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# ABSOLUTE INDEXED SUMMABILITY FACTOR OF AN INFINITE SERIES USING QUASI-F-POWER INCREASING SEQUENCES

S.K.PAIKRAY[1*], R.K.JATI[2], U.K.MISRA[3], N.C.SAHOO[4]

[1]Department of Mathematics, Ravenshaw University, Cuttack-753003, Odisha, India

[2]Department of Mathematics, DRIEMS, Tangi, Cuttack-754022, Odisha, India

[3]Department of Mathematics, NIST, Palur Hills, Berhampur-761008, Odisha, India

[4]Department of Mathematics, S.B.Women'sCollege(Auto), Cuttack-753001, Odisha, India

**Abstract:** A result concerning absolute indexed summability factor of an infinite series using Quasi - $f$ - power increasing sequences has been established.

**Keywords:**Quasi-increasing,Quasi - $\beta$ - power increasing, Quasi - $f$ - power increasing, index absolute summability, summability factor.

**2000 AMS Subject Classification:** 40A05, 40D15, 40F05

## 1. INTRODUCTION:

A positive sequence $(a_n)$ is said to be almost increasing if there exists a positive sequence $(b_n)$ and two positive constants $A$ and $B$ such that

**(1.1)** $Ab_n \leq a_n \leq Bb_n$ , for all $n$ .

The sequence $(a_n)$ is said to be quasi- $\beta$ -power increasing, if there exists a constant $K$ depending upon $\beta$ with $K \geq 1$ such that

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*Corresponding author

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**(1.2)** $K n^{\beta} a_n \ge m^{\beta} a_m$ ,

for all $n \ge m$ . In particular, if $\beta = 0$ , then $(a_n)$ is said to be quasi-increasing sequence. It is clear that every almost increasing sequence is a quasi- $\beta$ -power increasing sequence for any non-negative $\beta$ . But the converse is not true as $(n^{-\beta})$ is quasi- $\beta$ -power increasing but not almost increasing.

Let $f = (f_n)$ be a positive sequence of numbers. Then the positive sequence $(a_n)$ is said to be quasi- $f$ -power increasing, if there exists a constant $K$ depending upon $f$ with $K \ge 1$ such that

**(1.3)** $K f_n a_n \ge f_m a_m$,

for $n \ge m \ge 1$[4] . Clearly, if $(\alpha_n)$ is a quasi- $f$ -power increasing sequence, then the $(\alpha_n f_n)$ is a quasi- increasing sequence.

Let $\sum a_n$ be an infinite series with sequence of partial sums $\{s_n\}$. Let $(p_n)$ be a sequence of positive numbers such that

$$P_n = \sum_{\nu=0}^{n} p_\nu \to \infty, \text{ as } n \to \infty .$$

Then the sequence-to-sequence transformation

**(1.4)** $$T_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu s_\nu , \ P_n \ne 0,$$

defines the $(\overline{N}, p_n)$- mean of the sequence $(s_n)$ generated by the sequence of coefficients $\{p_n\}$. The series $\sum a_n$ is said to be summable $\left|\overline{N}, p_n\right|_k$ , $k \ge 1$[1], if

**(1.5)** $\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}\left|T_n - T_{n-1}\right|^k < \infty$ .

The series $\sum a_n$ is said to be summable $\left|\overline{N}, p_n; \delta\right|_k, k \geq 1, \delta \geq 0,$ if

**(1.6)** $\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left|T_n - T_{n-1}\right|^k < \infty$ .

The series $\sum a_n$ is said to be summable $\left|\overline{N}, p_n; \alpha_n(\delta)\right|_k, k \geq 1, \delta \geq 0,$ if

**(1.7)** $\sum_{n=1}^{\infty}(\alpha_n)^k\left(\frac{P_n}{p_n}\right)^{k-1}\left|T_n - T_{n-1}\right|^k < \infty$ .

Putting $\alpha_n = \left(\frac{P_n}{p_n}\right)^{\delta}$, $\left|\overline{N}, p_n; \alpha_n(\delta)\right|_k, k \geq 1, \delta \geq 0,$ reduces to $\left|\overline{N}, p_n; \delta\right|_k, k \geq 1, \delta \geq 0.$

## 2. PRELIMINARIES

Dealing with quasi- $\beta$ -power increasing sequence Bor and Debnath**[2]** have established the following theorem:

### 2.1. THEOREM:

Let $(X_n)$ be a quasi- $\beta$ -power increasing sequence for $0 < \beta < 1$ and $(\lambda_n)$ be a real sequence.

If the conditions

**(2.1.1)** $$\sum_{n=1}^{m}\frac{P_n}{n} = O(P_m),$$

**(2.1.2)** $$\lambda_n X_n = O(1),$$

**(2.1.3)** $$\sum_{n=1}^{m} \frac{|t_n|^k}{n} = O(X_m),$$

**(2.1.4)** $$\sum_{n=1}^{m} \frac{p_n |t_n|^k}{P_n} = O(X_m)$$

and

**(2.1.5)** $$\sum_{n=1}^{m} nX_n |\Delta^2 \lambda_n| < \infty$$

are satisfied, where $t_n$ is the $(C,1)$ mean of the sequence $(na_n)$ .Then the series $\sum a_n \lambda_n$ is summable $|\overline{N}, p_n|_k, k \geq 1$ .

Subsequently Leindler**[**3**]** established a similar result reducing certain condition of Bor. He established:

**2.2. THEOREM:**

Let the sequence $(X_n)$ be a quasi-$\beta$-power increasing sequence for $0 < \beta < 1$ **,** and the real sequence $(\lambda_n)$ satisfies the conditions

**(2.2.1)** $$\sum_{n=1}^{m} \lambda_n = O(m)$$

and

**(2.2.2)** $$\sum_{n=1}^{m} |\Delta \lambda_m| = O(m).$$

Further, suppose the conditions (2.1.3) ,(2.1.4) and

**(2.2.3)** $$\sum_{n=1}^{m} nX_n(\beta) |\Delta |\Delta \lambda_n|| < \infty,$$

hold, where $X_n(\beta)=\max\left(n^{\beta}X_n,\log n\right)$.Then the series $\sum a_n\lambda_n$ is summable $\left|\overline{N},p_n\right|_k, k\geq 1$ .

Recently, extending the above results to quasi- $f$ -power increasing sequence, Sulaiman**[5]** have established the following theorem:

**2.3. THEOREM:**

Let $f=(f_n)=\left(n^{\beta}\log^{\gamma} n\right), 0\leq\beta<1, \gamma\geq 0$ be a sequence. Let $\left(X_n\right)$ be a quasi- $f$ -power sequence and $\left(\lambda_n\right)$ a sequence of constants satisfying the conditions

**(2.3.1)** $$\lambda_n\rightarrow 0 \ as\, n\rightarrow\infty,$$

**(2.3.2)** $$\sum_{n=1}^{\infty} nX_n|\Delta|\ |\Delta\lambda_n|<\infty,$$

**(2.3.3)** $$|\lambda_n|X_n=O(1),$$

**(2.3.4)** $$\sum_{n=1}^{\infty}\frac{1}{nX_n^{k-1}}|t_n|^k=O\left(X_m\right)$$

and

**(2.3.5)** $$\sum_{n=1}^{\infty}\frac{p_n}{P_n}\frac{1}{X_n^{k-1}}|t_n|^k=O\left(X_m\right),$$

where $t_n$ is the $(C,1)$ mean of the sequence $\left(na_n\right)$ .Then the series $\sum a_n\lambda_n$ is summable $\left|\overline{N},p_n\right|_k, k\geq 1$ .

We prove the following theorem.

**3. MAIN RESULTS:**

Let $f=(f_n)=\left(n^{\beta}\log^{\gamma} n\right)$ be a sequence and $\left(X_n\right)$be a quasi- $f$ -power sequence. Let $\left(\lambda_n\right)$ a sequence of constants such that

**(3.1)** $$\lambda_n\rightarrow 0,\ as\ \ n\rightarrow\infty,$$

**(3.2)** $$\sum_{n=1}^{\infty} nX_n|\Delta|\ |\Delta\lambda_n|<\infty,$$

**(3.3)** $$|\lambda_n| X_n = O(1),$$

**(3.4)** $$\sum_{n=\nu+1}^{m} (\alpha_n)^k \left(\frac{p_n}{P_n}\right) \frac{1}{P_{n-1}} = O\left((\alpha_m)^k \frac{p_m}{P_m}\right),$$

**(3.5)** $$\sum_{n=1}^{m} (\alpha_n)^k \left(\frac{p_n}{P_n}\right) \frac{|t_\nu|^k}{X_\nu^{k-1}} = O(X_m),$$

**(3.6)** $$\sum_{n=1}^{m} (\alpha_n)^k \frac{|t_n|^k}{nX_n^{k-1}} = O(X_m).$$

Then the series $\sum a_n \lambda_n$ is summable $\left|\overline{N}, p_n; \alpha_n(\delta)\right|_k, k \geq 1, \delta \geq 0.$

In order to prove the theorem we require the following lemma.

**4. LEMMA:**

Let $f = (f_n) = (n^\beta \log^\gamma n), 0 \leq \beta < 1, \gamma \geq 0$ be a sequence and $(X_n)$ be a quasi - $f$ - power increasing sequence. Let $(\lambda_n)$ be a sequence of constants satisfying (3.1) and (3.2). then

**(4.1)** $$n X |\Delta\lambda_n| = O(1)$$

and

**(4.2)** $$\sum_{n=1}^{m} X_n |\Delta\lambda_n| < \infty.$$

**4.1. PROOF OF THE LEMMA**:

As $\Delta\lambda_n \to 0$ and $n^\beta \log^\gamma n X_n$ is non-decreasing, we have

$$n X_n |\Delta\lambda_n| = n^{1-\beta} \log^{-\gamma} n (n^\beta \log^\gamma n\, X_n) \sum_{\upsilon=n}^{\infty} \Delta |\Delta\lambda_\upsilon|$$

$$= O(1) n^{1-\beta} \log^{-\gamma} n \sum_{\upsilon=n}^{\infty} \nu^\beta \log^\gamma \nu\, X_\upsilon |\Delta| |\Delta\lambda_\upsilon|$$

$$= O(1) \sum_{\upsilon=n}^{\infty} \nu^{1-\beta} \log^{-\gamma} \nu\, \nu^\beta \log^\gamma \nu\, X_\upsilon |\Delta| |\Delta\lambda_\upsilon|$$

$$= O(1)\sum_{\upsilon=n}^{\infty} \nu \; X_{\upsilon}|\Delta|| \Delta\lambda_{\upsilon} | = O(1) .$$

This establishes (4.1). Next

$$\sum_{n=1}^{m} X_n \;|\Delta\lambda_n| = \sum_{n=1}^{m-1}\left(\sum_{r=1}^{n} X_r\right)\Delta|\Delta\lambda_r| + \left(\sum_{r=1}^{n} X_r\right)|\Delta\lambda_r|$$

$$= O(1)\sum_{n=1}^{m-1}\left(\sum_{r=1}^{n} r^{-\beta}\log^{-\gamma} r\, r^{\beta}\log^{\gamma} r\, X_r\right)|\Delta||\Delta\lambda_{\upsilon}|$$

$$+ O(1)\left(\sum_{r=1}^{m} r^{-\beta}\log^{-\gamma} r\, r^{\beta}\log^{\gamma} r\;X_r\right)|\Delta\lambda_m|$$

$$= O(1)\sum_{n=1}^{m-1}\left(n^{\beta}\log^{\gamma} n\;X_n\right)|\Delta||\Delta\lambda_{\upsilon}|\sum_{r=1}^{n} r^{-\beta-\in}\log_{r}^{-\gamma} r\, r^{\in}$$

$$+ O(1)\, m^{\beta}\;X_m|\Delta\lambda_m|\log^{\gamma} m\sum_{r=1}^{m} r^{-\beta-\in}\log^{-\gamma} r\, r^{\in}, \in < 1-\beta.$$

$$= O(1)\sum_{n=1}^{m-1}\left(n^{\beta}\log^{\gamma} n\;X_n\right)|\Delta||\Delta\lambda_{\upsilon}|n^{\in^{\in}}\;\log^{-\gamma} n\sum_{r=1}^{n} r^{-\beta-\in}$$

$$+ O(1)\, m^{\beta}\;X_m|\Delta\lambda_m|\log^{\gamma} m\, m^{\in}\log_{r}^{-\gamma} m\sum_{r=1}^{m} r^{-\beta-\in}$$

$$= O(1)\sum_{n=1}^{m} n^{\beta+\in} X_n|\Delta||\Delta\lambda_{\upsilon}|\left(\int_{1}^{n} u^{-\beta-\in} du\right) + O(1)\, m^{\beta+\in}\;X_n|\Delta\lambda_m|\left(\int_{1}^{m} u^{-\beta-\in} du\right)$$

$$= O(1)\sum_{n=1}^{m} nX_n|\Delta|\;|\Delta\lambda_{\upsilon}| + O(1)\, m\, X_m|\Delta\lambda_m|$$

$$= O(1) \;.$$

This establishes (4.2).

## 5. PROOF OF THE THEOREM:

Let $(T_n)$ be the sequence of $(\overline{N}, p_n)$ mean of the series $\sum_{n=1}^{\infty} a_n \lambda_n$, then

$$T_n = \frac{1}{P_n} \sum_{\upsilon=0}^{n} p_\upsilon \sum_{r=0}^{\upsilon} a_r \lambda_r$$

$$= \frac{1}{P_n} \sum_{\upsilon=0}^{n} (p_n - p_{\upsilon-1}) a_\nu \lambda_\nu$$

Hence for $n \geq 1$

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{\upsilon=1}^{n} p_{\upsilon-1} a_\nu \lambda_\nu$$

$$= \frac{p_n}{P_n P_{n-1}} \sum_{\upsilon=1}^{n} \nu a_\nu \left( \frac{1}{\nu} p_{\nu-1} \lambda_\nu \right)$$

$$= \frac{(n+1)}{n} \frac{p_n}{P_n} t_n \lambda_n + \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} p_{\nu-1} t_\nu \lambda_\nu \frac{\nu+1}{\nu} + \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} p_\nu t_\nu \frac{\nu+1}{\nu} \Delta\lambda_\nu$$

$$= \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} p_\nu t_\nu \frac{\lambda_{\nu+1}}{\nu}$$

$$= T_{n1} + T_{n2} + T_{n3} + T_{n4} \text{ (say).}$$

In order to prove the theorem, using Minkowski's inequality it is enough to show that

$$\sum_{n=1}^{\infty} (\alpha_n)^k \left( \frac{P_n}{p_n} \right)^{k-1} |T_{nj}| < \infty \quad , j = 1,2,3,4 .$$

Applying Holder's inequality, we have

$$\sum_{n=1}^{m} (\alpha_n)^k \left( \frac{P_n}{p_n} \right)^{k-1} |T_{n1}|^k = \sum_{n=1}^{m} (\alpha_n)^k \left( \frac{P_n}{p_n} \right)^{k-1} \left| \frac{n+1}{n} \frac{p_n}{P_n} t_n \lambda_n \right|^k$$

$$= O(1)\sum_{n=1}^{m}(\alpha_n)^k\left(\frac{P_n}{p_n}\right)^{-1}\frac{|t_n|^k}{X_n^{k-1}}\left(X_n|\lambda_n|\right)^{k-1}|\lambda_n|$$

$$= O(1)\sum_{n=1}^{m}(\alpha_n)^k\left(\frac{P_n}{p_n}\right)^{-1}\frac{|t_n|^k}{X_n^{k-1}}|\lambda_n|$$

$$= O(1)\sum_{n=1}^{m-1}\left(\sum_{\upsilon=1}^{n}(\alpha_n)^k\left(\frac{P_\nu}{p_\nu}\right)^{-1}\frac{|t_\nu|^k}{X_\nu^{k-1}}\right)\Delta|\lambda_\nu| + O(1)\sum_{\upsilon=1}^{m}(\alpha_n)^k\left(\frac{P_\upsilon}{p_\upsilon}\right)^{-1}\frac{|t_\nu|^k}{X_\nu^{k-1}}|\lambda_m|$$

$$= O(1)\sum_{n=1}^{m-1}X_n\Delta|\lambda_n| + O(1)X_m|\lambda_m|$$

$$= O(1).$$

Next,

$$\sum_{n=1}^{m}(\alpha_n)^k\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n2}|^k = \sum_{n=1}^{m}(\alpha_n)^k\left(\frac{P_n}{p_n}\right)^{k-1}\left|\frac{p_n}{P_nP_{n-1}}\sum_{\nu=1}^{n-1}p_{\nu-1}\,t_\nu\lambda_\nu\,\frac{\nu+1}{\nu}\right|^k$$

$$= O(1)\sum_{n=1}^{m}(\alpha_n)^k\left(\frac{P_n}{p_n}\right)^{-1}\frac{1}{P_{n-1}}\sum_{\nu=1}^{n-1}p_{\nu-1}|t_\nu|^k|\lambda_\nu|^k\left(\sum_{\nu=1}^{n-1}\frac{p_\nu}{P_{n-1}}\right)^{k-1}$$

$$= O(1)\sum_{\nu=1}^{m}p_{\nu-1}|t_\nu|^k|\lambda_\nu|^k\sum_{n=\nu+1}^{m}(\alpha_n)^k\left(\frac{P_n}{p_n}\right)^{-1}\frac{1}{P_{n-1}}.$$

$$= O(1)\sum_{\nu=1}^{n}(\alpha_\upsilon)^k\left(\frac{p_\nu}{P_\nu}\right)|t_\nu|^k|\lambda_\nu|^k$$

$= O(1)$, as in the case of $T_{n1}$.

Next,

$$\sum_{n=1}^{m}(\alpha_n)^k\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n3}|^k = \sum_{n=1}^{m}(\alpha_n)^k\left(\frac{P_n}{p_n}\right)^{k-1}\left|\frac{p_n}{P_nP_{n-1}}\sum_{\upsilon=1}^{n-1}P_\upsilon t_\upsilon\,\frac{\nu+1}{\nu}\Delta\lambda_\upsilon\right|^k$$

$$= O(1)\sum_{n=1}^{m}(\alpha_n)^k\left(\frac{P_n}{p_n}\right)^{-1}\frac{1}{P_{n-1}^k}\sum_{\upsilon=1}^{n-1}P_\nu^k\frac{|t_\upsilon|^k}{X_\nu^{k-1}}|\Delta\lambda_\upsilon|\left(\sum_{\upsilon=1}^{n-1}X_\upsilon|\Delta\lambda_\nu|\right)^{k-1}$$

$$= O(1)\sum_{\upsilon=1}^{m}P_\nu^k\frac{|t_\upsilon|^k}{X_\nu^{k-1}}|\Delta\lambda_\upsilon|\sum_{n=\upsilon+1}^{m+1}(\alpha_n)^k\left(\frac{p_n}{P_n}\right)\frac{1}{P_{n-1}^k}.$$

$$= O(1)\sum_{\upsilon=1}^{m}(\alpha_\upsilon)^k\frac{1}{\nu}\frac{|t_\upsilon|^k}{X_\nu^{k-1}}\left(\nu|\Delta\lambda_\nu|\right)$$

$$= O(1)\sum_{\upsilon=1}^{m-1}\sum_{r=1}^{\nu}(\alpha_r)^k\frac{1}{r}\frac{|t_r|^k}{X_r^{k-1}}\Delta\left(\nu|\Delta\lambda_\nu|\right)+O(1)\left(\sum_{r=1}^{m}(\alpha_r)^k\frac{1}{r}\frac{|t_r|^k}{X_r^{k-1}}\right)\left(m|\Delta\lambda_m|\right)$$

.

$$= O(1)\sum_{\upsilon=1}^{m-1}X_\upsilon\left(-|\Delta\lambda_\upsilon|+(\nu+1)|\Delta||\Delta\lambda_\upsilon|\right)+O(1)mX_m|\Delta\lambda_m|.$$

$$= O(1)\sum_{\upsilon=1}^{n}X_\upsilon|\Delta\lambda_\upsilon|+0(1)\sum_{\upsilon=1}^{n}\nu X_\upsilon|\Delta||\Delta\lambda_\upsilon|+O(1)mX_m|\Delta\lambda_m|$$

$$= O(1).$$

Finally,

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|T_{n4}|^k = \sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left|\frac{p_n}{P_nP_{n-1}}\sum_{\nu=1}^{n-1}P_\nu t_\nu\frac{\lambda_{\nu+1}}{\nu}\right|^k$$

$$= O(1)\sum_{n=1}^{m}(\alpha_n)^k\left(\frac{P_n}{p_n}\right)^{-1}\frac{1}{P_{n-1}^k}\sum_{\nu=1}^{n-1}\frac{P_\nu}{\nu}|t_\nu|^k|\lambda_\nu|^k\left(\sum_{\nu=1}^{n-1}\frac{P_\nu}{\nu}\right)^{k-1}$$

$$= O(1)\sum_{\nu=1}^{m}\frac{P_\nu}{\nu}|t_\nu|^k|\lambda_\nu|^k\sum_{n=\nu+1}^{m}(\alpha_n)^k\left(\frac{P_n}{p_n}\right)^{-1}\frac{1}{P_{n-1}}$$

$$= O(1)\sum_{\nu=1}^{m}(\alpha_\upsilon)^k\frac{|t_\nu|^k}{X_\nu^{k-1}}\left(X_\nu|\lambda_\nu|\right)^{k-1}|\lambda_\nu|$$

$$= O(1)\sum_{\nu=1}^{m}(\alpha_\nu)^k \frac{|t_\nu|^k |\lambda_\nu|}{\nu X_\nu^{k-1}}$$

$$= O(1)\sum_{\nu=1}^{m-1}\left(\sum_{r=1}^{\nu}(\alpha_r)^k \frac{|t_r|^k}{r X_r^{k-1}}\right)|\Delta\lambda_\nu| + O(1)\sum_{r=1}^{m}(\alpha_r)^k \frac{|t_r|^k}{r X_r^{k-1}}$$

$$= O(1)\sum_{\nu=1}^{m} X_\nu |\Delta\lambda_\nu| + O(1) X_m |\lambda_m|$$

$$= O(1).$$

This completes the proof of the theorem.

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