 30x_{n+3/2}^4 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+1} \\ f_n \\ f_{n+1/2} \\ f_{n+1} \\ f_{n+3/2} \\ f_{n+2} \end{pmatrix} \tag{16}$$

Using Gaussian elimination method, the unknown coefficients $a_j's$ can be obtained. Putting $a_j's$ back into (3) gives a method of the form:

$$y(x) = \sum_{j=0}^{k} \alpha_j(x) y_{n+j} + \sum_{j=0}^{k} \beta_j(x) f_{n+j}, \tag{17}$$

where $k = 2$ and $f_{n+j} = f(x_{n+j}, y_{n+j}, y'_{n+j}), 0 \le j \le 2$

The coefficients $\alpha_{i's}(t)$, $\beta_{j's}(t)$ are continuous coefficients obtained using the transformation $t = \frac{1}{h}(x - x_{n+k-1})$, $t \in (0,1]$

$$\frac{dt}{dx} = \frac{1}{h}.$$

Then simplifying the continuous $\alpha_{j's}$, $\beta_{j's}$ in (17)

$$\left.\begin{aligned}
\alpha_0 &= -t\\
\alpha_1 &= 1+t\\
\beta_0 &= \frac{-t^4}{72}+\frac{t^6}{45}+\frac{t}{72}+\frac{t^3}{36}-\frac{t^5}{30}\\
\beta_{\frac{1}{2}} &= \frac{2t^4}{9}+\frac{t^5}{15}-\frac{2t^3}{9}-\frac{4t^6}{45}+\frac{13t^5}{45}\\
\beta_1 &= \frac{-5t^4}{12}+\frac{2t^6}{15}+\frac{t^2}{2}+\frac{13t}{60}\\
\beta_{\frac{3}{2}} &= \frac{2t^4}{9}-\frac{t^5}{15}-\frac{4t^6}{45}+\frac{2t^3}{9}-\frac{t}{45}\\
\beta_2 &= \frac{t^5}{30}+\frac{t^6}{45}+\frac{t}{360}-\frac{3t^3}{36}-\frac{t^4}{72}
\end{aligned}\right\} \tag{18}$$

Differentiating equation (18) with respect to $'t'$ gives

$$\left.\begin{aligned}
\alpha_0' &= -1\\
\alpha_1' &= 1\\
\beta_0' &= \frac{-4t^3}{72}+\frac{6t^5}{45}+\frac{1}{72}+\frac{3t^2}{36}-\frac{5t^4}{30}\\
\beta_{\frac{1}{2}}' &= \frac{8t^7}{9}+\frac{5t^4}{15}-\frac{6t^2}{9}-\frac{24t^5}{45}+\frac{13}{45}\\
\beta_1' &= \frac{-20t^3}{12}+\frac{12t^5}{15}+\frac{2t}{2}+\frac{13}{20}\\
\beta_{\frac{3}{2}}' &= \frac{8t^3}{9}-\frac{5t^4}{15}-\frac{24t^5}{45}+\frac{6t^2}{9}-\frac{1}{45}\\
\beta_2' &= \frac{5t^4}{30}+\frac{6t^5}{45}+\frac{1}{360}-\frac{3t^2}{36}-\frac{4t^3}{72}
\end{aligned}\right\} \tag{19}$$

Simplifying equation (22), we have:

$$\left.\begin{aligned}
\beta_0' &= \frac{1}{360}\{-20t^3+48t^5+5+30t^2-60t^4\}\\
\beta_{\frac{1}{2}}' &= \frac{1}{45}\{40t^3+15t^4-30t^2-24t^5+13\}\\
\beta_1' &= \frac{1}{60}\{-100t^3+48t^5+60t+13\}\\
\beta_{\frac{3}{2}}' &= \frac{1}{45}\{40t^3-15t^4-24t^5+30t^2-1\}\\
\beta_2' &= \frac{1}{360}\{60t^4-48t^5-30t^2-20t^3+1\}
\end{aligned}\right\} \tag{20}$$

Putting $t=1$, which implies evaluating x at x_{n+2} gives

$$y_{n+2} - 2y_{n+1} + y_n = \frac{h^2}{60}\{ f_{n+2} + 16 f_{n+\frac{3}{2}} + 26 f_{n+1} + 16 f_{n+\frac{1}{2}} + f_n \} \quad (21)$$

with the order $C_6 = 6$, error constant $C_8 = -0.084082$, and interval of absolute stability $X(\Theta) = (-10.0,0)$

With first derivative :

$$y_{n+2}' - y_{n+1} + y_n = \frac{h^2}{360}\{59 f_{n+2} + 240 f_{n+\frac{3}{2}} + 126 f_{n+1} + 112 f_{n+\frac{1}{2}} + 3 f_n\} \quad (22)$$

Implementation of the method using Taylor series algorithm to evaluate

$y_{n+j}, y_{n+j}', y_{n+v}, y_{n+v}', f_{n+v}, f_{n+j}$,

where,

$j's = 1,2$ and $v's = \frac{1}{2}, \frac{3}{2}$ and,

$$f_{n+v} = f(x_{n+v}, y_{n+v}, y_{n+v}'),$$

such that

$$y_{n+v} = y_n + vhy_n' + \frac{(vh)^2}{2!} f_n + \frac{(vh)^3}{3!} f_n' + \frac{(vh)^4}{4!} f_n'' + ... \quad (23)$$

and,

$$y_{n+v}' = y_n' + vhf_n + \frac{(vh)^2}{2!} f_n' + \frac{(vh)^3}{3!} f_n'' + \frac{(vh)^4}{4!} f_n''' + ... \quad (24)$$

Also,

$$f_{n+j} = y''(x_n + jh) = f_n + jhf_n' + \frac{(jh)^2}{2!} f_n'' + ... \quad (25)$$

From, $f_n = f(x_n, y_n, y_n')$ $f^{(i)} = f^{(i)}(x_n, y_n, y_n'), i = 1,2$

Finding the partial derivative $f', f'',...$ as follows

$$\frac{df}{dx} = f' = \frac{\delta f}{\delta x} + \frac{\delta f}{\delta x} y' + \frac{\delta f}{\delta y'} f \quad (26)$$

$$f'' = \frac{d^2 f}{dx^2} = 2(Ay' + Bf) + Cfy' + D + E, \tag{27}$$

where,

$$A = \frac{\partial^2 f}{\partial x \partial y} + f\,\frac{\partial^2 f}{\partial y \partial y'} \tag{28}$$

$$B = \frac{\partial^2 f}{\partial x \partial y'} \tag{29}$$

$$C = \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + f\,\frac{\partial f}{\partial y'} \tag{30}$$

$$D = \frac{\partial^2 f}{\partial x^2} + (y')^2\frac{\partial^2 f}{\partial y^2} + f^2\,\frac{\partial^2 f}{\partial (y')^2} \tag{31}$$

$$E = f\,\frac{\partial f}{\partial y} \tag{32}$$

2.2 Analysis of the Properties of the Scheme

We shall consider the analysis of the basic properties of our methods which includes the order, the region of absolute stability and the zero stability of the methods.

Order of Accuracy of the Method

The local truncation error with $k-$step linear multistep method which is in line with the work of [16], is taken to be linear difference operator ℓ defined by

$$\ell[y(x);h] = \sum_{j=0}^{k}[\alpha_j y(x_n + j) - h\beta_j y(x_n + j)] \tag{33}$$

Thus,Expanding (21) as Taylor series about point x and comparing coefficients of h^k,

hence the method is of order $p = 6$ with error constant $C_{q+2} = -0.084082$

$$L[y(x),h] = C_0 y(x_n) + C_1 y'(x_n) + C_2 y''(x_n) + ... + C_p y^p(x_n), \tag{34}$$

where $C_p, p = 0,1,...,$ are the constant coefficients given as:

$$\left.\begin{aligned} C_0 &= \sum_{j=0}^{k}\alpha_j \\ C_1 &= \sum_{j=0}^{k} j\alpha_j \\ &and \\ C_p &= \frac{1}{p!}[\sum_{j=0}^{k} j\alpha_j - p(p-1)(\sum_{j=0}^{k} j^{p-1}\beta_j + \sum_{j=0}^{k} q^{p-1}\beta_{qj})] \end{aligned}\right\} \tag{35}$$

In line with [3], $k-$ step, linear multistep (21) has order p if $C_0 = C_1 = ... = C_{p-1} = C_p$ and $C_{p+1} \neq 0$,where, $C_{p+1} \neq 0$ is the error constant. Subjecting our schemes to equations 35, it is therefore established that linear multistep scheme is of order $p = 6$, relatively small error constant -0.084082.

Consistency of the Scheme

A linear multistep method is consistent if the following conditions are satisfied:

1. The order $p \geq 1$.
2. $p(1) = 0, p'(1) = \sigma(1)$.
3. $\sum_{j=0}^{k}\alpha_j = 0$.
4. $\sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k}\beta_j$.

Zero Stability of the Method

Equation (26) has its first characteristic polynomial to be :

$$\rho(r) = r^2 - 2r + 1 \tag{36}$$

Hence, the method is zero stable since the polynomial have roots $r = 1$ twice

Region of absolute stability of the method

In order to establish the region of absolute stability, we apply the boundary locus method as in [18] and which from the method implies that

$$h(\theta) = \frac{\rho(r)}{\delta(r)} \tag{37}$$

Where,

$$r = e^{i\theta} = cos\theta + isin\theta \tag{38}$$

From scheme (25), we have:

$$\rho(r) = r^2 - 2r + 1 \tag{39}$$

$$\sigma(r) = \frac{1}{60}[r^2 + 16r^{3/2} + 26r + 16r^{1/2} + 1] \tag{40}$$

so that;

$$h(\theta) = \frac{\rho(e^{i\theta})}{\delta(e^{i\theta})} \tag{41}$$

$$= \frac{60[r^2 - 2r + 1]}{r^2 + 16r^{3/2} + 26r + 16r^{1/2} + 1} \tag{42}$$

$$= \frac{670[cos2\theta + isin2\theta - 2cos\theta - 2isin\theta + 1]}{[47cos2\theta + 47isin2\theta + 810cos\frac{5\theta}{3} + 810isin\frac{5\theta}{3} + 1377cos\frac{4\theta}{3} + 1377isin\frac{4\theta}{3} + 2252cos\theta + A]} \tag{43}$$

Where from equation (43)

$$A = 2252isin\theta + 1377cos\frac{2\theta}{3} + 1377isin\frac{2\theta}{3} + 810cos\frac{2\theta}{3} + 810isin\frac{2\theta}{3} + 47$$

Meanwhile, considering the values of θ for $0 \le \theta \le 180$ at intervals of 30^{θ}, gives the region of absolute stability to be

$(-10,000,0)$

3. RESULTS/ NUMERICAL EXPERIMENTS

We test the accuracy of the proposed scheme on some numerical problems, and the results are compared with existing methods.

Problem 1:

$$y'' = x(y')^2, y(0) = 1, y'(0) = 0.5, h = \frac{0.1}{32} \tag{44}$$

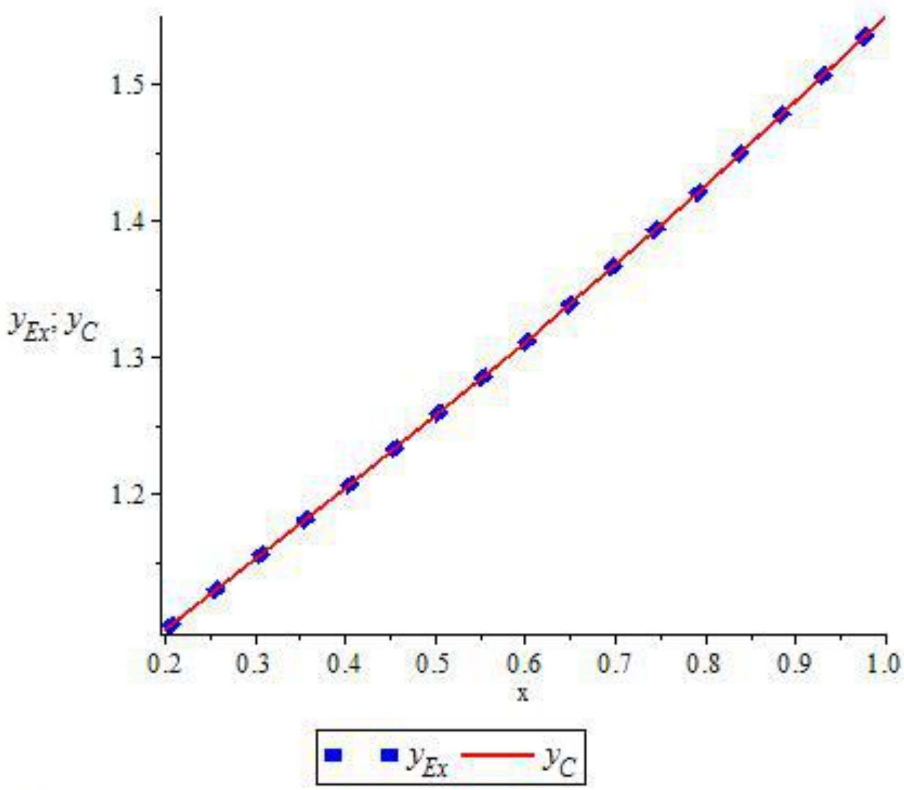
Exact solution

$$y(x) = 1 + \frac{1}{2} log(\frac{2+x}{2-x})$$

The numerical results of the problem is shown in tables 1, compared with the error in [6] which is of order 6.

Table 1: Results and errors for problem (42)

(x)	YEX	YC	ERRNew
0.2	1.100335347731075300	1.100335347731075300	$0.00000e+000$
0.4	1.20273255405481600	1.20273255405481700	1.110223×10^{-15}
0.6	1.309519604203111900	1.309519604203113800	2.886580×10^{-15}
0.8	1.423648930193603500	1.423648930193641200	4.751010×10^{-15}
1.0	1.549306144334058600	1.549306144334058800	2.310063×10^{-14}



1st Problem Graphical comparison between Exact solution & the method

Note: Y_{EX} = Y-exact, Y_C = Y-computed, ERRNew = Error in new method

Problem 2:

$$y'' = (y'), y(0) = 0, y'(0) = -1, h = 0.01 \tag{45}$$

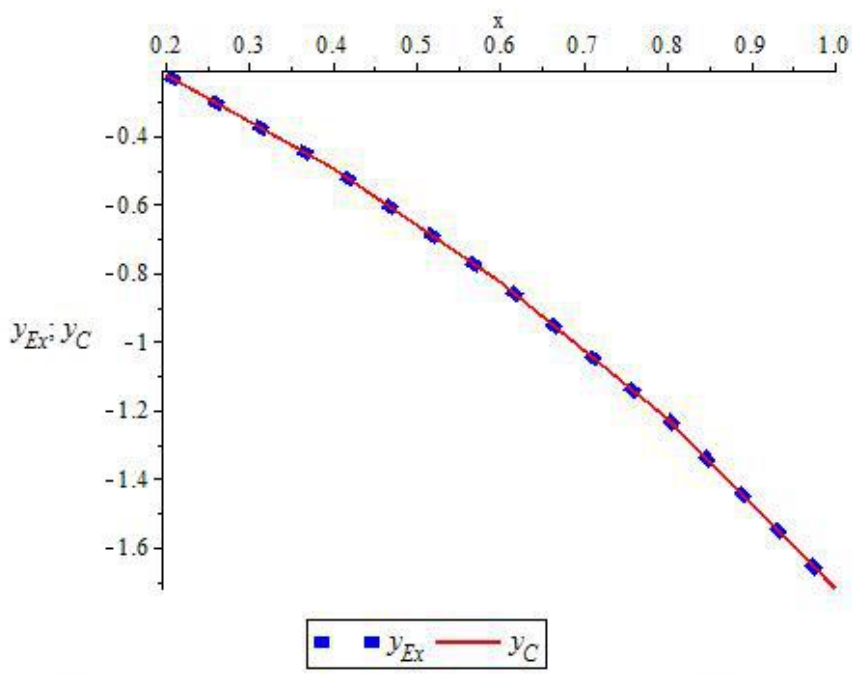
Exact solution

$y(x) = 1 - e^x$

The numerical results of the problem is shown in table 2, compared with the error in [1] of order 6.

Table 2: Results and errors for problem (43)

(x)	YEX	YC	ERRNew
0.2	-0.221402758000000000	-0.2214027558070370380	$8.979947\times^{-11}$
0.4	-0.491824697641270350	-0.491824696651586790	9.8968836×10^{-10}
0.6	-0.822118800390509110	-0.822118797459521540	2.930988×10^{-9}
0.8	-1.2255409228492467900	-1.225540922161721500	6.330746×10^{-9}
1.0	-1.718281828459045500	-1.7182818167194335800	1.173961×10^{-8}



2nd Problem Graphical comparison between Exact solution & the method

Note: Y_{EX} = Yexact, Y_C = Ycomputed, ERRNew = Error in new method

4. DISCUSION/CONCLUSION

A Linear Multistep method which implements a Taylor's series algorithm is developed for the direct solution of general second order initial value problems of ordinary differential equations without reduction to systems of first order differential equation. In this study also, the derivatives of continuous scheme to any order was computed, implementing Taylor's series algorithm.

The accuracy of the method developed was tested with two test problems, and their corresponding results were compared with those of Awoyemi (2005) in reference [6] and Adesanya (2011) in reference [1] each of order 6.

Moreovewr, the outcome of the comparison of the method to the results of exact solution using the examples of [12] and [17] showed that, the method is efficient and thus recommends it for similar purpose(s) in research.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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