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RUELLE-TAKENS CHAOS IN NON-AUTONOMOUS DYNAMICAL SYSTEMS

XIAOQIAN SONG*, JINKUI LIU, LIANGWEI WANG

College of Mathematics and Statistics, Chongqing Three-Gorges University, wanzhou, 404100, China

Abstract: The paper is devoted to a study of Ruelle-Takens chaos of non-autonomous discrete systems defined by a sequence of continuous maps $F=\{f_k\}_{k=1}^{\infty}$ acting on a metric space. First we introduce the notion of Ruelle-Takens chaos, defined in a way suitable for non-autonomous discrete systems. We then discuss the chaotic property of the product system; also the relations of two conjugate systems are researched.

Keywords: Non-autonomous discrete system; Chaos; Transitivity; Mixing.

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1. Introduction

The development of the theory of topological dynamics began in the earlier part of the last century. It focused, in particular, on problems related to autonomous discrete dynamical systems given by the pair (X, f), where X is a topological space and f a continuous map of X. The crucial problem was the study of properties of all orbits of all points in the space state X. For $x \in X$, the orbit of x by f is the sequence $(f^n(x))_{n=0}^{\infty}$, where $f^n = f(f^{n-1})$ for all $n \geq 1$ and $f^0 = id$ (identity on X). We now consider the situation in which the map describing the evolution of the dynamics is itself allowed to change with time. This admits the following formulation. Given a

*Corresponding author

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metric space X and a sequence of continuous self-maps $F=\{f_k\}_{k=1}^{\infty}$ from X into itself, the pair (X,F) will be called a non-autonomous discrete system where the orbit of a point $x_0 \in X$ is described by the sequence

$$x_1 = f_1(x_0), x_2 = f_2(x_1), \dots, x_{n+1} = f_{n+1}(x_n), \dots, n = 0,1,2,\dots,$$

we will use the notation $F_k = f_k \circ f_{k-1} \circ \cdots \circ f_1$, which was introduced in [1]. When all of the maps are the same, i.e. $f_k = f$ for $k = 1,2,\cdots$, then we have a autonomous discrete system or, simply, a discrete dynamical system.

While the study of non-autonomous discrete systems is usually more complex and demanding than the same studies in the setting of autonomous systems, such studies became more popular each year. In Ref. [2], Kolyada and Snoha defined the topological entropy of a non-autonomous discrete system, using the technique of open covers of X when the base space is compact as in the original paper [3]. In the case that X is metric, they also use separated and spanning sets as in [4]. In addition, Tian and Chen in Ref. [1] introduced the periodic points, transitivity, sensitivity and Devaney chaos of a time-varying map (i.e. sequence of maps) in a metric space. Mouron in Ref. [5] discussed the forward entropy and backwards entropy on non-autonomous interval maps, also the relation between these two entropies and the inverse limit space was considered. Huang et al. in Ref. [6] defined and studied pre-image entropy for the non-autonomous discrete systems given by a sequence $\{f_i\}_{i=1}^{\infty}$ of continuous self-maps of a compact topological space. Recently, Balibrea and Oprocha considered such properties as chaos in the sense of Li and Yorke, topological weak mixing and topological entropy for the non-autonomous discrete systems. And they showed that in general the dynamics of non-autonomous discrete systems is much richer and quite different than what is expected from the case of single map (see [7]). Zhu et al. studied the topological entropy and measure-theoretic entropy for non-autonomous dynamical systems; moreover, the bounds of them for several particular non-autonomous systems were obtained (see [8]). Along the same line, Ruelle-Takens chaos in non-autonomous systems is given in this paper, and the chaotic property of the product system and the relations of two conjugate systems are discussed.

2. Preliminaries

Throughout this paper, (X, F) represents a non-autonomous discrete system as illustrated above, where X is a metric space (not-necessarily compact) with metric d. The following two definitions can be found in [1].

Definition 2.1. *A sequence of maps* $F = \{f_k\}_{k=1}^{\infty}$ *is called to be topologically transitive if for any two nonempty open subsets* U *and* V *in* X, *there is a positive integer* k *such that* $F_k(U) \cap V \neq \varnothing$.

Definition 2.2 *A sequence of maps* $F = \{f_k\}_{k=1}^{\infty}$ *is called to be topologically mixing if for any two nonempty open subsets* U *and* V *in* X, *there is a positive integer* N *such that for every* $k > N, F_k(U) \cap V \neq \varnothing$.

Definition 2.3. *A sequence of maps* $F = \{f_k\}_{k=1}^{\infty}$ *is called to be sensitively dependent on initial conditions, if there is* $\delta > 0$ *such that for any point* $x \in X$ *and any neighborhood* U_x *of* x, *there exists* $y \in U_x$ *and a positive integer* k *such that* $d(F_k(x), F_k(y)) > \delta$.

Definition 2.4. *Let* h *be a map from a metric space* (X, d_X) *into a metric space* (Y, d_Y). *Then* h *is said to be a homeomorphism if it is one-to-one and onto, and both* h *and* h^{-1} *are continuous.*

Furthermore, if h is a homeomorphism, h is uniformly continuous (on X) and h^{-1} is uniformly continuous (on Y), then h is said to be a uniformly homeomorphism.

The definition of Ruelle-takens chaos of a sequence of maps in a metric space is given as follows.

Definition 2.5. *A sequence of maps* $F=\{f_k\}_{k=1}^{\infty}$ *is called to be chaotic in the sense of Ruelle-Takens if,*

i) F *is topologically transitive;*

ii) F *has sensitive dependence on initial conditions.*

Definition 2.6. *Let* (X,d_X) *and* (Y,d_Y) *be metric spaces,* $F=\{f_k\}_{k=1}^{\infty}$ *and* $G=\{g_k\}_{k=1}^{\infty}$ *be two sequences of maps on* X *and* Y *, respectively, and* $h:X\to Y$ *be a homeomorphism. If for any* $k\geq 1$, $g_k\circ h(x)=h\circ f_k(x), x\in X$ *, then* F *and* G *are said to be conjugate or* $h-$ *conjugate. In particularly, if* $h:X\to Y$ *is a uniformly homeomorphism, then* F *and* G *are said to be uniformly conjugate.*

Let (X,F) and (Y,G) be non-autonomous discrete systems and $X\times Y$ be product space with metric $d''((x_1,y_1),(x_2,y_2))=\max\{d_X(x_1,x_2),d_Y(y_1,y_2)\}$, where d_X and d_Y are metrics, $F=\{f_k\}_{k=1}^{\infty}$ and $G=\{g_k\}_{k=1}^{\infty}$ are sequence of maps on X and Y , respectively. Let the sequence of maps $F\times G=\{f_i\times g_i\}_{i=1}^{\infty}$ be defined by $\forall(x,y)\in X\times Y$, $(f_i\times g_i)(x,y)=(f_i(x),g_i(y)), i=1,2,\cdots$. And it is verified that

$$(F\times G)^i(x,y)=(f_i\times g_i)\circ(f_{i-1}\times g_{i-1})\circ\cdots\circ(f_1\times g_1)(x,y)=(F_i\times G_i)(x,y).$$

Now we obtain the product system $(X\times Y,F\times G)$ with metric d'' and the sequence of continuous maps $F\times G$.

3. Chaos in product system

Lemma 3.1. *Let* X *and* Y *be metric spaces with metrics* d_X *and* d_Y *, respectively,* $F=\{f_k\}_{k=1}^{\infty}:X\to X$ *and* $G=\{g_k\}_{k=1}^{\infty}:Y\to Y$ *be sequences of maps.*

i) If F *or* G *is sensitively dependent on initial conditions, then* $F\times G$ *is sensitively dependent on initial conditions.*

ii) If $F\times G$ *is sensitively dependent on initial conditions, then both* F *and* G *are sensitively dependent on initial conditions.*

Proof. i) Let us assume that F is sensitively dependent on initial conditions. Then for any $w=(x,y)\in X\times Y$ and any neighborhood W of w, there exist open neighborhoods U of x and V of y such that $U\times V\subset W$. Since F is sensitively dependent on initial conditions, let $\delta>0$ be the sensitive constant of F, so there exists $x'\in U_x$ and a positive integer k such that $d_X(F_k(x),F_k(x'))>\delta$. Obviously, $w'=(x',y)\in U\times V\subset W$, and

$$d((F\times G)^k(w),(F\times G)^k(w'))=d((F_k(x),G_k(y)),(F_k(x'),G_k(y)))$$

$$=\max\{d_X(F_k(x),G_k(x')),d_Y(F_k(y),G_k(y))\}>\delta.$$

This implies $F\times G$ is sensitively dependent on initial conditions.

ii) Let us assume that both F and G are not sensitively dependent on initial conditions. This means that for any $\delta>0$, there exist $x\in X$ and an open neighborhood U_x of x such that the inequality $d_X(F_k(x),G_k(x'))<\delta$ holds for any $x'\in U_x$ and any positive integer k. Similarly, there exist $y\in Y$ and an open neighborhood V_y of y such that the inequality $d_Y(F_k(y),G_k(y'))<\delta$ holds for any $y'\in V_y$ and any positive integer k. Thus

$$d((F\times G)^k(x,y),(F\times G)^k(x',y'))=d((F_k(x),G_k(y)),(F_k(x'),G_k(y)))$$

$$=\max\{d_X(F_k(x),G_k(x')),d_Y(F_k(y),G_k(y))\}<\delta$$

for any $(x',y')\in U_x\times V_y$, therefore $F\times G$ is not sensitively dependent on initial conditions, this contradict the hypothesis.

Corollary 3.1. *Let* X_i $(1\le i\le n)$ *be metric spaces with metrics* $d_i(1\le i\le n)$, *and* $F^{(i)}=\{f_k^{(i)}\}_{k=1}^{\infty}:X_i\to X_i$ *be sequences of maps.*

i) *If there is* $1\le i\le n$ *for which* $F^{(i)}=\{f_k^{(i)}\}_{k=1}^{\infty}$ *is sensitively dependent on initial conditions. then* $\prod_{i=1}^{\infty}F^{(i)}$ *is sensitively dependent on initial conditions.*

ii) *If* $\prod_{i=1}^{\infty} F^{(i)}$ *is sensitively dependent on initial conditions, then each of* $F^{(i)}$ $(1 \le i \le n)$ *is sensitively dependent on initial conditions.*

Lemma 3.2. *Let* $F = \{f_k\}_{k=1}^{\infty} : X \to X$ *and* $G = \{g_k\}_{k=1}^{\infty} : Y \to Y$ *be sequences of maps, then* $F \times G$ *is topologically transitive on* $X \times Y$ *implies that* F *and* G *are both topologically transitive on* X *and* Y*, respectively.*

Proof. For any non-empty open sets U and V in X , choose an open set W in Y, then the sets $U \times W$ and $V \times W$ are open in $X \times Y$, since $F \times G$ is topologically transitive on $X \times Y$, there is $k > 0$ for which $(F \times G)^k (U \times W) \cap (V \times W) \ne \emptyset$, from the equalities

$$(F \times G)^k (U \times W) \cap (V \times W) = (F_k \times G_k)(U \times W) \cap (V \times W)$$

$$= (F_k(U) \times G_k(W)) \cap (V \times W) = (F_k(U) \cap V) \times (G_k(W) \cap W),$$

it follows that $(F_k(U) \cap W) \times (G_k(V) \cap W) \ne \emptyset$, so $F_k(U) \cap V \ne \emptyset$, this implies F is topologically transitive . The transitivity of G can be showed similarly. This ends the proof.

Corollary 3.2. *Let* X_i $(1 \le i \le n)$ *be metric spaces with metrics* $d_i (1 \le i \le n)$ *, and* $F^{(i)} = \{f_k^{(i)}\}_{k=1}^{\infty} : X_i \to X_i$ *be sequences of maps, if* $\prod_{i=1}^{\infty} F^{(i)}$ *is topologically transitive, then all sequences of maps* $F^{(i)} (1 \le i \le n)$ *are topologically transitive.*

However the converse is not true, that is to say, the product of two topologically transitive sequences of maps need not be topologically transitive. We give the following example to illustrate this problem.

Example 1 *Let* $F = \{f, I, f, I, \cdots\}$ *, where* I *denotes the identity map and* $f : [0,2] \to [0,2]$ *be defined as follows:*

$$f(x) = \begin{cases} 2x + 1, 0 \le x \le 1/2 \\ -2x + 3, 1/2 \le x \le 1 \\ -x + 2, 1 \le x \le 2. \end{cases}$$

Then F *is topologically transitive, but* $F\times F$ *don't have the same property*.

Proof. Since $f^2\big|_{[0,1]}$ and $f^2\big|_{[1,2]}$ are both tent maps, it's enough to see that f is topologically transitive. By the construction of sequence of maps F, it is not hard to see the transitivity of F. Actually, for any non-empty open sets U and V in X, as f is topologically transitive, there is a positive integer n_0 for which $f^{n_0}(U)\cap V\neq\varnothing$, therefore $F_{2n_0}(U)\cap V=f^{n_0}(U)\cap V\neq\varnothing$, which shows F is topologically transitive.

Let $U=(0,1)\times(0,1)$, $V=(0,1)\times(1,2)$ then

$$(F_1\times F_1)(U)=(f\times f)(U)=(f\times f)(0,1)\times(0,1)=(1,2)\times(1,2),$$

$$(F_2\times F_2)(U)=(1,2)\times(1,2),$$

$$(F_3\times F_3)(U)=(0,1)\times(0,1)=U,\cdots$$

Therefore

$$(F_k\times F_k)(U)=U, k=2n-1 \text{ (for } n \text{ even)},$$

$$\text{or}\quad (F_k\times F_k)(U)=(1,2)\times(1,2), k=2n-1 \text{ (for } n \text{ odd)}.$$

So the equality $(F_k\times F_k)(U)\cap V=\varnothing$ holds for any k. This illustrate that $F\times F$ is not topologically transitive.

By lemma 3.1 and lemma 3.2, one can draw the following theorem.

Theorem 3.1. *Let* $F\times G$ *is Ruelle-takens chaotic, where* $F=\{f_k\}_{k=1}^{\infty}:X\to X$ *and* $G=\{g_k\}_{k=1}^{\infty}:Y\to Y$ *are sequences of maps, then both* F *and* G *are Ruelle-takens chaotic.*

The above results can be extended to the product of finite sequences. But the reverse may often not establish since the product of two transitive sequences doesn't keep the same property (see example 1). Therefore the following question arises:

Question. *What conditions should be satisfied by subsystems* F *and* G *to guarantee the Ruelle-takens chaos of product system? (in non-autonomous case)*

Lemma 3.3. *Let* $F=\{f_k\}_{k=1}^{\infty}:X\to X$ *and* $G=\{g_k\}_{k=1}^{\infty}:Y\to Y$ *be sequences of maps,* *and let us assume that F and G are both topologically mixing on X and Y respectively, then the product $F \times G$ is topologically mixing on $X \times Y$.*

Proof. For any open sets U and V in $X \times Y$, there exist open sets $U_1, V_1 \subset X$ and $U_1, V_1 \subset X$ such that $U_1 \times U_2 \subset U, V_1 \times V_2 \subset V$. Since F and G are both topologically mixing on X and Y respectively, there exist N_1, N_2 such that $F_n(U_1) \cap V_1 \neq \varnothing$ for $n \geq N_1$ and $G_n(U_2) \cap V_2 \neq \varnothing$ for $n \geq N_2$. Let $N = \max\{N_1, N_2\}$, then for any $n \geq N$, we have

$$(F \times G)^N (U) \cap V \supset (F \times G)^N (U_1 \times U_2) \cap (V_1 \times V_2)$$

$$= (F_N \times G_N)(U_1 \times U_2) \cap (V_1 \times V_2) = (F_N(U_1) \times G_N(U_2)) \cap (V_1 \times V_2)$$

$$= (F_N(U_1) \cap V_1) \times (G_N(U_2) \cap V_2) \neq \varnothing,$$

which implies $F \times G$ is topologically mixing on $X \times Y$.

Obviously, topologically mixing $\Rightarrow$ topologically transitive, so we get the following theorem.

Theorem 3.2. *If $F = \{f_k\}_{k=1}^{\infty} : X \to X$ and $G = \{g_k\}_{k=1}^{\infty} : Y \to Y$ are both Ruelle-Takens chaotic and topologically mixing, then $F \times G$ is Ruelle-Takens chaotic.*

4. Chaos in conjugation

Because any definition of chaos must face the obvious question: is it preserved under topological conjugation? That is to say, if F is chaotic and if we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{F} & X \\ h \downarrow & & \downarrow h \\ Y & \xrightarrow[G]{} & Y \end{array}$$

where Y is another metric space and h is a homeomorphism, then is G necessarily chaotic? Certainly, topologically transitivity is preserved as it is purely topological conditions.

Theorem 4.1. *Let* $F=\{f_k\}_{k=1}^{\infty}$ *and* $G=\{g_k\}_{k=1}^{\infty}$ *be two sequences of maps on* X *and* Y, *respectively. If there exists a homeomorphism* $h:X\to Y$ *such that* $F=\{f_k\}_{k=1}^{\infty}$ *and* $G=\{g_k\}_{k=1}^{\infty}$ *are* $h-$ *conjugate, then* $F=\{f_k\}_{k=1}^{\infty}$ *is topologically transitive on* X *if and only if* $G=\{g_k\}_{k=1}^{\infty}$ *is topologically transitive on* Y.

Proof. *Necessity*. If F is topologically transitive, then it is to prove that G is also topologically transitive.

For any nonempty open sets U and V in Y, as h is continuous, $h^{-1}(U)$ and $h^{-1}(V)$ are nonempty open sets in X. Hence, there exists a positive integer N such that $F_N(h^{-1}(U))\cap h^{-1}(V)\neq\emptyset$, in view of the given conditions,

$$F_N(h^{-1}(U))=f_N\circ f_{N-1}\circ\cdots\circ f_1\circ h^{-1}(U)=f_N\circ f_{N-1}\circ\cdots\circ h^{-1}\circ g_1(U)$$

$$=\cdots=h^{-1}\circ g_N\circ g_{N-1}\circ\cdots\circ g_1(U)=h^{-1}(G_N(U)),$$

thus $h^{-1}(G_N(U))\cap h^{-1}(V)=F_N(h^{-1}(U))\cap h^{-1}(V)\neq\emptyset$ and so $h^{-1}((G_N(U)\cap V)\neq\emptyset$. This implies G is topologically transitive.

Sufficiency. It can be proved similarly to the above proof.

However, sensitivity is a metric property and in general it is not preserved under topological conjugation, as the following example shows.

Example 2. Consider the sequences of maps $F=\{f,I,f,I,\cdots\}$ and $G=\{g,I,g,I,\cdots\}$ on $(1,\infty)$ and $\mathbb{R}^{+}$, equipped with the standard metric, respectively, and let $h:X\to Y$ be defined by $h(x)=\log x, x\in X$, $f(x)=2x, x\in(1,\infty)$, and $g(x)=x+\log 2, x\in\mathbb{R}^{+}$.

One can verify F and G are $h-$ conjugate since the following equations hold $h\circ f(x)=h(2x)=\log 2x=\log 2+\log x$ and $g\circ h(x)=g(\log x)=\log x+\log 2$ for any $x\in X$. Clearly, f has sensitive dependence on initial conditions, a similar approach as in example 1 can be used to illustrate that F is also sensitive. But g is just a translation, therefore G is also a translation according to its construction, and hence is not sensitive for the standard metric on □$^{+}$.

Nevertheless, when the map h is uniformly conjugate or h is just conjugate (not necessarily uniform) and X is a compact metric space, then sensitivity is preserved under two conjugate sequences of maps (see in [1] theorem 3.1).

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