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A GENERALIZATION OF A THEOREM OF MANDEL

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Abstract. A theorem of Mandel allows to determine the covector set of an oriented matroid from its set of topes by using the composition condition. We provide a generalization of that result, stating that the covector set of a conditional oriented matroid can also be determined by its set of topes, but by using the face symmetry condition. It permits to represent geometrical configurations in terms of conditional oriented matroids, more suitable for computer calculations. We treat apartments of hyperplane arrangements as example.

Keywords: conditional oriented matroid; tope; hyperplane arrangement.

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1. Introduction

Recently, Bandelt, Chepoi and Knauer [2] introduced the notion of conditional oriented matroids, or complexes of oriented matroids, which are common generalizations of oriented matroids and lopsided sets. As observed by Richter-Gebert and Ziegler [7], oriented matroids are abstractions for several mathematical objects including directed graphs and central hyperplane arrangements, while Bandelt, Chepoi, Dress and Koolen [1] pointed out that lopsided sets can be regarded as common generalizations of antimatroids and median graphs. We provide a generalization of a theorem of Mandel in Section 2 by proving that a conditional oriented matroid

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can completely determined from knowledge of its tope and by means of the face symmetry condition. Knauer and Marc [5], independently without the author having prior knowledge, gave another version of that generalization in their Theorem 4.9 by using tope graphs. In Section 3, we propose an algorithm to convert apartments of hyperplane arrangements to conditional oriented matroids. It gives the possibility to do computations, like f-polynomial computing, on these geometrical configurations.

2. Conditional Oriented Matroids

This section describes conditional oriented matroids, recalls oriented matroids and the theorem of Mandel, then establishes our generalization of that theorem.

A *sign system* is a pair $(E, \mathscr{L})$ containing a finite set E and a subset $\mathscr{L}$ of $\{-1, 0, 1\}^E$. For $X,Y \in \mathscr{L}$, the *composition* of X and Y is the element $X \circ Y$ of $\{-1, 0, 1\}^E$ defined, for every $e \in E$, by

$$(X \circ Y)_e := \begin{cases} X_e & \text{if } X_e \neq 0, \\ Y_e & \text{otherwise,} \end{cases}$$

and the *separation set* of X and Y is $\mathrm{S}(X, Y) := \big\{e \in E \,\big|\, X_e = -Y_e \neq 0\big\}$. A *conditional oriented matroid* is a sign system $(E, \mathscr{L})$ such that $\mathscr{L}$ satisfies the following conditions:

- **(FS)** if $X,Y \in \mathscr{L}$, then $X \circ -Y \in \mathscr{L}$,
- **(SE)** for each pair $X,Y \in \mathscr{L}$, and every $e \in \mathrm{S}(X, Y)$, there exists $Z \in \mathscr{L}$ such that

$$Z_e = 0 \quad \text{and} \quad \forall f \in E \setminus \mathrm{S}(X, Y),\ Z_f = (X \circ Y)_f = (Y \circ X)_f.$$

(FS) stands for face symmetry, and (SE) for strong elimination condition.

The elements of $\mathscr{L}$ are called *covectors*. A partial order $\preceq$ is defined on $\mathscr{L}$ by

$$\forall X,Y \in \mathscr{L}:\ X \preceq Y \iff \forall e \in E, X_e \in \{0, Y_e\}.$$

Write $X \prec Y$ if $X \preceq Y$ and $X \neq Y$. One says that Y *covers* X, denoted $X \prec\!\!\cdot\, Y$, if $X \prec Y$ and no $Z \in \mathscr{L}$ satisfies $X \prec Z \prec Y$. One calls Y a *tope* if no $Z \in \mathscr{L}$ covers Y.

An *oriented matroid* is a sign system $(E, \mathscr{L})$ such that $\mathscr{L}$ satisfies the following conditions:

- **(C)** if $X,Y \in \mathscr{L}$, then $X \circ Y \in \mathscr{L}$, (**Sym**) if $X \in \mathscr{L}$, then $-X \in \mathscr{L}$,

(**SE**) for each pair $X, Y \in \mathscr{L}$, and every $e \in \mathrm{S}(X,Y)$, there exists $Z \in \mathscr{L}$ such that

$$Z_e = 0 \quad \text{and} \quad \forall f \in E \setminus \mathrm{S}(X,Y),\ Z_f = (X \circ Y)_f = (Y \circ X)_f.$$

(C) stands for composition, and (Sym) for symmetry condition. For the sake of understanding, we provide a proof to the following known property.

Proposition 2.1. *An oriented matroid is a conditional oriented matroid* $(E, \mathscr{L})$ *that satisfies the zero vector condition*

(**Z**) *the zero element* $(0, \dots, 0)$ *belongs to* $\mathscr{L}$.

Proof. Every oriented matroid is a conditional oriented matroid since both satisfy (SE), and one gets (FS) by combining (C) with (Sym). Moreover, it is obvious that every oriented matroid satisfies (Z).

Now, suppose that $(E, \mathscr{L})$ is a conditional oriented matroid satisfying (Z):

- For every $X \in \mathscr{L}$, $(0, \dots, 0) \circ -X = -X \in \mathscr{L}$, so we get (Sym).
- If $X, Y \in \mathscr{L}$, then $-Y \in \mathscr{L}$, hence $X \circ -(-Y) = X \circ Y \in \mathscr{L}$, so we get (C).

□

We recall the Theorem of Mandel as stated by Theorem 4.2.13 in the book of Björner, Las Vergnas, Sturmfels, White and Ziegler [3]. One can look at Theorem 1.1 in the article of Cordovil [4] for a version using non-Radon partitions.

Theorem 2.2. *Let* $(E, \mathscr{L})$ *be an oriented matroid. Its set of topes* $\mathscr{T}$ *determines* $\mathscr{L}$ *via*

$$\mathscr{L} = \left\{ X \in \{-1, 0, 1\}^E \;\middle|\; \forall T \in \mathscr{T}, X \circ T \in \mathscr{T} \right\}.$$

Coming back to conditional oriented matroids, the *rank* of a covector X is 0 if it covers no elements in $\mathscr{L}$, otherwise it is

$$\mathrm{rk} X := \max\{ l \in \mathbb{N} \mid \exists X^1, X^2, \dots, X^l \in \mathscr{L}, X^1 \prec X^2 \prec \dots \prec X^l \prec X \}.$$

The rank of $\mathscr{L}$ is

$$\mathrm{rk} \mathscr{L} := \max\{ \mathrm{rk} X \mid X \in \mathscr{L} \}.$$

The *support* of X is $\underline{X} := \{e \in E \mid X_e \neq 0\}$. And for $A \subseteq E$, the *restriction* of X to $E \setminus A$ is the element $X \setminus A \in \{-1, 0, 1\}^{E \setminus A}$ such that $(X \setminus A)_e = X_e$ for all $e \in E \setminus A$.

Lemma 2.3. [2, Lem. 1] *Let $(E, \mathscr{L})$ be a conditional oriented matroid, and $A \subseteq E$.*

- *The* deletion $(E \setminus A, \mathscr{L} \setminus A)$ *of A, with $\mathscr{L} \setminus A = \{X \setminus A \mid X \in \mathscr{L}\}$,*
- *and the* contraction $(E \setminus A, \mathscr{L}/A)$ *of A, with $\mathscr{L}/A = \{X \setminus A \mid X \in \mathscr{L}, \underline{X} \cap A = \varnothing\}$,*

are conditional oriented matroids.

Lemma 2.4. *Let $(E, \mathscr{L})$ be a conditional oriented matroid, and take two topes $T^1, T^2 \in \mathscr{L}$. Then, $\underline{T^1} = \underline{T^2}$.*

Proof. Suppose that $\underline{T^1} \neq \underline{T^2}$. Then, $T^1 \circ -T^2 \in \mathscr{L}$ and $T^1 \prec T^1 \circ -T^2$. This implies that T^1 is not a tope, which is absurd. □

We can now state our generalization.

Theorem 2.5. *Let $(E, \mathscr{L})$ be a conditional oriented matroid. Its set of topes $\mathscr{T}$ determines $\mathscr{L}$ via*

$$\mathscr{L} = \big\{X \in \{-1, 0, 1\}^E \;\big|\; \forall T \in \mathscr{T}, X \circ -T \in \mathscr{T}\big\}.$$

Proof. It is clear that $\mathscr{L} \subseteq \big\{X \in \{-1, 0, 1\}^E \;\big|\; \forall T \in \mathscr{T}, X \circ T \in \mathscr{T}\big\}$ since, for every $X \in \mathscr{L}$ and all $T \in \mathscr{T}$, $X \circ -T \in \mathscr{L}$ and $(X \circ T)^0 = \varnothing$.

For the backward argument, we argue by induction on $\operatorname{rk} \mathscr{L}$ and $\#E$. If $\operatorname{rk} \mathscr{L} = 0$, $\mathscr{L}$ consists of a one-element set $\{T\} \subseteq \{-1, 0, 1\}^E$. Therefore, for an element $X \in \{-1, 0, 1\}^E$, the fact $X \circ -T = T$ implies $X = T$, hence $X \in \mathscr{L}$.

If $\operatorname{rk} \mathscr{L} = 1$ and $\#E = 1$, then $\mathscr{L} = \{-1, 0, 1\}$ and $\mathscr{T} = \{-1, 1\}$. So, we clearly have $X \in \mathscr{L}$ for all $X \in \{-1, 0, 1\}$.

Now, assume that $\operatorname{rk} \mathscr{L} = 1$ and $\#E > 1$. Take $X \in \{-1, 0, 1\}^E$ such that $X \circ -T \in \mathscr{T}$ for each tope $T \in \mathscr{T}$. Denote by F the subset of E such that $\underline{T} = F$ for every $T \in \mathscr{T}$. The case $\underline{X} = F$ is easily solved, since $X = X \circ -T \in \mathscr{T} \subseteq \mathscr{L}$. The case $\underline{X} \subsetneq F$ remains open. Pick an element $e \in X^0 \cap F$, and consider a tope $Y \setminus \{e\}$ of the deletion $(E \setminus \{e\}, \mathscr{L} \setminus \{e\})$. We have

$Y^0 = \{e\} \cap F$, and Y is covered in $\mathscr{L}$ by two topes $T^1, T^2 \in \mathscr{T}$ such that $\mathrm{S}(T^1, T^2) = \{e\}$. There exists $Z \in \mathscr{L}$ such that

$$Z_e = 0 \quad \text{and} \quad \forall f \in E \setminus \{e\},\ Z_f = \big((X \circ -T^1) \circ (X \circ -T^2)\big)_f = (X \circ -Y)_f.$$

The only possibility is $Z = X \circ -Y$, which means that $X \circ -Y \in \mathscr{L}$. Hence, for all topes $Y \setminus \{e\}$ in the deletion $(E \setminus \{e\}, \mathscr{L} \setminus \{e\})$, we have $X \setminus \{e\} \circ -(Y \setminus \{e\}) \in \mathscr{L} \setminus \{e\}$. By induction, we get $X \setminus \{e\} \in \mathscr{L} \setminus \{e\}$, and consequently $X \in \mathscr{L}$.

Finally, assume that $\mathrm{rk}\,\mathscr{L} > 1$. Take $X \in \{-1, 0, 1\}^E$ such that $X \circ -T \in \mathscr{T}$ for each tope $T \in \mathscr{T}$. The case $\underline{X} = F$ is easily solved like before. The case $\underline{X} \subsetneq F$ remains. Pick an element $e \in X^0 \cap F$, and consider a tope $Y \setminus \{e\}$ of the contraction $(E \setminus \{e\}, \mathscr{L}/\{e\})$. We have $Y^0 = \{e\} \cap F$, and Y is covered in $\mathscr{L}$ by two topes $T^1, T^2 \in \mathscr{T}$ such that $\mathrm{S}(T^1, T^2) = \{e\}$. There exists $Z \in \mathscr{L}$ such that

$$Z_e = 0 \quad \text{and} \quad \forall f \in E \setminus \{e\},\ Z_f = \big((X \circ -T^1) \circ (X \circ -T^2)\big)_f = (X \circ -Y)_f.$$

The only possibility is $Z = X \circ -Y$, which means that $X \circ -Y \in \mathscr{L}$. Hence, for all topes $Y \setminus \{e\}$ in the contraction $(E \setminus \{e\}, \mathscr{L}/\{e\})$, we have $X \setminus \{e\} \circ -(Y \setminus \{e\}) \in \mathscr{L}/\{e\}$. Since $\mathrm{rk}\,\mathscr{L}/\{e\} = \mathrm{rk}\,\mathscr{L} - 1$, then $X \setminus \{e\} \in \mathscr{L}/\{e\}$ by induction, and consequently $X \in \mathscr{L}$. □

3. Applications on Hyperplane Arrangements

This section describes the structure of apartments of hyperplane arrangements in term of conditional oriented matroids. Then, it proposes an algorithm to convert the former to the latter. We give the f-polynomial computing as extension example of this algorithm.

Let $a_1, \dots, a_n, b$ be $n+1$ real coefficients such that $(a_1, \dots, a_n) \neq (0, \dots, 0)$. A *hyperplane* of $\mathbb{R}^n$ is an affine subspace $H := \big\{(x_1, \dots, x_n) \in \mathbb{R}^n \;\big|\; a_1x_1 + \cdots + a_nx_n = b\big\}$ denoted by $\{a_1x_1 + \cdots + a_nx_n = b\}$. A *hyperplane arrangement* $\mathscr{A}$ is a finite set of hyperplanes. Denote by H^{-1} and H^1 both connected components $\{a_1x_1 + \cdots + a_nx_n < b\}$ and $\{a_1x_1 + \cdots + a_nx_n > b\}$ of $\mathbb{R}^n$, respectively. Moreover, set $H^0 = H$. The *sign map* of H is the function

$$\sigma_H : \mathbb{R}^n \to \{-1, 0, 1\}, \quad v \mapsto \begin{cases} -1 & \text{if } v \in H^{-1}, \\ 0 & \text{if } v \in H^0, \\ 1 & \text{if } v \in H^1. \end{cases}$$

The sign map of $\mathscr{A}$ is the function $\sigma_{\mathscr{A}} : \mathbb{R}^n \to \{-1, 0, 1\}^{\mathscr{A}},\ v \mapsto \big(\sigma_H(v)\big)_{H\in\mathscr{A}}$. And the *sign set* of $\mathscr{A}$ is the set $\sigma_{\mathscr{A}}(\mathbb{R}^n) := \big\{\sigma_{\mathscr{A}}(v) \;\big|\; v \in \mathbb{R}^n\big\}$. A *face* of $\mathscr{A}$ is a subset F of $\mathbb{R}^n$ such that

$$\exists x \in \sigma_{\mathscr{A}}(\mathbb{R}^n),\ F = \big\{v \in \mathbb{R}^n \;\big|\; \sigma_{\mathscr{A}}(v) = x\big\}.$$

A *chamber* of $\mathscr{A}$ is a face F such that $\sigma_{\mathscr{A}}(F) \in \{-1, 1\}^{\mathscr{A}}$. Denote by $F(\mathscr{A})$ and $C(\mathscr{A})$ the sets composed by the faces and the chambers of $\mathscr{A}$, respectively. An *apartment* of $\mathscr{A}$ is a chamber of a hyperplane arrangement contained in $\mathscr{A}$. Denote by $K(\mathscr{A})$ the apartment set of $\mathscr{A}$. The sets of faces and chambers in an apartment $K \in K(\mathscr{A})$ are, respectively,

$$F(\mathscr{A}, K) := \big\{F \in F(\mathscr{A}) \mid F \subseteq K\big\} \quad \text{and} \quad C(\mathscr{A}, K) := C(\mathscr{A}) \cap F(\mathscr{A}, K).$$

Let $K \in K(\mathscr{A})$, and $\mathscr{B} = \{H \in \mathscr{A} \mid H \cap K = \varnothing\}$. The sign system

$$\Big(\mathscr{A} \setminus \mathscr{B},\, \sigma_{\mathscr{A}}\big(F(\mathscr{A}, K)\big) \setminus \mathscr{B}\Big)$$

is a conditional oriented matroid. Bandelt, Chepoi and Knauer [2] called it realizable COMs, and presented it as motivating example for conditional oriented matroids.

We now present algorithms to do computations on apartments of hyperplane arrangements. The use of mathematics software system containing the following functions is assumed:

- **length** gives the length of a tuple,
- **RandomElement** returns randomly an element from a set,
- **poset** transforms a set, on which a partial order can be defined, to a poset,
- and **rank** computes the rank of a poset or that of its elements.

Algorithm 1. (Generating Conditional Oriented Matroid from Topes):

- *Input:* A tope set $\mathscr{T}$.
- *Output:* A covector set $\mathscr{L}$.
- *Remark:* It is an algorithmic version of Theorem 2.5.

```
function GeneratingCOM(T)
    L ← {}
    l ← length(RandomElement(T))
    for X in {−1, 0, 1}^l
        a ← true
        for Y in T
            a ← a and (X ∘ -Y in T)
        if a = true
            L ← L ⊔ {X}
    return L
```

We generate tope set by determining $\sigma_{\mathscr{A}}(v)$ for a random point of each chamber. Afterwards, we apply the previous algorithm to get the aimed conditional oriented matroid.

Algorithm 2. (Transforming Apartment to Conditional Oriented Matroid):

- *Input:* An affine function set $\mathscr{A}$ and a point set P.
- *Output:* A covector set $\mathscr{L}$.
- *Remark:* Each function in $\mathscr{A}$ corresponds to a hyperplane of the arrangement, and each point in P is included in a chamber of the arrangement.

```
function ApartmentToCOM(A, P)
    function ApartmentToTope(A, P)
        function covector(A, p)
            function sign(h, p)
                if h(p) < 0
                    return -1
                else
                    return 1
            return tuple(sign(h, p) for h in A)
        return set(covector(A, p) for p in P)
    return GeneratingCOM(ApartmentToTope(A, P))
```

Consider an apartment $K \in K(\mathscr{A})$ in $\mathbb{R}^n$. Let $f_i(K)$ be the number of i-dimensional faces in $F(\mathscr{A}, K)$, and x a variable. The *f-polynomial* of K is

$$f_K(x) := \sum_{i=0}^{n} f_i(K)\, x^{n-i}.$$

Algorithm 3. (f-Polynomial of Apartment):

- *Input:* An affine function set $\mathscr{A}$ and a point set P.
- *Output:* A f-polynomial $f_K(x)$.
- *Remark:* An apartment is still represented by a pair $(\mathscr{A}, E)$.

function fPolynomial(A, P)

  x **variable**

  f ← 0

  COM ← **poset**(**ApartmentToCOM**(A, P))

  for X **in** COM

    f ← f + $x^{\mathbf{rank}(COM)-\mathbf{rank}(X)}$

  return f

Other computer calculations of functions associated to apartments of hyperplane arrangements, like their Varchenko determinants [6, Th. 1.3], can also be implemented by means of their conversion to conditional oriented matroids.

Example. Consider the apartment on Figure 1 with arrangement composed by the hyperplanes $\{x_2 = 0\}$, $\{x_1 - x_2 = 0\}$, $\{x_1 + x_2 = 1\}$, $\{x_2 = 3\}$, and $\{x_2 = -2\}$. To be able to compute the corresponding conditional oriented arrangement, and its f-polynomial, we take the nine points $(0,4)$, $(0,1.5)$, $(0,0.5)$, $(0.5,0.2)$, $(1,0.2)$, $(-1,-0.2)$, $(0,-0.5)$, $(1.5,-0.2)$, and $(0,-3)$ into account. A computation with the mathematics software system SageMath gives us the following result.

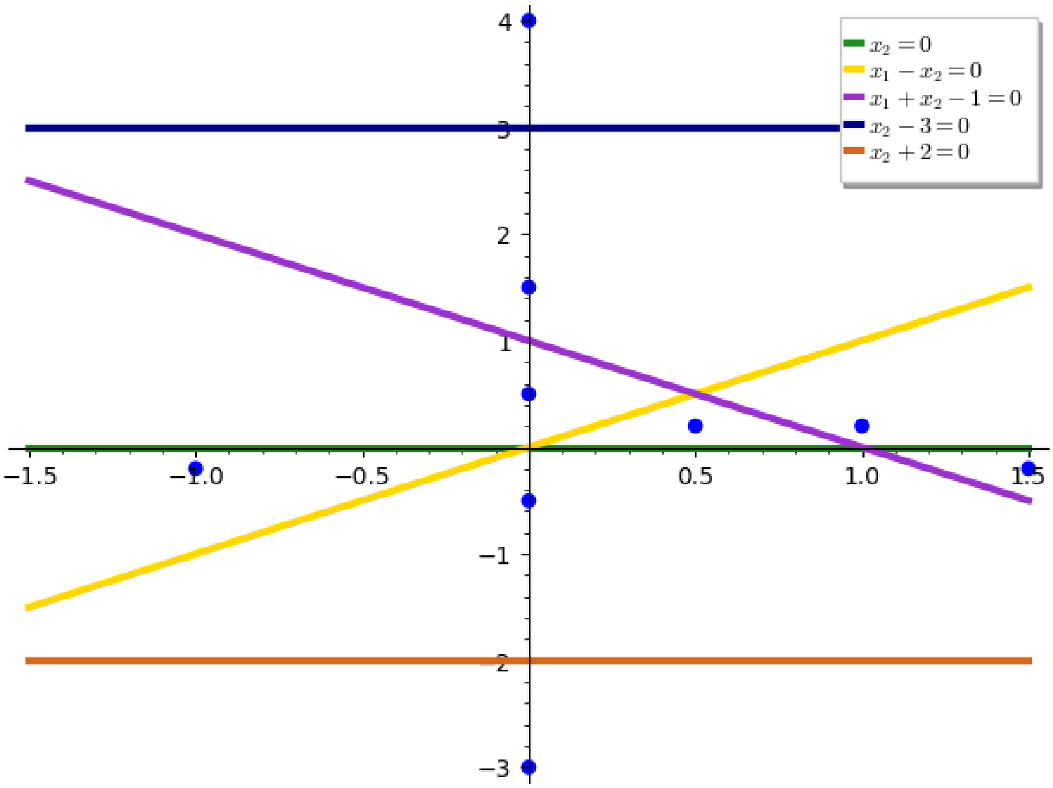

FIGURE 1. An Apartment of a Hyperplane Arrangement

```
sage: ApartmentToCOM(A, P)
{(1, -1, -1, -1, 1), (1, 0, -1, -1, 1), (1, 1, 0, -1, 1), (0, 1, 1, -1, 1),
(-1, 1, 0, -1, 1), (1, 0, 0, -1, 1), (1, -1, 1, 0, 1), (1, 1, 1, -1, 1),
(1, -1, 0, -1, 1), (0, 1, -1, -1, 1), (1, -1, 1, -1, 1), (1, 0, 1, -1, 1),
(-1, 1, 1, -1, 1), (-1, 1, -1, -1, 0), (0, -1, -1, -1, 1), (0, 0, -1, -1, 1),
(0, 1, 0, -1, 1), (1, -1, 1, 1, 1), (1, 1, -1, -1, 1), (-1, 0, -1, -1, 1),
(-1, 1, -1, -1, -1), (-1, -1, -1, -1, 1), (-1, 1, -1, -1, 1)}
sage: fPolynomial(A, P)
3*x^2 + 11*x + 9
```

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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