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A NEW ITERATIVE ALGORITHM FOR SUZUKI GENERALIZED NONEXPANSIVE MAPPING IN HYPERBOLIC SPACE

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Abstract. In this paper, we provide certain fixed point results for a Suzuki's generalized nonexpansive mapping, as well as a new iterative algorithm for approximating the fixed point of this class of mappings in the setting of hyperbolic spaces. Furthermore, we establish strong and Δ-converges theorem for Suzuki's generalized nonexpansive mapping in hyperbolic space. Finally, we present a numerical example to illustrate our main result and then display the efficiency of the proposed algorithm compared to different iterative algorithms in the literature. Our results obtained in this paper improve, extend and unify some related results in the literature.

Keywords: hyperbolic space; Suzuki's generalized nonexpansive mapping; strong and Δ-convergence theorems.
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1. INTRODUCTION

Once an existence of a solution for an operator equation is established then in many cases, such solution cannot be obtained by using ordinary analytical methods. To overcome such cases, one needs the approximate value of this solution. To do this, we first rearrange the operator equation in the form of fixed-point equation. We apply the most suitable iterative algorithm on the fixed point equation, and the limit of the sequence generated by this most suitable algorithm is in fact the value of the desired fixed point for the fixed point equation and the solution for the

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operator equation. The Banach Fixed Point Theorem [5] suggests the elementry Picard iteration $x_{n+1} = \mathfrak{G}x_n$ in the case of contraction mappings. Since for the class of nonexpansive mappings, Picard iterates do not always converge to a fixed point of a certain nonexpansive mapping, we, therefore use some other iterative processes involving different steps and set of parameters. Among the other things, Mann [23], Ishikawa [13], Noor [24], S iteration of Agarwal et al. [2], Abbas [1], Thakur [28] and Hussain [11] are the most studied iterative processes. In 2018, Ullah and Arshad introduced M [29] iteration process for Suzuki mappings and proved that it converges faster than all of these iteration processes.

Very recently, Dashputre et al. [10] introduced the novel iteration process, namely, SRJ iterative scheme for $x_1 \in K$, construct a sequence $\{x_n\}$ in K as follows:

$$
\begin{aligned}
&x_1 \in K \\
&z_n = \mathfrak{G}((1-\alpha_n)x_n + \alpha_n \mathfrak{G}x_n) \\
&y_n = \mathfrak{G}((1-\beta_n)z_n + \beta_n \mathfrak{G}z_n) \\
&x_{n+1} = \mathfrak{G}((1-\gamma_n)y_n + \gamma_n \mathfrak{G}y_n), n \geq 1
\end{aligned} \tag{1.1}
$$

where $\{\alpha_m\}$,$\{\beta_m\}$ and $\{\gamma_m\}$ are appropriate sequences in the interval (0,1).

They showed that the SRJ iteration (1.1) is stable and has a better rate of convergence when compared with the other iterations in the setting of generalized contractions.

Let K be a nonempty subset of a metric space (X,d) and $\mathfrak{G}\colon K \to K$ be a nonlinear mapping. The fixed point set of $\mathfrak{G}$ is denoted by $F(\mathfrak{G})$, that is, $F(\mathfrak{G}) = \{x \in K\colon x = \mathfrak{G}x\}$.

Remember that a selfmap $\mathfrak{G}$ on a metric space subset K is called nonexpansive if

$$
d(\mathfrak{G}x, \mathfrak{G}y) \leq d(x,y) \ \forall x,y \in K. \tag{1.2}
$$

Nowadays, the study of fixed points for nonexpansive operators is an important and active research field. One of earlier results states that nonexpansive operators always admit a fixed point on closed bounded and convex subsets in the framework of uniform convexity of Banach space. Suzuki [26] made a significant breakthrough in 2008 by introducing a weak notion of nonexpansive operators. It is worth noting that a selfmap $\mathfrak{G}$ of a metric space subset K is said to satisfy Condition (C) (also known as Suzuki map) if for any $x, y \in K$, we have

$$\frac{1}{2}d(x, \mathfrak{G}x) \leq d(x,y) \implies d(\mathfrak{G}x, \mathfrak{G}y) \leq d(x,y). \tag{1.3}$$

Remark 1.1. It is clear that every nonexpansive map is Suzuki nonexpansive. However, an example in [26] shows that there exists maps which are Suzuki nonexpansive but not nonexpansive.

2. Preliminaries

Throughout this paper, we consider the following definition of a hyperbolic space introduced by Kohlenbach [20].

Definition 2.1. A metric space (X,d) is said to be a hyperbolic space if there exists a map $W\colon X^2 \times [0,1] \to X$ satisfying
(i) $d(\rho, W(x,y,\alpha)) \leq \alpha d(\rho, x) + (1-\alpha)d(\rho, y)$,
(ii) $d(W(x,y,\alpha), W(x,y,\beta)) = |\alpha - \beta| d(x,y)$,
(iii) $W(x,y,\alpha) = W(y,x,(1-\alpha))$,
(iv) $d(W(x,z,\alpha), W(y,w,\alpha)) \leq \alpha d(x,y) + (1-\alpha)d(z,w)$,
for all $x,y,z,w \in X$ and $\alpha, \beta \in [0,1]$.

Definition 2.2. [27] A metric space is said to be convex, if a triple (X,d,W) satisfy only (i) in Definition 2.1.

Definition 2.3. [27] A subset K of a hyperbolic space X is said to be convex, if $W(x,y,\alpha) \in K$ for all $x, y \in K$ and $\alpha \in [0,1]$.

If $x,y \in X$ and $\lambda \in [0,1]$, then we use the notation $(1-\lambda)x \oplus \lambda y$ for $W(x,y,\lambda)$. The following holds even for more general setting of convex metric space [27] : for all $x,y \in X$ and $\lambda \in [0,1]$,

$$d(x, (1-\lambda)x \oplus \lambda y) = \lambda d(x,y)$$

and

$$d(y, (1-\lambda)x \oplus \lambda y) = (1-\lambda) d(x,y).$$

Thus

$$1x \oplus 0y = x,\ 0x \oplus 1y = y$$

and

$$(1-\lambda)x \oplus \lambda x = \lambda x \oplus (1-\lambda)x = x.$$

Definition 2.4. [21] A hyperbolic space (X, ∂, W) is said to be uniformly convex, if for any $\rho, x, y \in X,\ r > 0$ and $\varepsilon \in (0,2]$, there exists a $\delta \in (0,1]$ such that

$$d\left(\frac{1}{2}x \oplus \frac{1}{2}y, \rho\right) \le (1-\delta)r,$$

whenever $d(x,\rho) \le r,\ d(y,\rho) \le r$ and $d(x,y) \ge \varepsilon r$.

Definition 2.5. A map $\eta\colon (0,\infty)\times(0,2] \to (0,1]$ which provides such a $\delta = \eta(r,\varepsilon)$ for given $r > 0$ and $\varepsilon \in (0,2]$, is known as modulus of uniform convexity. We call η monotone if it decreases with r (for a fixed ε).

In [21], Luestean proved that every CAT(0) space is a uniformly convex hyperbolic space with modulus of uniform convexity $\eta(r,\varepsilon) = \frac{\varepsilon^2}{8}$ quadratic in ε.

Now we give the concept of Δ-convergence and some of its basic properties.

Let K be a nonempty subset of metric space (X,d) and $\{y_n\}$ be any bounded sequence in X while $diam(K)$ denotes the diameter of K. Consider a continuous functional $r_a(.,\{y_n\})\colon X \to R^+$ defined by

$$r_a(y,\{y_n\}) = \limsup_{n\to\infty} d(y_n, y),\ y \in X.$$

The infimum of $r_a(.,\{y_n\})$ over K is said to be an asymptotic radius of $\{y_n\}$ with respect to K and it is denoted by $r_a(K,\{y_n\})$. A point $z \in K$ is said to be an asymptotic center of the sequence $\{y_n\}$ with respect to K if

$$r_a(z,\{y_n\}) = inf\{r_a(y,\{y_n\})\colon y \in K\}.$$

The set of all asymptotic center of $\{y_n\}$ with respect to K is denoted by $AC(K,\{y_n\})$. The set $AC(K,\{y_n\})$ may be empty, singleton or have infinitely many points. If the asymptotic radius and asymptotic center are taken with respect to whole space X, then they are denoted by $r_a(X,\{y_n\}) = r_a(\{y_n\})$ and $AC(X,\{y_n\}) = AC(\{y_n\})$, respectively. We know that for $y \in X,\ r_a(y,\{y_n\}) = 0$ if and only if $\lim_{n\to\infty} y_n = y$ and every bounded sequence has a unique asymptotic center with respect to closed convex subset in uniformly convex Banach spaces.

Definition 2.6. The sequence $\{y_n\}$ in X is said to be Δ-convergent to $y \in X$, if y is unique asymptotic center of the every subsequence $\{u_n\}$ of $\{y_n\}$. In this case, we write $\Delta - \lim_{n\to\infty} y_n = y$ and call y is the Δ-limit of $\{y_n\}$.

Lemma 2.7. [22] *Let (X,d,W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset K of X.*

Consider the following lemma of Khan et al. [17] which we use in the sequel.

Lemma 2.8. *Let (X,d,W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{t_n\}$ be a sequence in $[a,b]$ for some $a,b \in (0,1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that*

$$\limsup_{n\to\infty} d(x_n,x) \leq c,$$

$$\limsup_{n\to\infty} d(y_n,x) \leq c$$

and

$$\limsup_{n\to\infty} d(W(x_n,y_n,t_n),x) = c$$

for some $c \geq 0$, then $\lim_{n\to\infty} d(x_n,y_n) = 0$.

Definition 2.9. Let K be a nonempty convex closed subset of a hyperbolic space X and $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ is said to be Fejér monotone with respect to M if for all $x \in K$ and $n \in \mathbb{N}$,

$$d(x_{n+1},x) \leq d(x_n,x).$$

Assume that K is a nonempty subset of a hyperbolic space (X,d) and $\mathfrak{G}\colon K \to K$ is a mapping and $F(\mathfrak{G}) = t \in K\colon \mathfrak{G}t = t$ is the set of all fixed points of the map $\mathfrak{G}$. The mapping $\mathfrak{G}\colon K \to K$ is called nonexpansive, if $||\mathfrak{G}t - \mathfrak{G}\rho|| \leq ||t-\rho||$ for all $t,\rho \in K$ and is called quasi-nonexpansive, if $F(\mathfrak{G}) \neq \emptyset$ and $||\mathfrak{G}t - q|| \leq ||t-q||$ for all $t \in K$ and $q \in F(\mathfrak{G})$.

We can easily prove the following Proposition.

Proposition 2.10. Let $\{x_n\}$ be a sequence in X and K be a nonempty subset of X. Let $\mathfrak{G}: K \to K$ be a nonexpansive mapping with $F(\mathfrak{G}) \neq \emptyset$. Suppose that $\{x_n\}$ is Fejér monotone with respect to K. Then we have the followings:

(1) $\{x_n\}$ is bounded.

(2) The sequence $\{d(x_n, p)\}$ is decreasing and converges for all $p \in F(\mathfrak{G})$.

(3) $\lim_{n\to\infty} D(x_n, F(\mathfrak{G}))$ exists, where $D(x, A) = \inf_{y\in A} d(x, y)$.

Definition 2.11. Assume that K is a nonempty subset of a hyperbolic space X and $\mathfrak{G}: K \to K$ is a Suzuki's generalized nonexpansive mapping with $F(\mathfrak{G}) \neq \emptyset$. Then $\mathfrak{G}$ is quasi-nonexpansive.

Lemma 2.12. [25] *Let X be complete uniformly convex hyperbolic space with monotone modulus of convexity η, K be a nonempty closed convex subset of X and $\mathfrak{G}: K \to K$ be a Suzuki's generalized nonexpansive mapping. If $\{x_n\}$ is a bounded sequence in K such that* $\lim_{n\to\infty} d(x_n, \mathfrak{G}x_n) = 0$*, then $\mathfrak{G}$ has a fixed point in K.*

Lemma 2.13. [25] *Let K be a nonempty, bounded, closed and convex subset of a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η and $\mathfrak{G}$ be a Suzuki's generalized nonexpansive mapping on K. Suppose that $\{x_n\}$ is a sequence in K, with $d(x_n, \mathfrak{G}x_n) \to 0$. If $AC(K, \{x_n\}) = \rho$, then ρ is a fixed point of $\mathfrak{G}$. Moreover, $F(\mathfrak{G})$ is closed and convex.*

3. Main Results

Now, we establish the convergence results for SRJ-iteration process for Suzuki's generalized nonexpansive mappings in hyperbolic spaces, as follows: Let K be a nonempty, closed and convex subset of a hyperbolic space X and $\mathfrak{G}$ be a Suzuki's generalized nonexpansive mapping on K. For any $x_1 \in K$ the sequence $\{x_n\}$ is defined by

$$
\begin{cases}
z_n = W(\mathfrak{G}\sigma_n, 0, 0), \\
\sigma_n = W(x_n, \mathfrak{G}x_n, \alpha_n), \\
y_n = W(\mathfrak{G}\nu_n, 0, 0), \\
\nu_n = W(z_n, \mathfrak{G}z_n, \beta_n), \\
x_{n+1} = W(\mathfrak{G}\rho_n, 0, 0), \\
\rho_n = W(y_n, \mathfrak{G}y_n, \gamma_n), \ \forall n \in \mathbb{N}
\end{cases} \tag{3.1}
$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in (0,1). This section establishes some significant strong and Δ-convergence results for operators with Suzuki's generalized nonexpansive mapping. Our results will generalize the results of Ullah et al. [29].

Theorem 3.1. *Let K be a nonempty, closed and convex subset of a hyperbolic space X and $\mathfrak{G}\colon K \to K$ be a Suzuki's generalized nonexpansive mapping. If $\{x_n\}$ is a sequence defined by (3.1), then $\{x_n\}$ is Fejér monotone with respect to $F(\mathfrak{G})$.*

Proof. Since $\mathfrak{G}$ is a Suzuki's generalized nonexpansive, for $\rho \in F(\mathfrak{G})$, we have

$$\frac{1}{2}d(\rho, \mathfrak{G}\rho) = 0 \le d(\rho, x_n),$$

$$\frac{1}{2}d(\rho, \mathfrak{G}\rho) = 0 \le d(\rho, y_n)$$

and

$$\frac{1}{2}d(\rho, \mathfrak{G}\rho) = 0 \le d(\rho, z_n),$$

for all $n \in \mathbb{N}$. Now, using (3.1) and Definition 2.11,

$$d(\mathfrak{G}\rho, \mathfrak{G}x_n) \le d(\rho, x_n),$$

$$d(\mathfrak{G}\rho, \mathfrak{G}y_n) \le d(\rho, y_n)$$

and

$$d(\mathfrak{G}\rho, \mathfrak{G}z_n) \le d(\rho, z_n). \tag{3.2}$$

Using Definition 2.11 and (3.1), we get

$$
\begin{aligned}
d(z_n, p) &= d(W(\mathfrak{G}\sigma_n, 0, 0)), \rho) \\
&= d(\mathfrak{G}\sigma_n, \rho) \\
&\leq d(\sigma_n, \rho) \\
&= d(W(x_n, \mathfrak{G}x_n, \alpha_n), \rho) \\
&\leq ((1-\alpha_n))d(x_n, \rho) + \alpha_n d(\mathfrak{G}x_n, \rho) \\
&\leq (1-\alpha_n)d(x_n, \rho) + \alpha_n d(x_n, \rho) \\
&\leq d(x_n, \rho).
\end{aligned} \tag{3.3}
$$

Using Definition 2.11, (3.1) and (3.3), we get

$$
\begin{aligned}
d(y_n, p) &= d(W(\mathfrak{G}\nu_n, 0, 0)), \rho) \\
&= d(\mathfrak{G}\nu_n, \rho) \\
&\leq d(\nu_n, \rho) \\
&= d(W(z_n, \mathfrak{G}z_n, \beta_n), \rho) \\
&\leq (1-\beta_n)d(z_n, \rho) + \beta_n d(\mathfrak{G}z_n, \rho) \\
&\leq (1-\beta_n)d(z_n, \rho) + \beta_n d(z_n, \rho) \\
&\leq (1-\beta_n)d(x_n, \rho) + \beta_n d(x_n, \rho) \\
&\leq d(x_n, \rho).
\end{aligned} \tag{3.4}
$$

Using Definition 2.11, (3.1), (3.3) and (3.4), we get

$$
\begin{aligned}
d(x_{n+1}, p) &= d(W(\mathfrak{G}\rho_n, 0, 0)), \rho) \\
&= d(\mathfrak{G}\rho_n, \rho) \\
&\leq d(\rho_n, \rho) \\
&= d(W(y_n, \mathfrak{G}y_n, \gamma_n), \rho) \\
&\leq (1-\gamma_n)d(y_n, \rho) + \gamma_n d(\mathfrak{G}y_n, \rho)
\end{aligned} \tag{3.5}
$$

$$\leq (1-\gamma_n)d(y_n,\rho)+\gamma_n d(y_n,\rho)$$

$$\leq (1-\gamma_n)d(x_n,\rho)+\gamma_n d(x_n,\rho)$$

$$\leq d(x_n,\rho).$$

Hence, $\{x_n\}$ is Fejér monotone with respect to $F(\mathfrak{G})$. □

Theorem 3.2. *Let K be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and $\mathfrak{G}$ be a Suzuki's generalized nonexpansive mapping on K. If $\{x_n\}$ is a sequence defined by (3.1), then $F(\mathfrak{G})$ is nonempty if and only if the sequence $\{x_n\}$ is bounded and* $\lim_{n\to\infty} d(x_n,\mathfrak{G}x_n)=0$.

Proof. Assume that $F(\mathfrak{G})$ is nonempty and let $\rho\in F(\mathfrak{G})$. From Theorem 3.1 and Proposition 2.10, we have $\{x_n\}$ is Fejér monotone with respect to $F(\mathfrak{G})$ and bounded such that $\lim_{n\to\infty} D((x_n,F(\mathfrak{G}))$ exists, let $\lim_{n\to\infty} d(x_n,\rho)=l$.

Case I. Let $l=0$. Then

$$d(x_n,\mathfrak{G}x_n)\leq d(x_n,\rho)+d(\rho,\mathfrak{G}x_n),$$

from Definition 2.11,

$$d(x_n,\mathfrak{G}x_n)\leq 2d(x_n,\rho).$$

On taking limit as $n\to\infty$ both sides of the inequality,

$$\lim_{n\to\infty} d(x_n,\mathfrak{G}x_n)=0.$$

Case II. Let $l>0$. Then, since K is a Suzuki's generalized nonexpansive mapping, by Definition 2.11, for $\rho\in F(\mathfrak{G})$,

$$d(\mathfrak{G}x_n,\rho)\leq d(x_n,\rho).$$

On taking $\limsup$ as $n\to\infty$ both sides of the inequality,

$$\limsup_{n\to\infty} d(\mathfrak{G}x_n,\rho)\leq l.$$

On taking $\limsup$ as $n\to\infty$ both sides of the (3.4),

$$\limsup_{n\to\infty} d(z_n,\rho)\leq l. \tag{3.6}$$

From (3.5),

$$
\begin{aligned}
d(x_{n+1},p) &= d(W(\mathfrak{G}\rho_n,0,0)),\rho) \\
&= d(\mathfrak{G}\rho_n,\rho) \\
&\leq d(\rho_n,\rho) \\
&= d(W(y_n,\mathfrak{G}y_n,\gamma_n),\rho) \\
&\leq (1-\gamma_n)d(y_n,\rho)+\gamma_n d(\mathfrak{G}y_n,\rho) \\
&\leq (1-\gamma_n)d(x_n,\rho)+\gamma_n d(y_n,\rho).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
d(x_{n+1},\rho)-d(x_n,\rho) &\leq \gamma_n(d(y_n,\rho)-d(x_n,\rho)) \\
d(x_{n+1},\rho)-d(x_n,\rho) &\leq \frac{d(x_{n+1},\rho)-d(x_n,\rho)}{\gamma_n} \\
&\leq d(y_n,\rho)-d(x_n,\rho)
\end{aligned}
$$

$$
d(x_{n+1},\rho) \leq d(y_n,\rho).
$$

On taking $\limsup$ as $n\to\infty$ both sides of the inequality,

$$
l \leq \liminf_{n\to\infty} d(y_n,\rho). \tag{3.7}
$$

From (3.6) and (3.7),

$$
\lim_{n\to\infty} d(y_n,\rho) = l.
$$

On taking $\limsup$ as $n\to\infty$ in (3.3),

$$
\limsup_{n\to\infty} d(z_n,\rho) \leq l. \tag{3.8}
$$

From (3.5),

$$
\begin{aligned}
d(x_{n+1},p) &= d(W(\mathfrak{G}\rho_n,0,0)),\rho) \\
&= d(\mathfrak{G}\rho_n,\rho) \\
&\leq d(\rho_n,\rho) \\
&= d(W(y_n,\mathfrak{G}y_n,\gamma_n),\rho)
\end{aligned}
$$

$$
\begin{aligned}
&\leq (1-\gamma_n)d(y_n,\rho)+\gamma_n d(\mathfrak{G}y_n,\rho)\\
&\leq (1-\gamma_n)d(z_n,\rho)+\gamma_n d(y_n,\rho)\\
&\leq (1-\gamma_n)d(z_n,\rho)+\gamma_n d(z_n,\rho)\\
&\leq d(z_n,\rho).
\end{aligned}
$$

On taking $\liminf$ as $n\to\infty$ both sides of the inequality,

$$
l \leq \liminf_{n\to\infty} d(z_n,\rho). \tag{3.9}
$$

From (3.8) and (3.9),

$$
\lim_{n\to\infty} d(z_n,\rho) = l.
$$

Therefore, by (3.3)

$$
\begin{aligned}
l &= \limsup_{n\to\infty} d(z_n,\rho)\\
&\leq \limsup_{n\to\infty} d(W(x_n,\mathfrak{G}x_n,\alpha_n),\rho)\\
&\leq \limsup_{n\to\infty} [(1-\alpha_n)d(x_n,\rho)+\alpha_n d(\mathfrak{G}x_n,\rho)]\\
&\leq \limsup_{n\to\infty} [(1-\alpha_n)d(x_n,\rho)+\alpha_n d(x_n,\rho)]\\
&\leq \limsup_{n\to\infty} d(x_n,\rho) = l.
\end{aligned}
$$

By Lemma 2.8, $\lim_{n\to\infty} d(x_n,\mathfrak{G}x_n)=0$.

Conversely, assume that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n,\mathfrak{G}x_n)=0$. Then, from Lemma 2.12, we have $\mathfrak{G}\rho=\rho$, that is, $F(\mathfrak{G})$ is nonempty. □

Theorem 3.3. *Let K be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $\mathfrak{G}\colon K\to K$ be a Suzuki's generalized nonexpansive mapping with $F(\mathfrak{G})\neq\emptyset$. Then the sequence $\{x_n\}$ defined in (3.1), is Δ-convergent to a fixed point of $\mathfrak{G}$.*

Proof. From Theorem 3.1, we observe that $\{x_n\}$ is a bounded sequence, therefore $\{x_n\}$ has a Δ-convergent subsequence. Now we will prove that every Δ-convergent subsequence of $\{x_n\}$ has a unique $\Delta-limit$ in $F(\mathfrak{G})$. For this, let y and z be $\Delta-limit$ of the subsequences $\{y_n\}$ and $\{z_n\}$ of $\{x_n\}$ respectively.

Now by Lemma 2.7, $AC(K,\{y_n\}) = \{y_n\}$ and $AC(K,\{z_n\}) = \{z_n\}$. By Theorem 3.2, we have $\lim_{n\to\infty} d(y_n, \mathfrak{G}y_n) = 0$.

Now we will prove that y and z are fixed points of $\mathfrak{G}$ and they are same. If not, then by the uniqueness of the asymptotic center

$$\begin{aligned}
\limsup_{n\to\infty} d(x_n,y) &= \limsup_{n\to\infty} d(y_n,y) \\
&< \limsup_{n\to\infty} d(y_n,z) \\
&= \limsup_{n\to\infty} d(x_n,z) \\
&= \limsup_{n\to\infty} d(z_n,z) \\
&< \limsup_{n\to\infty} d(z_n,y) \\
&= \limsup_{n\to\infty} d(x_n,y)
\end{aligned}$$

which is a contradiction. Hence $y = z$ and sequence $\{x_n\}$ is Δ-convergent to a unique fixed point of $\mathfrak{G}$. □

Theorem 3.4. *Let K be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and $\mathfrak{G}: K \to K$ be a Suzuki's generalized nonexpansive mapping with $F(\mathfrak{G}) \neq \emptyset$. Then the sequence $\{x_n\}$ which is defined by (3.1), converges strongly to some fixed point of $\mathfrak{G}$ if and only if $\liminf_{n\to\infty} D(x_n, F(\mathfrak{G})) = 0$, where $D(x_n, F(\mathfrak{G})) = \inf_{y\in F(\mathfrak{G})} d(x_n,y)$.*

Proof. Assume that $\{x_n\}$ converges strongly to $y \in F(\mathfrak{G})$. Therefore we have $\lim_{n\to\infty} d(x_n,y) = 0$. Since $0 \le D(x_n,F(\mathfrak{G})) \le d(x_n,y)$, we have

$$\liminf_{n\to\infty} D(x_n,F(\mathfrak{G})) = 0.$$

Next, we prove sufficient part. From Lemma 2.13, the fixed point set $F(\mathfrak{G})$ is closed. Suppose that

$$\liminf_{n\to\infty} D(x_n,F(\mathfrak{G})) = 0.$$

Then, from (3.5), we have

$$D(x_{n+1},F(\mathfrak{G})) \le D(x_n,F(\mathfrak{G})).$$

From Theorem 3.1 and Proposition 2.10, we have $\lim_{n\to\infty} d(x_n, F(\mathfrak{G}))$ exists. Hence

$$\lim_{n\to\infty} D(x_n, F(\mathfrak{G})) = 0.$$

Consider the subsequence $\{x_{n_k}\}$of $\{x_n\}$ such that $d(x_{n_k}, p_k) < \frac{1}{2^k}$ for all $k \geq 1$, where $\{p_k\}$ is in $F(\mathfrak{G})$. From (3.4), we have

$$d(x_{n_{k+1}}, p_k) \leq d(x_{n_k}, p_k) < \frac{1}{2^k},$$

which implies that

$$\begin{aligned} d(p_{k+1}, p_k) &\leq d(p_{k+1}, x_{n_{k+1}}) + d(x_{n_{k+1}}, p_k) \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}}. \end{aligned}$$

This shows that $\{p_k\}$ is a Cauchy sequence. Since $F(\mathfrak{G})$is closed, $\{p_k\}$ is a convergent sequence. Let $\lim_{k\to\infty} p_k = p$. Then we know that $\{x_n\}$ converges to y. Since

$$d(x_{n_k}, y) \leq d(x_{n_k}, p_k) + d(p_k, y),$$

we have

$$\lim_{k\to\infty} d(x_{n_k}, y) = 0.$$

Since $\lim_{n\to\infty} d(x_n, y)$ exists, the sequence $\{x_n\}$ converges to y. □

Recall that a mapping $\mathfrak{G}$ from a subset of a hyperbolic space X into itself with $F(\mathfrak{G}) \neq \emptyset$ is said to satisfy condition (I) if there exists a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0,\ f(t) > 0$ for $t \in (0,\infty)$ such that

$$d(x, \mathfrak{G}x) \geq f(D(x, F(\mathfrak{G}))),$$

for all $x \in K$.

Theorem 3.5. *Let K be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and $\mathfrak{G}\colon K \to K$ be a Suzuki's generalized nonexpansive mapping. Moreover, $\mathfrak{G}$ satisfies the condition (I) with $F(\mathfrak{G}) \neq \emptyset$. Then the sequence $\{x_n\}$ which is defined by (3.1), converges strongly to some fixed point of $\mathfrak{G}$.*

Proof. From Lemma 2.13, we have $F(\mathfrak{G})$ is closed. Observe that by Theorem 3.2, we have $\lim_{n\to\infty} d(x_n, \mathfrak{G}x_n) = 0$. It follows from the condition (I) that

$$\lim_{n\to\infty} f(D(x_n, F(\mathfrak{G}))) \leq \lim_{n\to\infty} d(x_n, \mathfrak{G}x_n).$$

Thus, we get $\lim_{n\to\infty} f(D(x_n, F(\mathfrak{G}))) = 0$. Since $f\colon [0,1) \to [0,1)$ is a nondecreasing mapping with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$, we have $\lim_{n\to\infty} D(x_n, F(\mathfrak{G})) = 0$. Rest of the proof follows in lines of Theorem 3.4. Hence the sequence $\{x_n\}$ is convergent to $p \in F(\mathfrak{G})$. This completes the proof. $\square$

4. NUMERICAL EXAMPLE

Example 4.1. Consider the mapping $\mathfrak{G}\colon [0,1] \to [0,1]$ defined by

$$\mathfrak{G}x = \begin{cases} 1 - x \; if \; x \in [0, \frac{1}{8}), \\ \frac{x+7}{8} \; if \; x \in [\frac{1}{8}, 1]. \end{cases}$$

Hence, $\mathfrak{G}$ is not a nonexpansive mapping but it satisfies condition(C). Using $\alpha_n = \frac{1}{\sqrt{n^3+4}}$, $\beta_n = \frac{2}{\sqrt{n^3+5}}$, $\gamma_n = \frac{3}{\sqrt{n^3+200}}$ in the given example with $x_0 = 0.5$ we get table 1, comparison of the convergence of our iteration process with M iteration, K iteration and Thakur New iteration processes are given, where $x_0 = 0.5$.

We can easily see that the SRJ iteration was the first converging one than the M iteration, the K iteration and the Thakur iteration processes.

Graphical representation is given in Fig.1. Also, We can easily see the efficiency of the New iteration process.

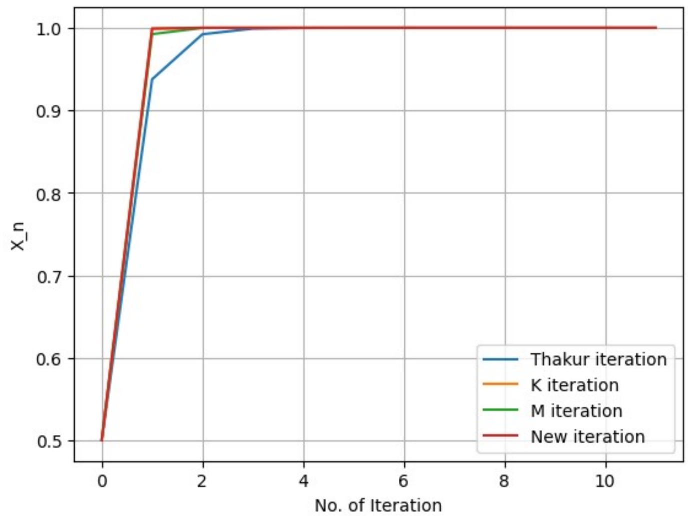

FIGURE 1. Convergence of New iteration, M, K and Thakur New iterations to the fixed point 1.

TABLE 1. Sequence generated by Thakur new, K, M and New iteration

S.No.	Thakur	K	M	New iteration
x0	0.500000000000000	0.500000000000000	0.500000000000000	0.500000000000000
x1	0.992188374608183	0.992242281082824	0.992242281082824	0.999063797455401
x2	0.999877957017476	0.999879635594404	0.999879635594404	0.999998247049591
x3	0.999998093291878	0.999998132493548	0.999998132493548	0.999999996717767
x4	0.999999970211021	0.999999971024820	0.999999971024820	0.999999999993854
x5	0.999999998534599	0.999999999550437	0.999999999550437	0.999999999999988
x6	0.999999999334588	0.999999999993025	0.999999999993025	1
x7	0.999999999134598	0.999999999999892	0.999999999999892	1
x8	0.999999999992729	0.999999999999998	1	1
x9	0.999999999999886	1	1	1
x10	0.999999999999998	1	1	1
x11	1	1	1	1

5. Conclusion

In this work, we present some fixed point results for a Suzuki's generalized nonexpansive mappings and also proposed a new iterative algorithm for approximating the fixed point of this class of mappings in the framework of hyperbolic spaces. Our numerical experiment shows that our iterative algorithm is better compare to some existing iterative algorithms in the literature.

Conflict of Interests

The authors declare that there is no conflict of interests.

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