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COMMON FIXED POINTS FOR WEAKLY COMMUTING MAPPINGS ON A MULTIPLICATIVE B-METRIC SPACE

ROHIT KUMAR VERMA[1], PRACHI SINGH[2], KULESHWARI[2,*]

[1]Department of Mathematics, Govt.Chandulal Chandrakar Arts and Science College Patan, Distt-Durg (C.G.) 491111, India

[2]Department of Mathematics, Govt. V.Y.T. PG Autonomous College, Distt-Durg (C.G.) 491001, India

Abstract. In this paper, we discuss the unique common fixed point of two pair of weakly commuting mappings on a complete multiplicative b-metric space, which satisfy the following inequality:

$$d(Sx,Ty) \leq \left[k\{max\{d(Ax,By),d(Ax,Sx),d(By,Ty),d(Sx,By),d(Ax,Ty)\}\}\right]^{\lambda},$$

where A and S are weak commutative, B and T also are weak commutative. Our result improve and generalize the results of X. He et al. [3].

Keywords: weak commutative mappings; multiplicative metric space; common fixed points.

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1. Introduction

The study for the fixed point of contractive mappings is a famous topic in metric spaces. Fixed point theory is, in fact, a simple, powerful, and useful tool for research area. In addition to an acceptable contraction condition, the metrical common fixed point theorems usually include

*Corresponding author

E-mail address: kuleshwarikamlesh@gmail.com

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constraints on commutativity, continuity, completeness, and appropriate containment of ranges of detailed maps. Since Banach [1] proved the Banach contraction principle in 1922.

Bashirov [2] introduced the usefullness of multiplicative calculus with some interesting applications. With the help of multiplicative absolute value function, they defined the multiplicative distance between two nonnegative real numbers as well as between two positive square matrices. In 1976, Jungck [4] introduced the notion of commuting maps to prove the existence of a common fixed point theorems on a metric space

In 2012, Ozavsar et al.[5] investigate the multiplicative metric space by remarking its topological properties and introduced the concept of multiplicative contraction mapping and some fixed-point theorem of multiplicative, contraction mappings on multiplicative metric space. They recently proved a common fixed-point theorem for four self-mappings in multiplicative metric spaces.

We present some definition and result in common fixed-point theorem for commuting mappings in complete multiplicative b-metric space. For, we have introduced the notion of multiplicative b-metric space.

2. Preliminaries

Definition 2.1. [3] Let X be a nonempty set. A multiplicative metric is a mapping $d : X \times X \to R^+$ satisfying the following conditions:

(i) $d(x,y) \geq 1,\ \forall x,y \in X \ and \ d(x,y) = 1 \ if \ and \ only \ if \ x = y$;

(ii) $d(x,y) = d(y,x),\ \forall x,y \in X$;

(iii) $d(x,y) \leq d(x,z)d(z,y),\ \forall x,y \in X$,

(multiplicative triangle inequality).

We use the following definition for our main result:

Definition 2.2. Let X be a nonempty set. A multiplicative b-metric is a mapping $d : X \times X \to R^+$ satisfying the following conditions:

$[B1]$ $d(x,y) \geq 1$ for all $x,y \in X$ and $d(x,y) = 1$ if and only if $x = y$;

$[B2]$ $d(x,y) = (y,x)$ for all $x,y \in X$;

$[B3]$ $d(x,y) \leq b.d(x,z).d(z,y)$ for all $x,y,z \in X$ (multiplicative triangle inequality), where $b \geq 1$.

Definition 2.3. [3] Let (X,d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$. If for every multiplicative open ball $B_\varepsilon(x) = \{y \mid d(x,y) < \varepsilon\}$, $\varepsilon > 1$, there exists a natural number N such that $n \geq N$, then $x_n \in B(x)$. The sequence $\{x_n\}$ is said to be multiplicative converging to x, denoted by $x_n \to x$ $(n \to \infty)$.

Definition 2.4. [3] Let (X,d) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. The sequence is called a multiplicative Cauchy sequence if it holds that for all $\varepsilon > 1$, there exists $N \in \mathbf{N}$ such that $d(x_n,x_m) < \varepsilon$ for all $m,n > N$.

Definition 2.5. [3] We call a multiplicative metric space complete if every multiplicative Cauchy sequence in it is multiplicative convergence to $x \in X$.

Definition 2.6. [3] Suppose that S,T are two self-mappings of a multiplicative metric space (X,d); S,T are called commutative mappings if it holds that for all $x \in X$, $STx = TSx$.

Definition 2.7. [3] Suppose that S,T are two self-mappings of a multiplicative metric space (X,d); S,T are called weak commutative mappings if it holds that for all $x \in X$, $d(STx,TSx) \leq d(Sx,Tx)$.

Definition 2.8. [3] Let (X,d) be a multiplicative metric space. A mapping $f : X \to X$ is called a multiplicative contraction if there exists a real constant $\lambda \in [0,1)$ such that $d(f(x_1), f(x_2)) \leq d(x_1,x_2)p^\lambda$ for all $x,y \in X$.

3. Main Results

In this section, we prove some common fixed point results for generalized contraction mappings satisfying commutative conditions:

Theorem 3.1. *Let S,T,A and B be self-mappings of a complete multiplicative metric space X; they satisfy the following conditions:*

(i) $SX \subset BX, TX \subset AX$;

(ii) A and S are weak commutative, B and T also are weak commutative;

(iii) One of S, T , A and B is continuous;

(iv) $d(Sx,Ty) \leq [k\{max\{d(Ax,By),d(Ax,Sx),d(By,Ty),d(Sx,By),d(Ax,Ty)\}\}]^{\lambda}$,

$\lambda \in (0,\frac{1}{2})\ \forall x,y \in X$,

where $b \geq 1$ *such that* $\lim_{m,n\to\infty}(kb)^{\frac{h}{1-h}^{(m-n)}} = 1$.

Then S,T,A *and* B *have a unique common fixed point.*

Proof. Since $SX \subset BX,$ *and* $T(X) \subset AX$, for an arbitrary chosen point x_0 in X we obtain x_1 in X. For this $x_1 \in X$, we may obtain $x_2 \in X$; etc. Continuing in this way we obtain a sequence $\{y_n\} \in X$,

$\exists x_2 \in X$ such that $Tx_1 = Ax_2 = y_1,\ldots$;

$\exists x_{2n+1} \in X$ such that $Bx_{2n+1} = y_{2n}$,

$\exists x_{2n+2} \in X$ such that $Tx_{2n+1} = Ax_{2n+2} = y_{2n+1},\ldots$; $\forall n = 0,1,2....\infty$.

define a sequence $\{y_n\} \in X$. Now

putting $x = x_{2n},\ y = x_{2n+1}$ in condition (iv) we obtain

In order to show $\{y_n\}$ a Cauchy sequence, let us put x_{2n} for x, and x_{2n+1} for y in condition (iv), and using (1) we have;

$$
\begin{aligned}
d(y_{2n},y_{2n+1}) &= d(Sx_{2n},Tx_{2n+1}) \\
&\leq [k\ (max\{d(Ax_{2n},Bx_{2n+1}),d(Ax_{2n},Sx_{2n}),d(Bx_{2n+1},Tx_{2n+1}),d(Sx_{2n},Bx_{2n+1}), \\
&\quad d(Ax_{2n},Tx_{2n+1})\}]^{\lambda} \\
&= [k\,(max\,\{d(y_{2n-1},y_{2n}),d(y_{2n-1},y_{2n}),d(y_{2n},y_{2n+1}),(y_{2n},y_{2n}),d(y_{2n-1},y_{2n+1})\})]^{\lambda} \\
&\leq [k\,(max\,\{d(y_{2n-1},y_{2n}),d(y_{2n-1},y_{2n}),d(y_{2n},y_{2n+1}), \\
&\quad 1,d(y_{2n-1},y_{2n})\ .\ d(y_{2n},y_{2n+1})\})]^{\lambda} \\
&\leq [k\,(max\,\{bd(y_{2n-1},y_{2n})\ .\ d(y_{2n},y_{2n+1}),bd(y_{2n-1},y_{2n})\ .d(y_{2n},y_{2n+1}), \\
&\quad bd(y_{2n-1},y_{2n})\ .\ d(y_{2n},y_{2n+1}),1,bd(y_{2n-1},y_{2n})\ .\ d(y_{2n},y_{2n+1})\})]^{\lambda} \\
&= [k\,(max\,\{bd(y_{2n-1},y_{2n})\ .\ d(y_{2n},y_{2n+1})\})]^{\lambda},\ (using\ B1, as\ d(x,y) \geq 1 \forall x \in X) \\
&\leq\ k^{\lambda}b^{\lambda}[d(y_{2n-1},y_{2n})]^{\lambda}\ .\ [d(y_{2n},y_{2n+1})]^{\lambda}
\end{aligned}
$$

$\implies d^{1-\lambda}(y_{2n},y_{2n+1}) \leq \ k^{\lambda}b^{\lambda}\ .\ d^{\lambda}(y_{2n-1},y_{2n})$

$\implies d(y_{2n},y_{2n+1}) \leq \ (kb)^{\frac{\lambda}{1-\lambda}} d^{\frac{\lambda}{1-\lambda}}(y_{2n-1},y_{2n}).$

Let $\frac{\lambda}{1-\lambda} = h,\ where\ \lambda \in (0,\frac{1}{2})\ then$

$d(y_{2n},y_{2n+1}) \leq (kb)^{h} d^{h}(y_{2n-1},y_{2n}).$

Similarly, putting $x = x_{2n+2},\ y = x_{2n+1}$ on (iv), we may obtain

$d(y_{2n+1},y_{2n+2})$

$= d(Sx_{2n+2},Tx_{2n+1})$

$\leq\ [k\max\{d(Ax_{2n+2},Bx_{2n+1}),d(Ax_{2n+2}Sx_{2n+2}),d(Bx_{2n+1},Tx_{2n+1}),d(Sx_{2n+2},Bx_{2n+1}),$

$d(Ax_{2n+2},Tx_{2n+1})\}\}]^{\lambda}$

$\leq [k\,(max\,\{d(y_{2n+1},y_{2n}),d(y_{2n+1},y_{2n+2}),d(y_{2n},y_{2n+1}),d(y_{2n+2},y_{2n}),d(y_{2n+1},y_{2n+1})\})]^{\lambda}$

$\leq [k\,(max\,\{d(y_{2n},y_{2n+1}),d(y_{2n+1},y_{2n+2}),d(y_{2n},y_{2n+1}).d(y_{2n},y_{2n+1}),$

$d(y_{2n+1},y_{2n+2}),1)\})]^{\lambda}$

$\leq [k\,(max\,\{bd(y_{2n},y_{2n+1}).d(y_{2n+1},y_{2n+2}),bd(y_{2n},y_{2n+1}).d(y_{2n+1},y_{2n+2}),bd(y_{2n},y_{2n+1}).$

$d(y_{2n+1},y_{2n+2}),bd(y_{2n},y_{2n+1}).d(y_{2n+1},y_{2n+2}),1)\})]^{\lambda}$

$= [k\,(max\,\{bd(y_{2n},y_{2n+1}).d(y_{2n+1},y_{2n+2})\})]^{\lambda}$

$\leq\ \ k^{\lambda}b^{\lambda}[d(y_{2n},y_{2n+1})]^{\lambda}.[d(y_{2n+1},y_{2n+2})]^{\lambda}.$

This implies that $d^{1-\lambda}(y_{2n+1},y_{2n+2}) \leq\ k^{\lambda}b^{\lambda}\ .\ d^{\lambda}(y_{2n+1},y_{2n})$

$d(y_{2n+1},y_{2n+2}) \leq\ (kb)^{\frac{\lambda}{1-\lambda}} d^{\frac{\lambda}{1-\lambda}}(y_{2n+1},y_{2n}).$

Let $\frac{\lambda}{1-\lambda} = h,\ where\ \lambda \in (0,\frac{1}{2})$ then

$$d(y_{2n},y_{2n+1}) \leq (kb)^{h}.d^{h}(y_{2n-1},y_{2n}), \tag{3.1}$$

$$d(y_{2n+1},y_{2n+2}) \leq ((kb)^{h}.d^{h}(y_{2n},y_{2n+1}). \tag{3.2}$$

From (3.1) and (3.2), we obtain $d(y_n,y_{n+1}) \leq (kb)^{h}d^{h}(y_{n-1},y_n),\ n = 1,2,3,...$ which inductively implies that

$$
\begin{aligned}
d(y_n, y_{n+1}) &\leq (kb)^h[(kb)^h d^h(y_{n-2}, y_{n-1})]^h \\
&= (kb)^{h+h^2}[d^{h^2}(y_{n-2}, y_{n-1})] \\
&\leq (kb)^{h+h^2}[(kb)^h d^h(y_{n-3}, y_{n-2})]^{h^2} \\
&= (kb)^{h+h^2+h^3}[d^{h^3}(y_{n-3}, y_{n-2})] \\
&\vdots \\
&\leq (kb)^{h+h^2+h^3+...+h^n}[d^{h^n}(y_0, y_1)] \\
&\leq (kb)^{\frac{h}{1-h}}[d^{h^n}(y_0, y_1)],\ h+h^2+h^3+...+h^n \leq \frac{h}{1-h}.
\end{aligned}
$$

Let $m, n \in \mathbb{N}$ such that $m \geq n$, then for Cauchy sequence, we get

$$
\begin{aligned}
& d(y_m, y_n) \\
& \leq d(y_m, y_{m-1}).d(y_{m-1}, y_{m-2})...d(y_{n+1}, y_n) \\
& \leq (kb)^{\frac{h}{1-h}} d^{h^{m-1}}(y_0, y_1).(kb)^{\frac{h}{1-h}} d^{h^{m-2}}(y_0, y_1)...(kb)^{\frac{h}{1-h}} d^{h^n}(y_0, y_1)] \\
& \leq \{(kb)^{\frac{h}{1-h}}\}^{(m-n)}\{d^{h^{[(m-1)+(m-2)+...+n]}}(y_0, y_1)\} \\
& = \{(kb)^{\frac{h}{1-h}}\}^{(m-n)}\{d^{h^{(m-n)[(m-1)-\frac{1}{2}(m-n-1)]}}(y_0, y_1)\} \\
& \leq \{(kb)^{\frac{h}{1-h}}\}^{(m-n)} d^{h^{m(m-n)}}(y_0, y_1),\ since\ (m-1)+(m-2)+...+n \leq m(m-n)\ where\ m > n, \\
& = \mathscr{B} d^{h^{m(m-n)}}(y_0, y_1),\ where\ \mathscr{B} = \{(kb)^{\frac{h}{1-h}}\}^{(m-n)} \to 1\ as\ n \to \infty.
\end{aligned}
$$

This implies that $d(y_m, y_n) \to 1$ *as* $m, n \to \infty$. Hence $\{y_n\}$ is a multiplicative Cauchy sequence in X.

By the completeness of X, there exists $z \in X$ such that $y_n \to z$ *as* $n \to \infty$.

We claim that z is a coincidence point of the pair A, S for, putting $x = z$ and $y = x_{2n+1}$ in the inequality (1) we have

Moreover, since

$$\{Sx_{2n}\} = \{Bx_{2n+1}\} = \{y_{2n}\}\ and\ \{Tx_{2n+1}\} = \{Ax_{2n+2}\} = \{y_{2n+1}\},$$

are subsequence of $\{y_n\}$, so we obtain

$$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Ax_{2n+2} = z.$$

Taking condition (ii) and (iii) we obtain following cases;

Case 1: Suppose that A is continuous then

$$\lim_{n\to\infty} ASx_{2n} = \lim_{n\to\infty} A^2x_{2n} = Az.$$

Since A and S are weakly commuting , then

$$d(ASx_{2n}, SAx_{2n}) \leq d(Sx_{2n}, Ax_{2n}).$$

Let $n \to \infty$, we get $\lim_{n\to\infty} d(SAx_{2n}, Az) \leq d(z,z) = 1,\ i.e., \lim_{n\to\infty} SAx_{2n} = Az.$

Putting Ax_{2n} and x_{2n+1}, respectively for x and y in condition (iv) of Theorem 3.1, and using the continuity of A, we respectively obtain,

$$\begin{aligned} d(SAx_{2n}, Tx_{2n+1}) \leq [k\{max\{&d(A^2x_{2n}, Bx_{2n+1}), d(A^2x_{2n}, SAx_{2n}), \\ &d(Bx_{2n+1}, Tx_{2n+1}), d(SAx_{2n}, Bx_{2n+1}), d(A^2x_{2n}, Tx_{2n+1})\}\}]^{\lambda}. \end{aligned}$$

Let $n \to \infty$, we can obtain

$$\begin{aligned} d(Az,z) &\leq [k\{max\{d(Az,z), d(Az,Az), d(z,z), d(Az,z), d(Az,z)\}\}]^{\lambda} \\ &= [k\{max\{d(Az,z),1\}\}]^{\lambda} \\ &= k^{\lambda} d^{\lambda}(Az,z). \end{aligned}$$

This implies that $d(Az,z) = 1, i.e., Az = z.$

Putting $x = z$, and $y = x_{2n+1}$, we obtain

$$\begin{aligned} &d(Sz, Tx_{2n+1}) \\ &\leq [k\{max\{d(Az,,Bx_{2n+1}), d(Az,Sz), d(Bx_{2n+1}, Tx_{2n+1}), d(Sz,Bx_{2n+1}), d(Az, Tx_{2n+1})\}\}]^{\lambda}. \end{aligned}$$

Let $n \to \infty$ we can obtain

$$\begin{aligned} d(Sz,z) &\leq [k\{max\{d(Az,z),d(z,Sz),d(z,z),d(Sz,z),d(z,z)\}\}]^{\lambda} \\ &= [k\{max\{d(Sz,z),1\}\}]^{\lambda} \\ &= k^{\lambda} d^{\lambda}(Sz,z), \end{aligned}$$

which implies that $d(Sz,z) = 1$, i.e., $sz = z$,

$z = Sz \in SX \subseteq BX$, so $\exists z^* \in X$ such that $z = Bz^*$

$$\begin{aligned} d(z,Tz^*) &= d(Sz,Tz^*) \\ &\leq [k\{max\{d(Az,Bz^*),d(Az,Sz),d(Bz^*,Tz^*),d(Sz,Bz^*),d(Az,Tz^*)\}\}^{\lambda} \\ &= [k\{max\{d(z,Tz^*),1\}\}]^{\lambda} \\ &= k^{\lambda} d^{\lambda}(z,Tz^*), \end{aligned}$$

which implies $d(Sz,z) = 1$ i.e., $Tz^* = z$.

Since B and T are weakly commuting mappings then

$$d(Bz,Tz) = d(BTz^*,TBz^*) \leq d(Bz^*,Tz^*) = d(z,z) = 1,$$

so $Bz = Tz$,

$$\begin{aligned} d(Sx_{2n},Tz) \leq [k\{max\{&d(Ax_{2n},Bz),d(Ax_{2n},Sx_{2n}),d(Bz,Tz),d(Sx_{2n},Bz), \\ &d(Ax_{2n},Tz),d^*(Ax_{2n},Bz),d^*(Sx_{2n},Tz)\}\}]^{\lambda}. \end{aligned}$$

$$\begin{aligned} d(z,Tz) &= d(Sz,Tz) \\ &\leq [k\{max\{d(Az,Bz),d(Az,Sz),d(Bz,Sz),d(Sz,Bz),d(Az,Tz)\}\}]^{\lambda} \\ &= [k\{max\{d(z,Tz),1\}\}]^{\lambda} \\ &= k^{\lambda} d^{\lambda}(z,Tz), \end{aligned}$$

which implies $d(Tz,z) = 1$ i.e., $Tz = z$.

Case 2: Suppose that B is continuous , we can obtain the same result by the way of case 1.

Case 3: Suppose that S is continuous then $\lim_{n\to\infty} SAx_{2n} = \lim_{n\to\infty} S^2x_{2n} = Sz$.

Since A and S are weak commutative, then $d(ASx_{2n}, SAx_{2n}) \leq d(Sx_{2n}, Ax_{2n})$.

Let $n \to \infty$ then $\lim_{n\to\infty} (ASx_{2n}, Sz) \leq d(z,z) = 1,\ i.e.,\ \lim_{n\to\infty} ASx_{2n} = Sz$,

$$d(S^2x_{2n}, Tx_{2n+1}) \leq [k\{max\{d(ASx_{2n}, Bx_{2n+1}), d(ASx_{2n}, S^2x_{2n}), d(Bx_{2n+1}, Tx_{2n+1}),$$
$$d(S^2x_{2n}, Bx_{2n+1}), d(ASx_{2n}, Tx_{2n+1})\}\}]^{\lambda}.$$

Let $n \to \infty$ we can obtain

$$\begin{aligned} d(Sz,z) &\leq [k\{max\{d(Sz,z), d(Sz,Sz), d(z,z), d(Sz,z), d(Sz,z)\}\}]^{\lambda} \\ &= [k\{max\{d(Sz,z), 1\}\}]^{\lambda} \\ &= k^{\lambda} d^{\lambda}(Sz,z), \end{aligned}$$

which implies $d(Sz,z) = 1$ i.e., $Sz = z$. Now

$z = Sz \in SX \subseteq BX,\ so\ \exists z^* \in X$ such that $z = Bz^*$

$$d(S^2x_{2n}, Tz^*)$$
$$\leq [k\{max\{d(ASx_{2n}, Bz^*), d(ASx_{2n}, S^2x_{2n}), d(Bz^*, Tz^*), d(S^2x_{2n}, Bz^*), d(ASx_{2n}, Tz^*)\}\}]^{\lambda}.$$

Letting $n \to \infty$ using $z = Sz = Bz^*$, we can obtain

$$\begin{aligned} d(z, Tz^*) &= d(Sz, Tz^*) \\ &\leq [k\{max\{d(Sz,z), d(Sz,Sz), d(z,Tz^*), d(Sz,z), d(sz,Tz^*)\}\}]^{\lambda} \\ &= [k\{max\{d(z,Tz^*), 1\}\}]^{\lambda} \\ &= k^{\lambda} d^{\lambda}(z, Tz^*), \end{aligned}$$

which implies that $d(z,Tz^*) = 1,\ i.e.,\ Tz^* = z$.

Since T and B are weak commutative, then

$d(Tz, Bz) = d(TBz^*, BTz^*) \leq d(Tz^*, Bz^*) = d(z,z) = 1$, so $Bz = Tz$,

z is a fixed point of T. For, we have on using condition (iv).

$$d(Sx_{2n}, Tz) \leq [k\{max\{d(Ax_{2n}, Bz), d(Ax_{2n}, Sx_{2n}), d(Bz, Tz), d(Sx_{2n}, Bz), d(Ax_{2n}, Tz)\}\}]^{\lambda}.$$

Let $n \to \infty$ we can obtain

$$\begin{aligned} d(z,Tz) &\leq [k\{max\{d(z,Tz), d(z,z), d(Tz,Tz), d(z,Tz), d(z,Tz)\}\}]^{\lambda} \\ &= [k\{max\{d(z,Tz), 1\}\}]^{\lambda} \\ &= k^{\lambda} d^{\lambda}(z,Tz), \end{aligned}$$

which implies $d(z,Tz) = 1$ i.e., $Tz = z$.

$z = Tz \in TX \subseteq AX$, so $\exists z^{**} \in X$, such that $z = Az^{**}$

$$\begin{aligned} d(Sz^{**}, z) &= d(Sz^{**}, Tz) \\ &\leq [k\{max\{d(Az^{**}, Bz), d(Az^{**}, Sz^{**}), d(Bz, Tz), d(Sz^{**}, Bz), d(Az^{**}, Tz)\}\}]^{\lambda} \\ &= [k\{max\{d(z,z), d(z, Sz^{**}), d(Bz, Bz), d(Sz^{**}, z), d(z,z)\}\}]^{\lambda} \\ &= [k\{max\{d(Sz^{**}, z), 1\}\}]^{\lambda} \\ &= k^{\lambda} d^{\lambda}(Sz^{**}, z). \end{aligned}$$

This implies that $d(Sz^{**}, z) = 1$ i.e., $Sz^{**} = z$.

Since S and A are weak commutative, then

$d(Az, Sz) = d(ASz^{**}, SAz^{**}) \leq d(Az^{**}, Sz^{**}) = d(z,z) = 1$, so $Az = Sz$.

We obtain $Sz = Tz = Az = Bz = z$, so z is common fixed point of S, T, A and B.

Case 4: Suppose that T is continuous, we can obtain the same result by the way of case 3. In addition we prove that S, T, A and B have a unique common fixed point. suppose that $w \in X$ is also a common fixed point of S, T, A and B then we obtain

$$\begin{aligned} d(z,w) &= d(Sz, Tw) \\ &\leq [k\{max\{d(Az, Bw), d(Az, Sz), d(Bw, Tw), d(Sz, Bw), d(Az, Tw)\}\}]^{\lambda} \\ &= [k\{max\{d(z,w), 1\}\}]^{\lambda} \\ &= k^{\lambda} d^{\lambda}(z,w). \end{aligned}$$

This is a contradiction as $d(z,w) > 1$, when $z \neq w$.

Thus z is a unique common fixed point of $A, B, S, T \subset X$. $\square$

Corollary 3.2. *Let T be a mappings of a complete multiplicative metric space (X,d) into itself satisfying the following condition:*

$$d(Tx,Ty) \leq [d(x,y)]^{\lambda}$$

for all $x,y \in X$, where $\lambda \in (0,\frac{1}{2})$.

Corollary 3.3. *Let S,T,A and B be self-mappings of a complete multiplicative metric space X; they satisfy the following conditions:*

(i) $SX \subset BX, TX \subset AX$;

(ii) A and S are weak commutative, B and T also are weak commutative;

(iii) One of S, T , A and B is continuous;

(iv) $d(S^p x, T^q y) \leq [k\{max\{d(Ax,By), d(Ax,S^p x), d(By,T^q y), d(S^p x,By), d(Ax,T^q y)\}\}]^{\lambda}$,

$\lambda \in (0,\frac{1}{2})$ $\forall x,y \in X$,

where $b \geq 1$.

Then S,T,A and B have a unique common fixed point.

Conflict of Interests

The authors declare that there is no conflict of interests.

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