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ON GENERALIZED MELLIN-HARDY INTEGRAL TRANSFORMATIONS

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Abstract. In this paper certain testing function spaces are constructed and classical Mellin-Hardy integral transformations is extended to generalized functions.

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1. Introduction

The classical integral transforms find several applications to the problem of mathematical physics, particularly for solving the differential equations which arise in certain physical problems. There are physical situations governed by differential equations whose boundary conditions are not enough smooth and are generalized functions. In such situations the classical integral transforms of generalized functions or generalized integral transformations. Therefore it is of importance to extend the classical integral transforms to generalized functions. There is good number of integral transforms available in the literature. The well-known classical integral transforms such as Laplace, Mellin, Hankal and Hardy etc., have been extended to generalized functions. The conventional Hardy transformation is very general in the sense that the includes several other important integral

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transforms (see for details [6]), which is extended to generalized functions by Pathak and Panday [6]. Recently the extension of a pair of classical integral transforms to generalized functions is initiated, and Ahirro and More [2] have extended the classical Laplace-Hardy integral transformation to generalized functions. The aim of the present paper is to extend the classical Mellin-Hardy integral transformations to generalized functions by using an approach similar to Zemanin [9]. The rest of the paper is organized as follows. In this section below we give the definitions of classical Mellin and Hardy transformations. In section we construct certain function spaces and finally in section, we discuss the distributional Mellin-Hardy integral transformations.

The classical Hardy transformations [4] which is also called C_ν-transformations is defined by

$$f(y) = \int_0^\infty F_\nu(ty)t\,dt \int_0^\infty C_\nu(tx)xf(x)\,dx, \tag{1.1}$$

where

(a) $C_\nu(z) = \cos(p\pi)J_\nu(Z) + \sin(p\pi)Y_\nu(Z)$.

(b) The function $F_\nu(z)$ is given by

$$\begin{aligned} F_\nu(z) &= \sum_{m=0}^{\infty} \frac{(-1)^m(\frac{1}{2}z)^{\nu+2p+2m}}{\Gamma(p+m+1)\Gamma(p+m+\nu+1)} \\ &= \frac{2^{2-\nu-2p}S_{\nu+2p-1,\nu}(z)}{\Gamma(p)\Gamma(\nu+p)}. \end{aligned}$$

(c) $J_\nu(Z)$ and $Y_\nu(z)$ are Bessel functions of first and second kind respectively, and

(d) $S_{\mu,\nu}(z)$ is the Lommel function [3, 8].

The inversion formula (1.1) is valid under the following conditions [2, page 384].

(i) $p > -1, p+\nu > -1, |\nu+2p| < \frac{3}{2}$,

(ii) $t^\alpha f(t)$ is a integrable over $(0,\delta)$ where, $\sigma = \min\{1-\nu-2p, 1-|\nu|, \frac{1}{2}\}$, $\delta > 0$,

(iii) $t^{\frac{1}{2}}f(t)$ is a integrable over (δ,∞), and

(iv) $f(t)$ is of bounded variation in a neighborhood of the point $t = x$.

The theory of expansion formula (1.1) has been given by Cooke [3]. The kernal functionC_ν is the solution of Bessel equation

$$\Delta_x f(x) = 0$$

where

$$\Delta_x = D_x^2 + \frac{1}{x} D_x - \frac{\nu^2}{x^2} \quad \left(D_x = \frac{d}{d_x} \right)$$

The classical Mellin integral transformation is given by

$$F(s) = \int_0^\infty t^{s-1} f(t)\, dt$$

where $f(t)$ is a suitably restricted conventional function on the real line $(0, \infty)$. Thus this integral transformation maps $f(t)$ into function $F(s)$ of the complex variables. Panday and Pathak [6] extended the classical Hardy transformations to generalized functions and Zemanian [9] extended the classical Mellin transformations to generalized functions. In this paper we discuss the distributional Mellin-Hardy transformations.

2. Testing Function Spaces

Throughout this paper, let $\mathbb{R}$ and $\mathbb{N}$ denote the sets of real numbers and nonnegative integers respectively. Let $a, b, c, d, t, \in \mathbb{R}$ and $s \in \mathbb{C}$, where $\mathbb{C}$ denotes the set of complex numbers. consider the functions $\kappa(t, x)$ defined by

$$\kappa(t, x) = \begin{cases} t^{-a} x^{\alpha}, & 0 \le t < 1,\ 0 < x \le 1, \\ \\ t^{-b} x^{3-\alpha}, & 1 < t < \infty,\ x > 1, \end{cases} \tag{2.1}$$

where α is a fixed positive number satisfying $|\nu| \le \alpha \le \frac{1}{2}$.

Now for each number$k = (k_1, k_2) \in \mathbb{N}_+ \times \mathbb{N}_+$ we define a space $\mathcal{MH}_\alpha(\Omega)$ consisting of all infinitely differentiable functions $\phi(t, x)$ over the domain

$$\Omega = \{(t, x) \,\big|\, 0 \le t < \infty, 0 \le x < \infty\}, \tag{2.2}$$

satisfying the semi-norm

$$\gamma^{\alpha}_{a,b,k}(\phi(t, x)) = \sup_{\substack{0 \le t \le \infty \\ 0 \le x < \infty}} \left| \kappa_{a,b}(t, x) D_t^{k_1} \Delta_x^{k_2} \phi(t, x) \right| < \infty, \tag{2.3}$$

where Δ_x is the Bessel differentiable operator as defined in section 1 and $D_t = \dfrac{\partial}{\partial t}$.

Obviously, one can prove that $\mathcal{MH}_\alpha(\Omega)$ is a linear space under the pointwise addition of functions and their multiplication by complex numbers. Clearly, $\gamma^\alpha_{a,b,0}$ is a norm. for if $\gamma^\alpha_{a,b,0}(\phi) = 0$, then

$$\sup_{\substack{0\leq t\leq\infty \\ 0\leq x\leq\infty}} \Big|\kappa_{a,b}(t,x)\phi(t,x)\Big| = 0.$$

This further implies that $\phi(t,x) = 0$, for all $(t,x) \in \Omega$, showing that $\gamma^\alpha_{a,b,0}$ is a norm on $\mathcal{MH}_\alpha(\Omega)$. Therefore, the collection of the semi-norms $\{\gamma^\alpha_{a,b,k}\}^\infty_{k_1,k_2=0}$ is a multi-norm for $\mathcal{MH}_\alpha(\Omega)$ and we assign to $\mathcal{MH}_\alpha(\Omega)$, the topology [9, page 9] generated by the multi-norm $\{\gamma^\alpha_{a,b,k}\}^\infty_{k_1,k_2=0}$ Then the linear space $\mathcal{MH}_\alpha(\Omega)$ becomes a multi-normed space. As $k = (k_1,k_2) \in \mathbb{N}_+ \times \mathbb{N}_+$ transverses the nonnegative integers in $\mathbb{N}_+$, we obtain a countable multi-norm $\{\gamma^\alpha_{a,b,k}\}^\infty_{k_1,k_2=0}$ defined on $\mathcal{MH}_\alpha(\Omega)$ and hence $\mathcal{MH}_\alpha(\Omega)$ is a countably multi-normed space. we say that a sequence a sequence $\{\phi_n(t,x)\}^\infty_{n=1}$ converges to $\phi(t,x)$ in $\mathcal{MH}_\alpha(\Omega)$ if for each fixed a, b, k and α; and $n \to \infty$, where each $\phi_n \in \mathcal{MH}(\Omega)$. Similarly, a sequence $\{\phi_n\}$ in $\mathcal{MH}(\Omega)$is said to be Cauchy if $\gamma^\alpha_{a,b,k}(\phi_m - \phi_n) \to 0$ as $m, n \to \infty$. The space $\mathcal{MH}(\Omega)$ is said to be complete if every Cauchy sequence in $\mathcal{MH}(\Omega)$ converges to a point in $\mathcal{MH}(\Omega)$.

3. Countably Union Spaces

Let w denote either a finite or $-\infty$ and let z denote a finite or $+\infty$. consider two monotonic sequences $\{a_n\}^\infty_{n=1}$ and $\{b_n\}^\infty_{n=1}$ of real numbers such that $a_n \to w^+$ and $b_n \to z^-$. Let $\{\mathcal{MH}_{\alpha,a_n,b_n}\}$ be a sequence of countably multi-normed spaces that

$$a_n < a_{n+1}, b_n < b_{n+1}, \ldots,$$

etc., such that

$$\mathcal{MH}_{\alpha,a_1,b_1} \subset \mathcal{MH}_{\alpha,a_2,b_2} \subset \mathcal{MH}_{\alpha,a_1,b_1} \subset \cdots \subset \mathcal{MH}_{\alpha,a_n,b_n} \subset \cdots \tag{3.1}$$

Further we assume that the topology of each $\mathcal{MH}_{\alpha,a_n,b_n}$ is stronger than the topology induced on it by $\mathcal{MH}_{\alpha,a_{n+1},b_{n+1}}$ Let $\mathcal{MH}_\alpha(w,z)$ denote the union of all these spaces, i.e.,

$$\mathcal{MH}_\alpha(w,z) = \bigcup_{n=1}^{\infty} \mathcal{MH}_{\alpha,a_n,b_n}. \tag{3.2}$$

A sequence converges in $\mathcal{MH}_\alpha(w,z)$ if and only if it converges in one of the $\mathcal{MH}_{\alpha,a_n,b_n}$ spaces and this definition does not depend upon the choices of the sequences $\{a_n\}$ and $\{b_n\}$ since each of the space $\mathcal{MH}_{\alpha,a_n,b_n}$ is complete, $\mathcal{MH}_\alpha(w,z)$ is also complete [9, Theorem 1.8.3].

Lemma 3.1. Let $w \le u$ and $v \le z$, then

$$\mathcal{MH}_\alpha(u,v) \subset \mathcal{MH}_\alpha(w,z),$$

and the converges in $\mathcal{MH}_\alpha(u,v)$ implies the converges in $\mathcal{MH}_\alpha(w,z)$.

Proof. To prove the theorem , it suffices to show that if $a_n \le c_n$ and $d_n \le b_n$, then

$$\mathcal{MH}_{\alpha,c_n,d_n} \subset \mathcal{MH}_{\alpha,a_n,b_n},$$

and the topology of $\mathcal{MH}_{\alpha,c_n,d_n}$ is stronger than the topology induced on $\mathcal{MH}_{\alpha,c_n,d_n}$ by $\mathcal{MH}_{\alpha,a_n,b_n}$ For, we note that

$$0 < \kappa_{a_n,b_n}(t,x) \le \kappa_{c_n,d_n}(t,x)$$

on Ω. Therefore,

$$\begin{aligned} & \left|\kappa_{a_n,b_n}(t,x) D_t^{k_1} \Delta_{k_2} \phi(t,x)\right| \le \left|\kappa_{c_n,d_n}(t,x) D_t^{k_1} \Delta_x^{k_2} \phi(t,x)\right| \\ \Longrightarrow \quad & \gamma^\alpha_{a_n,b_n,k}(\phi(t,x)) \le \gamma^\alpha_{c_n,d_n}(\phi(t,x)). \end{aligned} \tag{3.3}$$

Hence by Lemma 1.6.3 of [9, page12,13] the conclusion of the lemma follows.

Theorem 3.1. $\mathcal{MH}_\alpha(\Omega)$ is complete and therefore, a Frenchet space.

Proof. Let $\{\phi_n(t,x)\}$ be a Cauchy sequence in $\mathcal{MH}_\alpha(\Omega)$. Then for each fixed a, b, k and α, the functions $\{\kappa_{a,b}(t,x) D_t^{k_1} \Delta_x^{k_2} \phi_n(t,x)\}$ comprise a uniform Cauchy sequence on Ω. Therefore, by Cauchy criterion [1, Page 345], it converges uniformly on Ω as $n \to \infty$, for each fixed a, b, k and α. Suppose that

$$\phi_n(t,x) = \kappa_{a,b}(t,x) D_t^{k_1} \Delta_x^{k_2} \phi_n(t,x), \ (t,x) \in \Omega. \tag{3.4}$$

Now by Cauchy criterion $\{D\phi_n\}$ converges uniformly on Ω for each fixed a, b, k and α. Hence by a standard theorem [1, page 345], there exists a smooth functions $\phi(t, x)$ defined on Ω such that $\phi_n(t, x) \to \phi(t, x)$ uniformly on every compact subset I of I' , where I' is an open subset of Ω, and $D\phi_n \to D\phi$ as $n \to \infty$, where the function ϕ is given by

$$\phi(t, x) = \kappa_{a,b}(t, x) D_t^{k_1} \Delta_x^{k_2} \phi(t, x). \tag{3.5}$$

Moreover, again the fact that $\{\phi_n(t, x)\}_{n=0}^{\infty}$ is a Cauchy sequence for each fixed a, b, k and α. Hence, for every $\epsilon > 0$, therefore, there exists an integer $n_0 \in \mathbb{N}$ such that

$$\gamma_{a,b,k}^{\alpha}(\phi_m - \phi_n) < \epsilon \tag{3.6}$$

for all $m, n \geq n_0$. Taking the limit as $m \to \infty$ in (3.6) we get

$$\gamma_{a,b,k}^{\alpha}(\phi_n - \phi) < \epsilon$$

for all $n \geq n_0$. This proves that $\phi_n \to \phi$ in $\mathcal{MH}(\Omega)$ and the proof of the theorem is complete.

Theorem 3.2. $\mathcal{MH}(\Omega)$ is a testing function space.

Proof. The proof is simple and can be obtained by giving the arguments similar to Lemma 2.5 of Ahirrao and More [2]. Before stating the main result of this section, we prove a useful lemma.

Lemma 3.2. For $t > 0, x > 0$, one has

(i) $|\kappa_{a,b}(t, x) C_\nu(t, x)| < \infty$.
(ii) $|\kappa_{a,b}(t, x) x C'_\nu(t, x)| < \infty$.
(iii) $|\kappa_{a,b}(t, x) x^2 C''_\nu(t, x)| < \infty$.

Proof. It is well-known that

$$\sup_{0<y<\infty} \big| y^\alpha C_\nu(y) \big| < \infty. \tag{3.7}$$

We prove that assertion (i) of the lemma. Suppose that $0 < t < 1$ and $0 < x < 1$, then we have

$$\begin{aligned}
\left|\kappa_{a,b}(t,x)C_\nu(t,x)\right| &= \left|t^{-a}x^{\alpha}C_\nu(t,x)\right| \\
&= \left|t^{-a}(t,x)^{\alpha}C_\nu(t,x)t^{-\alpha}\right| \\
&= \left|t^{-a-\alpha}(t,x)^{\alpha}C_\nu(t,x)\right| \\
&= t^{-(a+\alpha)}\left|(tx)^{\alpha}C_\nu(tx)\right| \\
&< \infty.
\end{aligned}$$

Now suppose that $1 < t < \infty, 1 < x < \infty$, then

$$\begin{aligned}
\left|\kappa_{a,b}(t,x)C_\nu(t,x)\right| &= \left|t^{-b}x^{3-\alpha}(tx)^{\alpha}C_\nu(tx)t^{-\alpha}x^{-\alpha}\right| \\
&= \left|t^{-b-\alpha}x^{3-\alpha}(tx)^{\alpha}C_\nu(t,x)\right| \\
&= \left|t^{-b-\alpha}x^{3-\alpha}\right|\left|(tx)^{\alpha}C_\nu(tx)\right| \\
&< \infty.
\end{aligned}$$

The proofs of the assertions (ii) and (iii) of the lemma are similar to case (i) above and hence we omit the details.

The following result is useful for extending the conventional Millin-Hardy integral transformation to generalized functions.

Theorem 3.3. If $|\nu| \le \alpha \le \frac{1}{2}$ and $t > 0, x > 0$, then for fixed α, $t^{s-1}C_\nu \in \mathcal{MH}_\alpha(\Omega)$, where Re(s) < 0 and $C_\nu(t,x)$ is given as in (1.1).

Proof. By (3.7), $|y^{\alpha}C_\nu(y)| < \infty$, for all $y > 0$. Now we consider the following estimate.

$$\begin{aligned}
&\sup_{\substack{0<t<\infty\\0<x<\infty}}\left|\kappa_{a,b}(t,x)D_t^{k_1}\Delta_x^{k_2}t^{s-1}C_\nu(tx)\right| \\
&\quad= \sup_{\substack{0<t<\infty\\0<x<\infty}}\left|\kappa_{a,b}(t,x)D_t^{k_1}(t^{3-1})t^{2k_2}C_\nu(tx)\right| \\
&\quad= \sup_{\substack{0<t<\infty\\0<x<\infty}}\left|\kappa_{a,b}(t,x)\big[t^{2k_2+s-1}D_t^{k_1}C_\nu(tx)\right.
\end{aligned}$$

$$+ C_\nu(tx)\prod_{j=1}^{k_1}(2k_2+j+s-1)]t^{2k_2+s-k_1-1}\Big|$$

$$= \sup_{\substack{0<t<\infty\\0<x<\infty}}\Big|\kappa_{a,b}(t,x)t^{2k_2+s-1}D_t^{k_1}C_\nu(tx)\Big|$$

$$+ \sup_{\substack{0<t<\infty\\0<x<\infty}}\Big|\kappa_{a,b}(t,x)C_\nu(tx)\prod_{j=1}^{k_1}(2k_2+j+s-1)t^{2k_2+s-k_1-1}\big|.$$

Now for the first term on the right hand side of the inequality (2.11), we have

$$\begin{aligned}
&\sup_{\substack{0<t<\infty\\0<x<\infty}}\Big|\kappa_{a,b}(t,x)t^{2k_2+s-1}D_t^{k_1}C_\nu(tx)\Big|\\
&\leq \sup_{\substack{0<t<1\\0<x<1}}\Big|\kappa_{a,b}(t,x)t^{2k_2+s-1}D_t^{k_1}C_\nu(tx)\Big|\\
&\quad + \sup_{\substack{1<t<\infty\\1<x<\infty}}\Big|\kappa_{a,b}(t,x)t^{2k_2+s-1}D_t^{k_1}C_\nu(t,x)\Big|\\
&= \sup_{\substack{0<t<1\\0<x<1}}\Big|t^{-a}x^{\alpha}D^{k_1}C_\nu(tx)\Big|\\
&+ \sup_{\substack{1<t<\infty\\1<x<\infty}}\Big|t^{-b}x^{3-\alpha}t^{2k_2+s-1}D_t^{k_1}C_\nu(tx)\Big|\\
&= \sup_{\substack{0<t<1\\0<x<1}}\Big|t^{-a}x^{a}t^{-k_1}x^{-k_1}(tx)^{k_1}D_t^{k_1}C_\nu(tx)t^{2k_2+s-1}\Big|\\
&\quad + \sup_{\substack{1<t<\infty\\1<x<\infty}}\Big|t^{-b+2k_2+s-k_1-1}x^{s-\alpha-k_1}(tx)^{k_1}D_t^{k_1}C_\nu(tx)\big|\\
&= \sup_{\substack{0<t<1\\0<x<1}}\Big|t^{-(a+k_1)}x^{\alpha-k_1}\Big|\Big|(tx)^{k_1}D_t^{k_1}C_\nu(tx)\Big|\Big|t^{2k_2+s-1}\Big|\\
&\quad + \sup_{\substack{1<t<\infty\\1<x<\infty}}\Big|t^{-b+2k_2-k_1+s-1}x^{3-\alpha-k}\Big|\Big|(tx)^{k_1}D_t^{k_1}C_\nu(tx)\Big|\\
&< \infty \qquad [\text{by Lemma 3.2}]
\end{aligned} \tag{3.9}$$

Again for the second term on the right hand side of the inequality (3.7), we have

$$\sup_{\substack{0<t<\infty \\ 0<x<\infty}} \left| \kappa_{a,b}(t,x) C_\nu(tx) \prod j = 1^{k_1}(2k_2 + j + s + -1) t^{2k_2+s-k_1-1} \right|$$

$$\leq \sup_{\substack{0<t<\infty \\ 0<x<1}} \left| \kappa_{a,b}(t,x) C_\nu(t,x) \prod_{j=1}^{k_1}(2k_2 + +j + s - 1) t^{2k_2+s-k_1-1} \right|$$

$$+ \sup_{\substack{1<t<\infty \\ 0<x<\infty}} \left| \kappa_{a,b}(t,x) C_\nu(t,x) \prod_{j=1}^{k_1}(2k_2 + j + s - 1) t^{2k_2+s-k_1-1} \right|$$

$$= \sup_{\substack{0<t<1 \\ 0<x<1}} \left| t^{-a} x^{\alpha} C_\nu(tx) \prod_{j=1}^{k_1}(2k_2 + j + s - 1) t^{2k_2+s-k_1-1} \right|$$

$$+ \sup_{\substack{0<t<1 \\ 0<x<1}} \left| t^{-b} x^{3-\alpha} C_\nu(tx) \prod_{j=1}^{k_1}(2k + j + s - 1) t^{2k_2+s-k_1-1} \right| \tag{3.10}$$

$$= \sup_{\substack{0<t<1 \\ 0<x<1}} \left| \Pi_{j=1}^{k_1}(2k_2 + j + s - 1) t^{-(a+\alpha)} \right| \left| (tx)^{\alpha} C_\nu(tx) \right| \left| t^{2k_2+3-k_1-1} \right|$$

$$+ \sup_{\substack{1<t<\infty \\ 1<x<\infty}} \left| \prod_{j=1}^{k_1}(2k_2 + j + s - 1) t^{-b-\alpha} x^{3-2\alpha} \right| \left| (tx)^{\alpha} C_\nu(tx) \right| \left| t^{2k_2+s-k_1-1} \right|$$

$$< \infty. \qquad [\text{by Lemma3.2}]$$

From (3.9) and(3.10), we get

$$\sup_{\substack{0,t<\infty \\ 0<x<\infty}} \left| \kappa_{a,b}(t,x) D_t^{k_1} \Delta_x^{k_2} t^{s-1} C_\nu(tx) \right| < \infty.$$

$$i.e. \qquad \gamma_{a,b,k}^{\alpha}\big(t^{s-1}\, C_\nu(tx)\big) < \infty.$$

Hence $t^{s-1} C_\nu(tx) \in \mathcal{MH}(\Omega)$ and the proof of the lemma is complete.

4. Conventional and Generalized Mellin-Hardy Integral Transformations

In this section we extend the conventional Mellin-Hardy integral transformation to generalized functions. Let $\phi(t,x)$ be a conventional function defined over the domain Ω, where Ω is given by

$$\Omega = \big\{(t,x) : 0 < t < \infty,\, 0 < x < \infty\big\} \tag{4.1}$$

Then the conventional Mellin-Hardy integral transformation of $\phi(t,x)$ is defined as

$$\begin{aligned}\mathcal{F}(s,y) &= \mathcal{MH}_\alpha(\phi(t,x)) \\ &= \int_0^\infty t^{s-1}\,dt \int_0^\infty C_\nu(xy)\phi(t,x)\,dx \\ &= \int_0^\infty \int_0^\infty t^{s-1}c_\nu(xy)\phi(t,x)\,dx\,dt.\end{aligned} \tag{4.2}$$

We note that the conventional or classical Mellin-Hardy integral transformation $\mathcal{F}(s,y)$ is a mapping from Ω into the set of complex numbers. A result concerning the existence of $\mathcal{F}(s,y)$ is given in the following.

Theorem 4.1. The Mellin-Hardy conventional transformation exists for $b+\mathrm{Re(s)}<0$ and $|\nu|\le\alpha\le\frac{1}{2}$.

Proof. It is well known [6, page 36] that for an appropriate $M>0$, one has

$$|C_\nu(x,y)| \le M(xy)^{\frac{-1}{2}} \tag{4.3}$$

for all $x>0, y>0$. Therefore,

$$\begin{aligned}|\mathcal{F}(s,y)| &= \left|\int_0^\infty \int_0^\infty t^{s-1}C_\nu(xy)\phi(t,x)\,dx\,dt\right| \\ &\le \int_0^\infty \int_0^\infty \big|C_\nu(xy)|t^{s-1}\phi(t,x)\big|\,dx\,dt \\ &\le \int_0^\infty \int_0^\infty M(xy)^{\frac{-1}{2}}t^{s-1}|\phi(t,x)|\,dx\,dt \\ &= \int_0^1 \int_0^1 \frac{M(xy)^{\frac{-1}{2}}}{t^{-a}x^\alpha}t^{s-1}\gamma^\alpha_{a,b,0}(\phi(t,x))\,dx\,dt \\ &\quad + \int_1^\infty \int_1^\infty \frac{M(xy)^{\frac{-1}{2}}}{t^{-b}x^{3-\alpha}}t^{s-1}\gamma^\alpha_{a,b,0}(\phi(t,x))\,dx\,dt \\ &= y^{\frac{-1}{2}}M\gamma^\alpha_{a,b,0}(\phi(t,x))\int_0^1 \int_0^1 x^{\frac{-1}{2}}t^{s+a-1}x^{-\alpha}\,dx\,dt \\ &\quad + y^{\frac{-1}{2}}M\gamma a,b,0^\alpha(\phi(t,x))\int_0^\infty \int_1^\infty x^{\frac{-1}{2}}t^{-b}x^{\alpha-3}t^{s-1}\,dx\,dt\end{aligned} \tag{4.4}$$

Now for the first term in the inequality (3.4), we have

$$\begin{aligned}&\left(\int_0^1 x^{\frac{1}{2}-\alpha}\,dx\right)\left(\int_0^1 t^{Re(s)+a-1}\,dt\right)\\&=\left[x^{\frac{1}{2}-\alpha}\right]_0^1\left[t^{\mathrm{Re(s)+a}}\right]_0^1\\&<\infty.\end{aligned}\tag{4.5}$$

Similarly,

$$\begin{aligned}&\Big(\int_1^\infty\int_1^\infty x^{\frac{-1}{2}}t^{-b}x^{\alpha-3}t^{Re(s)-1}dxdt\Big)\\&=\Big(\int_1^\infty x^{\frac{-1}{2}+\alpha-3}dx\Big)\Big(\int_1^\infty t^{b+Re(s)-1}dt\Big)\\&=\Big(\int_1^\infty x^{\frac{-7}{2}+\alpha}dx\Big)\Big(\int_1^\infty t^{b+Re(s)-1}dt\Big)\\&\le\Big(\int_1^\infty x^{-3}dx\Big)\Big(\int_1^\infty t^{b+Re(s)-1}dt\Big)\\&=\lim_{n\to\infty}\left\{\left[\frac{x^{-2}}{-2}\right]_1^n\times\left[\frac{t^b+Re(s)}{b+Re(s)}\right]_1^n\right\}\\&\le\frac{1}{2|b+\mathrm{Re(s)}|}\lim_{n\to\infty}\left\{\left[x^{-2}\right]_1^n\times\left[t^b+\mathrm{Re(s)}\right]_1^{\mathrm{n}}\right\}\\&\le\frac{1}{2|b+\mathrm{Re(s)}|}\lim_{n\to\infty}\left\{(n^{-2}-1)(n^{b+\mathrm{Re(s)}-1})\right\}\\&=\frac{1}{2|b+\mathrm{Re(s)}|}\qquad[\because b+\mathrm{Re(s)}<0]\end{aligned}\tag{4.6}$$

From (4.4) and (4.5) and (4.6), it is clear that the integral on the right hand of 4.2 exists for $b+\mathrm{Re(s)}<0$. This is complete the proof.

Remark 4.1. Now let $\mathcal{MH}(\Omega)$ denote the dual of the testing function space $\mathcal{MH}(\Omega)$. Then for any $f(t,x)\in\mathcal{MH}(\Omega)$, we define its distributional Mellin-Hardy integral transformation by

$$\mathcal{F}(s,y)=\mathcal{MH}(f(t,x))=\big\langle f(t,x),t^{s-1}C_\nu(x,y)\big\rangle\tag{4.7}$$

where y is a non-zero real number, $x>0,t>0,|\nu|\le\alpha\le\frac{1}{2},b+\mathrm{Re(s)}<0$ and $C_\nu(xy)$ is same as defined in section 1. By Theorem 2.3, we know that for a fixed $y\neq 0$,

$t^{s-1}C_\nu(xy) \in \mathcal{MH}(\Omega)$ and therefore, the relation (4.7) is meaningful. Below we prove some order properties of the distributional Mellin-Hardy integral transformation.

Theorem 4.2 Let $\mathcal{F}(s,y)$ be the distributional Mellin-Hardy integral transformation of $f(t,x) \in \mathcal{MH}'(\Omega)$. Then,

$$\left.\begin{aligned} &|\mathcal{F}(s,y)| = 0(|sy|^{-\alpha}) \quad \text{as } y \to 0 \\ \text{and} \qquad &|\mathcal{F}(\text{s},\text{y})| = 0(|sy|^{4r-\alpha}) \quad \text{as } y \to \infty, \end{aligned}\right\} \tag{4.8}$$

where r is some nonnegative integer and α is a fixed positive number satisfying $|r| \leq \alpha \leq y_2$ and, $b + Re(s) < 0$.

Proof. In view of the result [9, page 18,19], there exists a constant $C > 0$ and a nonnegative integer r such that

$$\begin{aligned} |\mathcal{F}(s,y)| &= \left|\left\langle f(t,x), t^{s-1}C_\nu(xy)\right\rangle\right| \\ &\leq C \max_{\substack{0\leq k_1\leq r\\ 0\leq k_2\leq r}} \sup_{\Omega} \left|k_{a,b}(t,x) D_t^{k_1}\Delta_x^{k_2} t^{s-1}C_\nu(xy)\right| \\ &= C \max_{\substack{0\leq k_1\leq r\\ 0\leq k_2\leq r}} \sup_{\Omega} \left|k_{a,b}(t,x) D_t^{k_1} t^{s-1}\Delta_x^{k_2} C_\nu(xy)\right| \\ &= C \max_{\substack{0\leq k_1\leq r\\ 0\leq k_2\leq r}} \sup_{\Omega} \left|k_{a,b}(t,x)\, (\prod_{j=1}^{k_1}(s-j))t^{s-k-1}(-1)^{k_2}y^{2k_2}C_\nu(xy)\right| \\ &= C \max_{\substack{0\leq k_1\leq r\\ 0\leq k_2\leq r}} \sup_{\Omega} \left\{|(xy)^\alpha C_\nu(cy)x^{-\alpha}y^{2k_2-\alpha}k_{a,b}(t,x) \prod_{j=1}^{k_1}(s-j)t^{s-k-1}\Big|\right\} \\ &= C \max_{\substack{0\leq k_1\leq r\\ 0\leq k_2\leq r}} \sup_{\Omega} \left\{\Big|(xy)\, C_\nu(xy)\Big|\, \Big|sy\Big|^{2(k1+k_2)-\alpha}\right. \\ &\qquad\qquad \left.\times \left|x^{-\alpha}s^{-2k_2-2k_1+\alpha}y^{-2k_1}\, k_{a,b}(t,x)s\,\Pi_{j=1}^{k_1}(s-j)t^{s-k_1}-1\right|\right\}. \end{aligned} \tag{4.9}$$

Now $|(xy)^\alpha C_\nu(xy)|$ is bounded and

$$\left|x^{-\alpha}s^{-2k_2-2k_1} + y^{-2k_1}\kappa_{a,b}(t,x)\prod\nolimits_{j=1}^{k_1}(s-j)t^{s-k-1}\right|$$

is bounded and suppose B is their bound. Hence we get

$$|\mathcal{F}(s,y)| \leq CB \max_{\substack{0\leq k_1\leq r\\ 0\leq k_1\leq r}} |sy|^{2(k_1+k_2)-\alpha}$$

$$|\mathcal{F}(s,y)| = 0(|sy|)^{-\alpha} \quad \text{as } y \to 0.$$

and

$$|\mathcal{F}(s,y)| = 0(|sy|^{4r-\alpha}) \quad \text{as } y \to \infty.$$

The proof of theorem is complete.

Next we prove that the boundedness theorem for the distributional Melli-Hardy integral transformation.

Theorem If $\mathcal{F}(s,y) = \mathcal{MH}(f(t,x))$, $(t,y) \in \Omega_f'$, then $F(s,y)$ is bounded on any subset $\Omega_f' = \left\{(s,y) : b + \mathrm{Re(s)} < 0,\, 0 < \mathrm{y} < \infty\right\}$ of Ω_f according to

$$|\mathcal{F}(s,y) \leq |y|^{-\alpha} P(|sy|) \tag{4.10}$$

where, $P(|sy|)$ is a polynomial depending upon a, b and α, $|\nu| \leq \alpha \leq y_2$.

Proof. In view of Theorem 1.8.1 of Zemanian [9], there exists a constant $C > 0$ and a nonnegative integer r such that

$$\begin{aligned}
|\mathcal{F}(s,y)| &= \left|\left\langle f(t,x), t^{s-1}C_\nu(xy)\right\rangle\right| \\
&\leq C \max_{\substack{0\leq k_1\leq r\\ 0\leq k_2\leq r}} \sup_{\Omega} \left|\kappa_{a,b}(t,x) D_t^{k_1} \Delta_x^{k_2} t^{s-1} C_\nu(xy)\right| \\
&\leq C \max_{\substack{0\leq k_1\leq r\\ 0\leq k_2\leq r}} \sup_{\Omega} \left|\kappa_{a,b}(t,x)(-1)^{k_2} y^{2k_2} C_\nu(xy) \prod_{j=1}^{k_1}(s-j-1)t^{s-k-1}\right| \\
&= C \max_{\substack{0\leq k_1\leq r\\ 0\leq k_2\leq r}} \sup_{\Omega} \left|\kappa_{a,b}(t,x)\, C_\nu(xy) y^{2k_2} \prod_{j=1}^{k_1}(s-j-1)t^{s-k-1}\right|.
\end{aligned}$$

It can be readily seen that for an appropriate $m > 0$,

$$|C_\nu(xy)| \leq M\,(xy)^{\frac{-1}{2}},$$

for all $x > 0, y > 0$ and $|\nu| \leq \alpha \leq \frac{1}{2}$ ([6, page 251]). Hence,

$$|\mathcal{F}(s,y)| \leq CM \max_{\substack{0 \leq k_1 \leq r \\ 0 \leq k_2 \leq r}} \Big| \prod_{j=1}^{k_1(s-j-1)} y^{2k_2 - y_2} \Big| \Big| \kappa_{a,b}(t,x) x^{\frac{-1}{2}} t^{s-k_1-1} \Big|.$$

Now, $\sup |\kappa_{a,b}(t,x) t^{s-k_1-1} x^{\frac{-1}{2}}|$ is bounded for $b + \mathrm{Re(s)} < 0$ and for every $k_1 \in \mathbb{N}_+$ say by the constant $C' > 0$. Therefore, from (3.12), we obtain

$$\begin{aligned} |\mathcal{F}(s,y)| &\leq CMC' \max_{\substack{0 \leq k_1 \leq r \\ 0 \leq k_2 \leq r}} \Big| \prod_{j=1}^{k_1} (s-j-1) y^{2k_2 - \frac{1}{2}} \Big| \\ &= |y|^{\frac{-1}{2}} P(|sy|) \\ &\leq |y|^{-\alpha} P(|sy|), \end{aligned} \tag{4.11}$$

where $P(|sy|)$ is a polynomial depending upon a, b and α. This completes the proof.

Finally we mention that some related results of distributional Mellin-Hardy integral transformation will be reported elsewhere.

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