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COMMON FIXED POINT THEOREM IN S-MULTIPLICATIVE METRIC SPACES

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Abstract. The aim of this paper is to prove a unique common fixed point theorem in S multiplicative metric space for four self mappings.

Keywords: multiplicative metric spaces; S-metric space; weakly compatible mappings; property E.A.

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1. Introduction

Research on common fixed points of mappings that satisfy specific contractive requirements has been continuing and extremely active. The idea of the presence of weakly compatible mappings was initially introduced by Jungck and Rhoades [2] in 1988. They further weakened the concept of compatibility by ignoring the continuity of the mappings involved in the metric spaces. In order to prove common fixed point theorems and to broaden the concepts of noncompatible mappings in metric spaces, Aamri and Moutawakil[3] introduced a new concept of the property E.A. in 2002. Several mathematicians since developed a number of widely accepted fixed point theorems for contraction mappings in metric spaces applying various concepts, such

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as compatible mappings, weakly compatible mappings, and property E.A.In this paper we prove a unique common fixed point in S-multiplicative metric space.

2. PRELIMINARIES

Definition 2.1. [3] *Let $\mathscr{X}$ be a non empty set and a function $d : \mathscr{X}^2 \to [0,\infty)$ completing the resulting requirements :*

$(i) d(a,b) \geq 0 \; \forall \; a,b \in \mathscr{X}$;

$(ii) d(a,b) = 0 \; iff \; a = b \; \forall \; a,b \in \mathscr{X}$;

$(iii) d(a,b) = d(b,a), \forall a,b \in \mathscr{X}$*;*

$(iv) d(a,b) \geq d(a,c) + d(c,b) \; \forall \; a,b,c \in \mathscr{X}$.

d is called a metric on $\mathscr{X}$ and ($\mathscr{X}$, d) is called a rectangular metric space.

Definition 2.2. [6] *Let $\mathscr{X}$ be a nonempty set and $\overline{S} : \mathscr{X}^3 \to R^+$, a function completing the resulting requirements :*

$(i) \mathscr{S}(a,b,c) = 0 \; iff \; a = b = c$;

$(ii) \mathscr{S}(a,b,c) \leq \mathscr{S}(a,a,d) + \mathscr{S}(b,b,d) + \mathscr{S}(c,c,d) \; \forall \; a,b,c,d \in \mathscr{X}$ *(rectangle inequality).*

Then, $(\mathscr{X}, \mathscr{S})$ is called a $\mathscr{S}$-metric-metric space.

The following is the definition of $\mathscr{S}_p$-metric spaces, a generalization of both $\mathscr{S}$-metric spaces and $\mathscr{S}_b$-metric spaces.

Definition 2.3. [5] *For a non empty set $\mathscr{X}$ and $S : \mathscr{X}^3 \to R+$, a function with a strictly increasing continuous function, $\Omega : [0,\infty) \to [0,\infty)$ such that $t \leq \Omega(t)$ for all $t > 0$ and $\Omega(0) = 0$, completing the resulting requirements :*

$(i) \overline{\mathscr{S}}(a,b,c) = 0 \; iff \; a = b = c$;

$(ii) \overline{\mathscr{S}}(a,b,c) \leq \Omega(S(a,a,d) + S(b,b,d) + S(c,c,d)) \; \forall \; a,b,c,d \in \mathscr{X}$ *(rectangle inequality).*

Then, $(\mathscr{X}, \mathscr{S})$ is called an $\mathscr{S}_p$-metric-metric space.

(i) If $\Omega(c) = c$, $\mathscr{S}_p$-metric space reduces to $\mathscr{S}$-metric space.

(ii) If $\Omega(c) = bc, \mathscr{S}_p$-metric space reduces to $\mathscr{S}_b$-metric space.

In 2008, Bashirov et al. [2], introduced multiplicative metric spaces in the following way.

Definition 2.4. [2] *For a non empty set $\mathscr{X}$ and a function $d : \mathscr{X}^2 \to [0,\infty)$ completing the resulting requirements :*

$(i) d(a,b) \geq 1 \; \forall \; a,b \in \mathscr{X}$;

$(ii) d(a,b) = 1 \; iff \; a = b \; \forall \; a,b \in \mathscr{X}$;

$(iii) d(a,b) = d(b,a), \; \forall \; a,b \in \mathscr{X}$;

$(iv) d(a,b) \leq d(a,c).d(c,b), \; \forall \; a,b,c \in \mathscr{X}$.

d is called a multiplicative metric on $\mathscr{X}$ and $(\mathscr{X},d)$ is called a multiplicative metric space. By taking logarithms of (iv), the multiplicative metric space is equivalent to the standard metric space.

Definition 2.5. [7] *For a non empty set $\mathscr{X}$ and a function $S : \mathscr{X}^3 \to [0,\infty)$ is said to be an $\mathscr{S}$-metric on $\mathscr{X}$, if for each $a,b,c,d \in \mathscr{X}$,*

$1. \mathscr{X}(a,b,c) \geq 0$;

$2. \mathscr{X}(a,b,c) = 0 \; iff \; a = b = c$;

$3. \mathscr{X}(a,b,c) \leq \mathscr{S}(a,a,d) + \mathscr{S}(b,b,d) + \mathscr{S}(c,c,d)$

The pair $(\mathscr{X},\mathscr{S})$ is called an $\mathscr{S}$-metric space.

Definition 2.6. [7] *Let $(\mathscr{X},\mathscr{S})$ be an $\mathscr{S}$-metric space and $A \subset \mathscr{X}$.*

(1.) A sequence a_n in $\mathscr{S}$ converges to a if $\mathscr{S}(a_n,a_n,a) \to 0$ as $n \to \infty$, that is for every $\varepsilon > 0$ there exists $n_0 \in N$ such that for $n \geq n_0$,$\mathscr{S}(a_n,a_n,a) < \varepsilon$. We denote this by $\lim_{n\to\infty} a_n = a$ and we say that a is the limit of a_n in$\mathscr{X}$.

(2.) A sequence a_n in $\mathscr{X}$ is said to be Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in N$ such that $\mathscr{S}(a_n,a_n,a_m) < \varepsilon$ for each $n,m \geq n_0$.

(3.) The $\mathscr{S}$-metric space $(\mathscr{X},\mathscr{S})$ is said to be complete if every Cauchy sequence is convergent.

Lemma 2.1. [7] *Let $(\mathscr{X},\mathscr{S})$ be an $\mathscr{X}$-metric space. Then, we have $\mathscr{S}(a,a,b) = \mathscr{S}(b,b,a), \; a, \; b \in \mathscr{X}$.*

Lemma 2.2. [7] *Let $(\mathscr{X},\mathscr{S})$ be an $\mathscr{S}$-metric space. If there exist sequences a_n and a_n such that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then $\lim_{n\to\infty} \mathscr{S}(a_n,a_n,b_n) = \mathscr{S}(a,a,b)$.*

Definition 2.7. [4] *Let $(\mathscr{X},\mathscr{S})$ be an $\mathscr{S}$-metric space and f and g be two self mappings on $\mathscr{X}$. If f and g commute at coincidence points, they are considered weakly compatible.*

Definition 2.8. [1] *Let*$(X\mathscr{X},\mathscr{S})$ *be an* $\mathscr{X}$*-metric space and* f *and* g *be two self mappings on* $\mathscr{X}$*.It is then claimed that* f *and* g *fulfill the property E.A. if there exists a sequence* a_n *in* $\mathscr{X}$ *such that* $\lim_{n\to\infty} fa_n = \lim_{n\to\infty} ga_n = t\ \forall\ t \in \mathscr{X}$.

3. Main Result

Theorem 3.1. *Let* $\mathscr{X}$ *be an* $\mathscr{S}$*-multiplicative metric space and* $A,B,f,g:\mathscr{X}\to\mathscr{X}$ *be four self mapping such that*

(1) $A(\mathscr{X})\subseteq g(\mathscr{X})$ *and* $B(\mathscr{X})\subseteq f(\mathscr{X})$.
(2) $(\mathscr{A}, f)$ *and* $(\mathscr{B}, g)$ *satisfies property E.A.*
(3) $\overline{\mathscr{S}}(\mathscr{A}a,\mathscr{A}a,\mathscr{B}b)\leq\phi[max\{\overline{\mathscr{S}}(fa,fa,gb),\overline{\mathscr{S}}(fa,fa,\mathscr{B}b)\}^k]$.
(4) *One of* $A(\mathscr{X}),B(\mathscr{X}),f(\mathscr{X})$ *and* $g(\mathscr{X})$ *is complete subset of* $\mathscr{X}$.
(5) $If(\mathscr{A},f)$ *and* $(\mathscr{B},g)$ *are weakly compatible. Then* $\mathscr{A},\mathscr{B},f$ *and ghave a unique common fixed point in* $\mathscr{X}$.

Proof. Suppose X be an $\mathscr{X}$-multiplicative metric space and $\mathscr{A},\mathscr{B},f,g:\mathscr{X}\to\mathscr{X}$ be four self mapping satisfying above (5) conditions. since $(\mathscr{B},g)$ satisfies the property E.A., then there exists a sequence $a_n in\mathscr{X}$, such that

$$\lim_{n\to\infty}\mathscr{B}a_n=\lim_{n\to\infty}ga_n=t\ for\ some\ t\in\mathscr{X}. \tag{3.1}$$

Since $B(\mathscr{X})\subseteq f(\mathscr{X})$, then there exists a sequence $b_n in\mathscr{X}$, such that
$\mathscr{B}a_n = fb_n$.
so,

$$\lim_{n\to\infty}fb_n=\lim_{n\to\infty}\mathscr{B}a_n=t \tag{3.2}$$

from equation ((3.1) and (3.2))

$$\lim_{n\to\infty}fb_n=\lim_{n\to\infty}\mathscr{B}a_n=\lim_{n\to\infty}ga_n=t \tag{3.3}$$

but from condition (2) in theorem, $(\mathscr{A},f)$ satisfies the property E.A.,
so,

$$\lim_{\mathscr{A}}b_n=\lim_{n\to\infty}fb_n=\lim_{n\to\infty}\mathscr{B}a_n=\lim_{n\to\infty}ga_n=t \tag{3.4}$$

Suppose that $f(\mathscr{X})$ is a complete subspace of $\mathscr{X}$. Then $t = fu$ for some $u \in \mathscr{X}$, Now we will show that $\mathscr{B}u = fu = t$, from condition (3), we have

$$\overline{S}(\mathscr{A}a, \mathscr{A}a, \mathscr{B}b) \leq \varphi[max\{\overline{\mathscr{S}}(fa, fa, gb), \overline{\mathscr{S}}(fa, fa, \mathscr{B}b)\}^k]$$

$$\overline{S}(\mathscr{A}u, \mathscr{A}u, \mathscr{B}a_n) \leq \varphi[max\{\overline{S}(fu, fu, ga_n), \overline{\mathscr{S}}(fu, fu, \mathscr{B}a_n)\}^k]$$

$$taking\ n \to \infty,$$

$$\overline{\mathscr{S}}(\mathscr{A}u, \mathscr{A}u, t) \leq \varphi[max\{\overline{\mathscr{S}}(fu, fu, t), \overline{\mathscr{S}}(fu, fu, t)\}^k]$$

$$\begin{aligned}\overline{\mathscr{S}}(\mathscr{A}u, \mathscr{A}u, fu) &\leq \varphi[max\{\overline{\mathscr{S}}(t,t,t), \overline{\mathscr{S}}(t,t,t)\}^k] \\ &\leq \varphi[max\{1,1\}^k] \\ &\leq \varphi[max\{1^k\}] \\ &\leq \varphi[max 1] \\ &\leq \varphi(1) \\ &\leq 1,\end{aligned}$$

it is contradictory. Hence $\mathscr{A}$u = fu

so,

$$\mathscr{A}u = fu = t, \tag{3.5}$$

since ($\mathscr{A}$, f) is weakly compatible so,

$$\mathscr{A}\mathscr{A}u = \mathscr{A}fu = f\mathscr{A}u = ffu. \tag{3.6}$$

Since $\mathscr{A}(\mathscr{X}) \subseteq g(\mathscr{X})$.

So, there exists $v \in \mathscr{X}$, such that

$$\mathscr{A}u = gv \tag{3.7}$$

Now we will prove that $gv = Bv$ from condition (3),

$$\overline{\mathscr{S}}(\mathscr{A}a, \mathscr{A}a, \mathscr{B}b) \leq \varphi[max\{\overline{\mathscr{S}}(fa, fa, gb), \overline{\mathscr{S}}(fa, fa, Bb)\}^k]$$

$$\overline{\mathscr{S}}(\mathscr{A}u, \mathscr{A}u, \mathscr{B}v) \leq \varphi[max\{\overline{\mathscr{S}}(fu, fu, gv), \overline{\mathscr{S}}(fu, fu, Bv)\}^k]$$

$$\leq \varphi[max\{\overline{\mathscr{S}}(fu,fu,fu),\overline{\mathscr{S}}(fu,fu,Bv)\}^k]$$

$$\leq \varphi[max\{1,\overline{\mathscr{S}}(\mathscr{A}u,\mathscr{A}u,\mathscr{B}v)\}^k]$$

Case 1 : If max =1, then

$$\overline{\mathscr{S}}(\mathscr{A}u,\mathscr{A}u,\mathscr{B}v) \leq \varphi(1)$$

$$\overline{\mathscr{S}}(\mathscr{A}u,\mathscr{A}u,\mathscr{B}v) \leq \varphi(1)$$

$$\leq 1, \ which\ is\ contradiction.\ So,\ \mathscr{A}u = \mathscr{B}v\ implies\ gv = \mathscr{B}v$$

Case 2 : If max $=\overline{\mathscr{S}}(\mathscr{A}u,\mathscr{A}u,\mathscr{B}v)$

then,

$$\overline{\mathscr{S}}(\mathscr{A}u,\mathscr{A}u,\mathscr{B}v) \leq \varphi[\overline{\mathscr{S}}(\mathscr{A}u,\mathscr{A}u,\mathscr{B}v)^k]$$

$$\overline{\mathscr{S}}(\mathscr{A}u,\mathscr{A}u,\mathscr{B}v) < [\overline{\mathscr{S}}(\mathscr{A}u,\mathscr{A}u,\mathscr{B}v)],$$

it is contradictory. So, $\mathscr{A}$u = $\mathscr{A}$v and hence gv =$\mathscr{A}$v

clearly, $\mathscr{A}$u = fu = gv = $\mathscr{B}$v. If$\mathscr{A}$ and g are weakly compatible, then

$\mathscr{B}\mathscr{B}v = \mathscr{B}gv = g\mathscr{B}v = ggv$

Suppose $\mathscr{A}\mathscr{A}u \neq \mathscr{A}$u, By condition (3),

$$\overline{\mathscr{S}}(\mathscr{A}x,\mathscr{A}x,\mathscr{B}y) \leq \varphi[max\{\overline{\mathscr{S}}(fx,fx,gy),\overline{\mathscr{S}}(fx,fx,By)\}^k]$$

$$\overline{\mathscr{S}}(\mathscr{A}\mathscr{A}u,\mathscr{A}\mathscr{A}u,\mathscr{A}u) \leq \varphi[max\{\overline{\mathscr{S}}(\mathscr{A}\mathscr{A}u,\mathscr{A}\mathscr{A}u,\mathscr{B}v)$$

$$\leq \varphi[max\{\overline{\mathscr{S}}(f\mathscr{A}u,f\mathscr{A}u,gv),\overline{\mathscr{S}}(\mathscr{A}\mathscr{A}u,\mathscr{A}\mathscr{A}u,\mathscr{B}v)\}^k]$$

$$\leq \varphi[\overline{\mathscr{S}}(\mathscr{A}\mathscr{A}u,\mathscr{A}\mathscr{A}u,\mathscr{B}v)\}^k]$$

$$< \overline{\mathscr{S}}(\mathscr{A}\mathscr{A}u,\mathscr{A}\mathscr{A}u,\mathscr{B}v)$$

it is contradictory, hence $\mathscr{A}\mathscr{A}u = \mathscr{A}u$

so, $\mathscr{A}\mathscr{A}$u = f$\mathscr{A}$u =$\mathscr{A}$u

which implies $\mathscr{A}u = u$ and $fu = u$.

Hence $\mathscr{A}u$ is common fixed point of $\mathscr{A}$ and f.

Similarly, $\mathscr{B}v$ is common fixed point of $\mathscr{A}$ and g. But $\mathscr{A}$u = $\mathscr{B}$v. So, $\mathscr{A}$, $\mathscr{B}$, f and g have common fixed point $\mathscr{A}$u. □

Uniqueness of common fixed point - Let l and m are two common fixed points of $\mathscr{A}$,$\mathscr{A}$, f and g.

So,

$$\begin{aligned}
\overline{\mathscr{S}}(l,l,m) &\leq \{\overline{\mathscr{S}}(Al,Al,Bm) \\
&\leq \varphi[max\{\overline{\mathscr{S}}(fl,fl,gm),\overline{\mathscr{S}}(fl,fl,Bm)\}^k] \\
&\leq \varphi[\overline{\mathscr{S}}(l,l,m),(l,l,m)\}^k] \\
&< \varphi[\overline{\mathscr{S}}(l,l,m)]^k \\
&< \overline{\mathscr{S}}(l,l,m)^k \\
&< \overline{\mathscr{S}}(l,l,m)
\end{aligned}$$

which is contradiction, hence $l = m$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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