Available online at http://scik.org
Engineering Mathematics Letters, 2 (2013), No. 1, 42-55
ISSN 2049-9337

RELIABILITY EQUIVALENCE FACTOR OF A PARALLEL SYSTEM SUBJECT TO TIMEVARYING FAILURE RATES

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Abstract. The reliability equivalence factors of n-independent and identical component of parallel system such that the failure rates of the system's components are assumed to be time varying failure rate. Three different methods to improve the quality of a system are considered by Råde [5, 6].The mean time to failure of the original and improved system are obtained .Comparison between The mean time to failure of the improved system obtained via different methods used is presented . Numerical studies are presented to Compare the different reliability factors.

Keywords: Non-repairable system ; Weibull distribution; time varying failure rate ; hot duplications; cold duplications; Reduction method

2000 AMS Subject Classication: 60K25

1. Introduction

The concept of reliability equivalence has been introduced by Råde [2]. Råde [3,4] and Sarhan [8] have applied such concept to various reliability system. Three different methods to improve the quality of a system are considered by Råde [5, 6]. He assumed that the reliability function of the system can be improved by:

(i) Improving the quality of one or several components by decreasing their failure rates ,

(ii) Adding a hot component to the system, and

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Received October 2, 2012

(iii) Adding a cold redundant component to the system.

Sarhan [7] has considered more general method to improve the reliability of a system, in these methods, it is assumed that the quality of the

system can be improved according to one of the following four different methods:

(i) Improving the quality of some component by reducing their failure rates by a factor ρ, $0<\rho<1$,

(ii) Assuming hot duplications of some of the system's components

(iii) Assuming cold duplications of some of the system's components

iv) Assuming cold redundant stand by connected with same system components (one for each) via random switches.

Råde has calculated the reliability equivalence factors for a single component and for two independent and identical component, series and parallel system, Sarhan [7,9] has obtained ,the reliability equivalence factor of *n*-independent and identical components series and parallel system. He used the reliability function and mean time to failure (*MTTF*) as characteristics to compare different system design such that the failure rates of the system's components are assumed to be constant.

Our aim in this paper is to derive the reliability equivalence factors of *n*-independent and identical component of parallel system such that the failure rates of the system's components are assumed to be time varying failure rate. and we use, as Sarhan[9], the reliability function and *MTTF* to compare different system designs. We shall make equivalence between

the improved systems obtained according to the reduction method and the hot and cold duplication method based on the value of the reliability function.

2. *n*-Component Parallel System

The system considered here consists of $n-$independent and identical components connected in parallel. Each component has time varying failure rate. That is, the life time of component i, say T_i, is a Weibull distribution random variable with the following reliability function

$$R_i(t) = e^{-(\lambda t)^{\beta}}, \quad t \geq 0,\ \lambda,\ \beta > 0 \tag{1}$$

The reliability of the system can be improved according to one of the following three different methods.

(1) Reducing the failure rates of r, $1 \leq r \leq n$, of the components by the same factor same ρ, $0 < \rho < 1$, this methods is called the reduction methods.

(2) Assuming hot duplications of m, $1 \leq m \leq n$, of the components. This means that each of m of the components is duplicated by a hot redundant standby component. This method will be called the hot duplication method.

(3) Assuming cold duplication of m, $1 \leq m \leq n$, of the components, It means that each of m of the components is duplicated by a cold redundant standby component. This method is called by the cold duplication method.

3. The Reliability Functions and *MTTF's*.

We present the reliability function and the *MTTF's* of the original and improving systems.

3.1. The original system:

The reliability function of the original system which consists of *n*-independent and identical components, denoted by $R(t)$, can be get as :

$$R(t) = 1 - \prod_{i=1}^{n}[1 - R_i(t)] = 1 - [1 - R(t)]^n.$$

Using (1), we have

$$R(t) = 1 - [1 - \exp(-(\lambda t)^{\beta})]^n . \tag{2}$$

Using the binomial expansion, one can write $R(t)$ in the following form

$$R(t) = \sum_{i=1}^{n} \binom{n}{i} (-1)^{i+1} \exp[-i(\lambda t)^{\beta}].$$

Then ,we obtain the system *MTTF* as follows

$$MTTF = \int_0^{\infty} R(t)\, dt = \sum_{i=1}^{n} \binom{n}{i} (-1)^{i+1} \int_0^{\infty} \exp[-i(\lambda t)^{\beta}]\, dt$$

$$= \frac{\Gamma(\frac{1}{\beta}+1)}{\lambda} \sum_{i=1}^{n} \binom{n}{i} (-1)^{i+1} i^{-\frac{1}{\beta}} \quad . \qquad \textbf{(3)}$$

3.2. The reduction method

Assuming that the system is improved by improving $r, 1 \le r \le n$, of its components according to the reduction methods that is, the time varying failure rates of r components are reduced from $\lambda\beta(\lambda t)^{\beta-1}$ to $\rho\lambda\beta(\lambda t)^{\beta-1}$, $0 < \rho < 1$. Let $R_{(r),\rho}(t)$ denote the reliability function of the system improved by reducing the time varying failure rates of *r* of its components by the factor. One can obtain $R_{(r),\rho}(t)$ to be

$$R_{(r),\rho}(t) = 1 - [1 - \exp(-\rho(\lambda t)^{\beta})]^{r} [1 - \exp(-(\lambda t)^{\beta}]^{n-r} \quad . \qquad (4)$$

Using the binomial expansion, then we can write $R_{(r),\rho}(t)$ in the following from

$$R_{(r),\rho}(t) = \sum_{j=0}^{n-r} \sum_{i=1}^{r} \binom{n-r}{j} \binom{n}{i} (-1)^{i+j+1} \exp\{-(j+i\rho)(\lambda t)^{\beta}\}$$

$$+ \sum_{j=1}^{n-r} \binom{n-r}{j} (-1)^{j+1} \exp\{-j(\lambda t)^{\beta}\} \qquad (5)$$

One can obtain the $MTTF_{(r),\rho}$ as follows :

$$MTTF_{(r),\rho} = \frac{\Gamma(\frac{1}{\beta}+1)}{\lambda} \sum_{j=0}^{n-r} \sum_{i=1}^{r} \binom{n-r}{j} \binom{n}{i} (-1)^{i+j+1} \frac{1}{(j+i\rho)^{1/\beta}}$$

$$+\frac{\Gamma(\frac{1}{\beta}+1)}{\lambda}\sum_{j=1}^{n-r}\binom{n-r}{j}(-1)^{j+1}\frac{1}{j^{1/\beta}} \quad (6)$$

It's clear that, the time varying failure rate of r of the system's component by the factor ρ increases the $MTTF$ of the system by the amount $\Delta\ MTTF_{(r),\rho}$ which is given by

$$\Delta\ MTTF_{(r),\rho} = MTTF_{(r),\rho} - MTTF\ . \quad (7)$$

Since $R_{(r),\rho}(t) \succ R(t) \Rightarrow MTTF_{(r),\rho} \succ MTTF$

3.3. The hot duplication methods

Let us assume that the system is improved by improving $m, 1 \le m \le n$, of its components according to the hot duplication method. Let $R^H_{(m)}(t)$ represent the reliability function of the system improved by improving *m* of its components by hot duplication, then, we can get $R^H_{(m)}(t)$ as follow:

$$R^H_{(m)}(t) = 1 - [1 - \exp(-(\lambda t)^{\beta})]^{n+m} \quad (8)$$

Using the binomial expansion, we can write $R^H_{(m)}(t)$ as follows

$$R^H_{(m)}(t) = \sum_{i=1}^{n+m}\binom{n+m}{i}(-1)^{i+1}\exp(-i(\lambda t)^{\beta})\ . \quad (9)$$

The system $MTTF$, say $MTTF^H_{(m)}$, is given by

$$MTTF^H_{(m)} = \sum_{i=1}^{n+m}\binom{n+m}{i}(-1)^{i+1}\frac{\Gamma(\frac{1}{\beta}+1)}{\lambda i^{1/\beta}}\ . \quad (10)$$

The process of improving the original system by improving *m* of its components according to a hot duplication method increases the *MTTF* of the system by the amount $\Delta MTTF^H_{(m)}$ which is given by :

$$\Delta MTTF_{(m)}^{H} = MTTF_{(m)}^{H} - MTTF \tag{11}$$

3.4. The cold duplication method

Let us consider that the system is improved by improving $m, 1 \le m \le n$ of its components according to the cold duplication methods. Let $R_{(m)}^{C}(t)$ denote the reliability function of the system improved by improving *m* of its components according to cold duplication methods. The function $R_{(m)}^{C}(t)$ can be obtained as follows:

$$R_{(m)}^{C}(t) = 1 - [1 - R_1^C(t)]^m \, [1 - R_1(t)]^{n-m} \quad , \tag{12 }$$

where $R_1^C(t)$ denote to the reliability of a system's component after it was improved according to cold duplication method [1], one can obtain $R_1^C(t)$ to be

$$R_1^C(t) = [1 + (\lambda t)^{\beta}]\exp[-(\lambda t)^{\beta}].$$

Therefore, $R_{(m)}^{C}(t)$ becomes :

$$R_{(m)}^{C}(t) = 1 - \left[1 - (1 + (\lambda t)^{\beta})\exp[-(\lambda t)^{\beta}]\right]^m \left[1 - \exp(-(\lambda t)^{\beta})\right]^{n-m}. \tag{13}$$

Using the binomial expansion, one can write $R_{(m)}^{C}(t)$ in the following form

<u>$1 \le m \prec n$</u>

$$R_{(m)}^{C}(t) = \sum_{i=0}^{n-m} \sum_{j=1}^{m} \sum_{k=0}^{j} \binom{n-m}{i}\binom{m}{j}\binom{j}{k}(-1)^{i+j+1}(\lambda t)^{\beta k}$$

$$\times \exp\{-(i+j)(\lambda t)^{\beta}\} + \sum_{i=1}^{n-m} \binom{n-m}{i}(-1)^{i+1}\exp\{-i(\lambda t)^{\beta}\}, \tag{14.1}$$

<u>$m = n$</u>

$$R_{(m)}^{C}(t) = \sum_{j=1}^{m} \sum_{k=0}^{j} \binom{m}{j}\binom{j}{k}(-1)^{j+1}(\lambda t)^{\beta k} \times \exp\{-j(\lambda t)^{\beta}\} \;. \tag{14.2}$$

The MTTF of this system, say $MTTF^{C}_{(m)}$, can be given as follow:

$\underline{1 \le m \prec n}$

$$MTTF^{C}_{(m)} = \frac{1}{\lambda\beta}\sum_{i=0}^{n-m}\sum_{j=1}^{m}\sum_{k=0}^{j}\binom{n-m}{i}\binom{m}{j}\binom{j}{k}(-1)^{i+j+1}$$

$$\times \frac{\Gamma(\frac{1}{\beta}+k)}{(i+j)^{\frac{1}{\beta}+k}} + \frac{\Gamma(\frac{1}{\beta}+1)}{\lambda}\sum_{i=1}^{n-m}\binom{n-m}{i}(-1)^{i+1}\frac{1}{i^{\frac{1}{\beta}}} \quad , \qquad (15.1)$$

$\underline{m = n}$

$$MTTF^{C}_{(m)} = \frac{1}{\lambda\beta}\sum_{j=1}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k}(-1)^{j+1} \times \frac{\Gamma(\frac{1}{\beta}+k)}{(j)^{\frac{1}{\beta}+k}} \quad . \qquad (15.2)$$

and the amount $\Delta MTTF^{C}_{(m)}$ given as :

$$\Delta\, MTTF^{C}_{(m)} = MTTF^{C}_{(m)} - MTTF \qquad (16)$$

Figure (1)-(3) show the reliability function of the original parallel system consisting of n=3 independent components and the improving system if m of the systems components are improved according to hot and cold duplication methods, when m=1,2,3,respectively, the shape parameter $\beta = 4$ and the scale parameter $\lambda = 1$ of each component.

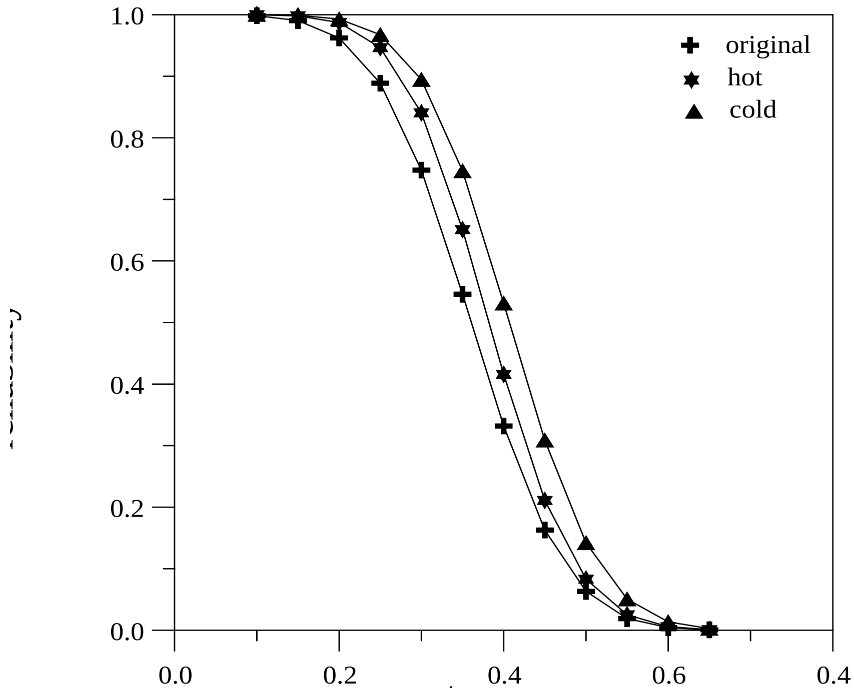

Fig 1 $(m=1)$

$R(t),\ R^{H}_{(1)}(t),\ R^{C}_{(1)}$ against t

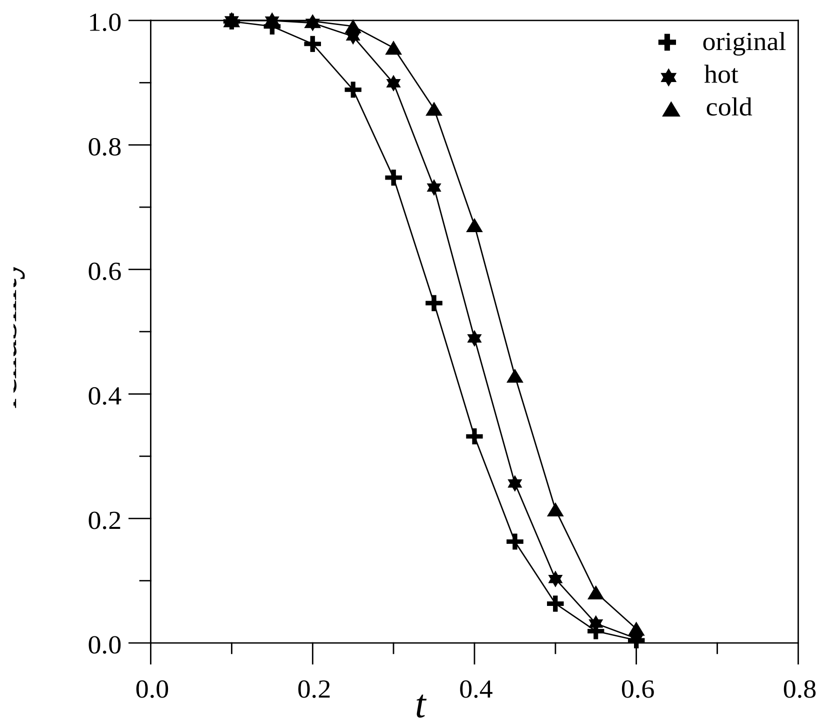

Fig .2. $(m=2)$

$R(t),\ R^{H}_{(2)}(t),\ R^{C}_{(2)}$ against t

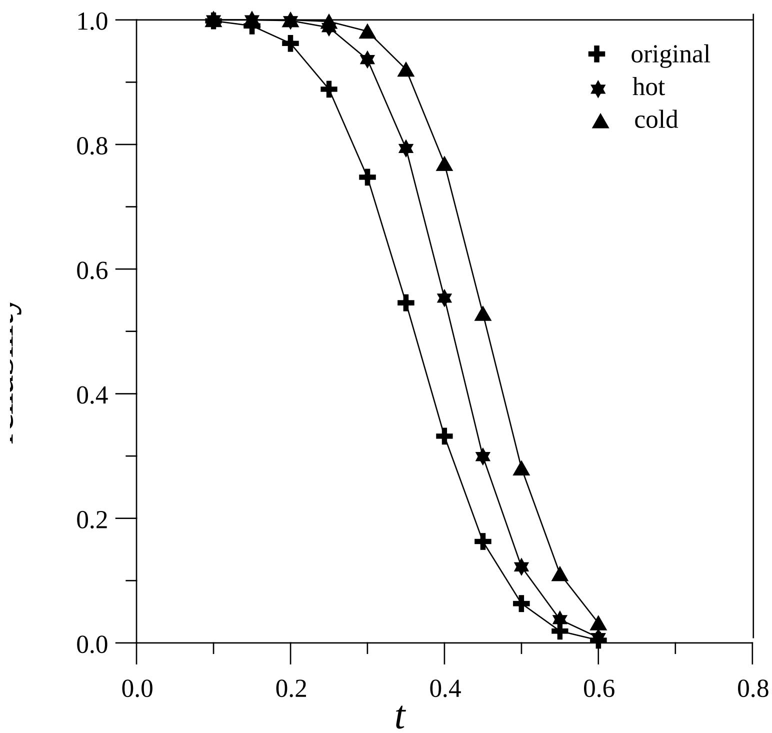

Fig5.3. $(m = 3)$

$R(t)$, $R^H_{(3)}(t)$, $R^C_{(3)}$ against t

It seems from Figs (1)-(3) that $R(t)^C_{(m)} \succ R(t)^H_{(m)} \succ R(t)$ for all $t \succ 0$ and for any m=1,2,3. It means that cold duplication method gives a modified system with a higher reliability than that of the system modified according to the hot duplication method.

4. Reliability Equivalence Factor

Generally, the reliability equivalence factor is defined as that factor by which the time varying failure rates of some of the system's components should be reduced in order to reach equality of the reliability of another better system [8].

4.1. Hot reliability equivalence factor $\rho^H_{(m),(r)}(\alpha)$

The hot reliability equivalence factor, say $\rho = \rho^H_{(m),(r)}(\alpha)$, is defined as that factor ρ by which the time varying failure rates of r of the system's components should be reduced in order to reach the reliability of that system which improved by improving m of the original system's components according to hot duplication method that is, $\rho = \rho^{H}_{(m),(r)}(\alpha)$ is the solution of the following system of two equations.

$$R_{(r),\rho}(t) = \alpha, \qquad R^{H}_{(m)}(t) = \alpha \tag{17}$$

Substituting from (8) into the second equation in (17), we have:

$$1 - \{1 - \exp(-(\lambda t)^{\beta})\}^{n+m} = \alpha$$

$$\Rightarrow \exp(-(\lambda t)^{\beta}) = 1 - (1-\alpha)^{1/(n+m)} \tag{18}$$

Substituting from (4) into the first equation in (17), we have:

$$1 - [1 - \exp(-\rho(\lambda t)^{\beta})]^{r}[1 - \exp(-\lambda t)^{\beta})]^{n-r} = \alpha \tag{19}$$

Using (5.18) and (5.19) , we find.

$$1 - \alpha = \{1 - [1 - (1-\alpha)^{1/(n+m)}]^{\rho}\}^{r}\{(1-\alpha)^{(n-r)/(n+m)}\} \tag{20}$$

Solving equation (20) with respect to ρ, the hot reliability factor $\rho = \rho^{H}_{(m),(r)}(\alpha)$ becomes

$$\rho = \rho^{H}_{(m),(r)}(\alpha) = \frac{\ln[1-(1-\alpha)^{(m+r)/r(n+m)}]}{\ln[1-(1-\alpha)1/n+m]} \tag{21}$$

4.2. Cold reliability equivalence factor, $\rho^{C}_{(m),(r)}(\alpha)$

The cold reliability equivalence factor, say $\rho = \rho^{C}_{(m),(r)}(\alpha)$, is defined as the factor ρ by which the time varying failure rates of r of the system's components should be reduced in order to reach the reliability of that system which improved by improving *m* of the original system's components have cold duplicates that is, $\rho = \rho^{C}_{(m),(r)}(\alpha)$ is the solution of the following system of two equations :

$$R_{(r),\rho}(t) = \alpha, \qquad R^{C}_{(m)}(t) = \alpha \ . \tag{22}$$

Substituting from (13) into the second equation in (22), one gets that

$$1-[1-(1+(\lambda t)^{\beta})\exp(-(\lambda t)^{\beta})]^{m}[1-\exp(-(\lambda t)^{\beta})]^{n-m}=\alpha \quad . \tag{23}$$

Substituting from (5.4) into the first equation in (22), we get

$$1-[1-\exp(-\rho(\lambda t)^{\beta})]^{r}[1-\exp(-(\lambda t)^{\beta})]^{n-r}=\alpha \, . \tag{24}$$

Solving (23) and (24), we can derive the cold reliability factor $\rho=\rho^{C}_{(m),(r)}(\alpha)$ as [35] :

$$\rho=\rho^{C}_{(m),(r)}(\alpha)=\frac{1}{\ln x}\ln[1-\frac{(1-\alpha)^{1/r}}{(1-x)^{(n-r)/r}}] \tag{25}$$

Where x is a non-negative real solution of the following equation

$$(1-x-x\ln x)^{m}(1-x)^{n-m}+\alpha-1=0 \tag{26}$$

Equation (26) has no closed form solution therefore, some numerical technique is needed to get x and then calculate $\rho^{C}_{(m),(r)}(\alpha)$.

5. α -Fractiles

The α -fractiles of the original and modified system are presented in this section. Let $L(\alpha)$ be the α -fractile of the original system. Also, let $L^{H}_{(m)}(\alpha) and\ L^{C}_{(m)}(\alpha)$ denote, respectively , the α -fractile of the modified system obtained by improving *m* of the system's components according to hot and cold duplication methods. The fractile $L(\alpha)$ can be found by solving the following equation with respect to L

$$R(\frac{L^{1/\beta}}{\lambda})=\alpha. \tag{27}$$

Using (2) and (27), we can find that

$$T(\alpha)=[\ln\{1/[1-(1-\alpha)^{\frac{1}{n}}]\}]^{\frac{1}{\beta}}/\lambda \, . \tag{28}$$

The fractile $L^{H}_{(m)}(\alpha)$ can be deduced by solving the following equation with respect to L .

$$R^{H}_{(m)}(\frac{L^{1/\beta}}{\lambda}) = R^{H}_{(m)}(t) = \alpha. \quad (29)$$

Using (8) and (29),we can obtain

$$T^{H}_{(m)}(\alpha) = [\ln\{1/[1-(1-\alpha)^{\frac{1}{(n+m)}}]\}]^{\frac{1}{\beta}} / \lambda \quad . \quad (30)$$

Also, $L^{C}_{(m)}(\alpha)$ can be obtained by solving the following equation with respect to L

$$R^{C}_{(m)}(L^{1/\beta}/\lambda) = \alpha\,. \quad (31)$$

Using (13) and (31), we have

$$1-[1-(1+L)\exp(-L)]^{m}[1-\exp(-L)]^{n-m} - \alpha = 0\,. \quad (32)$$

The above equation has no closed solution in *L*. Thus, to find out

L= $L^{C}_{(m)}(\alpha) = (\lambda T^{C}_{(m)}(\alpha))^{\beta}$ we have to use a numerical technique method to solve (32)

The *MTTF* of the original system consisting of n=3 components when $\lambda = 1$ and $\beta = 2$ is 1.29

Table (1) gives the $MTTF^{H}_{(m)}$ and $MTTF^{C}_{(m)}$ when n=3, $\lambda = 1$, $\beta = 2$ and $m = 1,2,3$. It is clear that, the $MTTF^{C}_{(m)} > MTTF^{H}_{(m)}$ for all $m = 1,2,3$.

Table 1 : value of system MTTF for various additions of spare

m	$MTTF^{H}_{(m)}$	$MTTF^{C}_{(m)}$
1	1.39	1.51
2	1.49	1.65
3	1.52	1.74

Table(2) gives the $\alpha-$fractiles and the reliability equivalence factors of the system studied here. Table (2) represent the $\alpha-$fractiles, $\rho^{H}_{(m),(r)}(\alpha)$ and $\rho^{C}_{(m),(r)}(\alpha)$

when n=3, $\lambda = 1$, $\beta = 2$, $m = 1$ and $r = 1,2,3$,respectively in these calculations the level is chosen to be $\alpha = 0.1,\ 0.2, \ldots,\ 0.9$.

Table 2.The α -fractiles , $\rho^{H}_{(1),(r)}(\alpha)$ and $\rho^{C}_{(1),(r)}(\alpha)$

α	$T(\alpha)$	$T^{H}_{(1)}(\alpha)$	$T^{C}_{(1)}(\alpha)$	$\rho^{H}_{(1),(1)}(\alpha)$	$\rho^{H}_{(1),(2)}(\alpha)$	$\rho^{H}_{(1),(3)}(\alpha)$	$\rho^{C}_{(1),(1)}(\alpha)$	$\rho^{C}_{(1),(2)}(\alpha)$	$\rho^{C}_{(1),(3)}(\alpha)$
0.1	1.835	1.910	2.065	0.814	0.891	0.922	0.610	0.730	0.790
0.2	1.624	1.707	1.849	0.772	0.866	0.904	0.565	0.703	0.771
0.3	1.479	1.569	1.701	0.736	0.844	0.889	0.530	0.681	0.756
0.4	1.362	1.456	1.581	0.702	0.824	0.874	0.499	0.662	0.742
0.5	1.256	1.356	1.472	0.668	0.802	0.859	0.468	0.642	0.728
0.6	1.155	1.259	1.369	0.631	0.779	0.842	0.437	0.621	0.713
0.7	1.052	1.161	1.262	0.589	0.751	0.822	0.402	0.597	0.695
0.8	0.938	1.051	1.144	0.537	0.717	0.796	0.361	0.567	0.672
0.9	0.790	0.909	0.989	0.460	0.663	0.755	0.303	0.522	0.637

It seems from Table (2) that, for $\alpha = 0.1$:

Hot duplication of a one component will increase the 0.1-fractile from 1.835 to 1.910 .The same increase can be obtained by making one of the following

(i) reducing the failure rate of one component by the factor $\rho^{H}_{(1),(1)}(0.1) = 0.814$,

(ii) reducing the failure rates of any two components by the factor $\rho^{H}_{(1),(2)}(0.1) = 0.891$, and

(iii) reducing the failure rates of each of three components by the factor $\rho^{H}_{(1),(3)}(0.1) = 0.922$.

Cold duplication of a one component will increase the 0.1-fractile from 1.385 to 2.065. The same increase can be obtained by making one of the following :

(i) reducing the failure rate of one component by the factor $\rho^{C}_{(1),(1)}(0.1) = 0.610$,

(ii) reducing the failure rates of any two components by the factor $\rho^{C}_{(1),(2)}(0.1) = 0.730$, and

(iii) reducing the failure rates of each of three components by the factor $\rho^{C}_{(1),(3)}(0.1) = 0.790$.

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