Available online at http://scik.org
Eng. Math. Lett. 2025, 2025:3
https://doi.org/10.28919/eml/9045
ISSN: 2049-9337

THE GENERALIZED INTERPOLATIVE KANNAN TYPE CONTRACTIONS COMMON FIXED POINTS FOR TWO MAPPINGS

ROHIT KUMAR VERMA[1], JAYNENDRA SHRIVAS[2,*], PRACHI SINGH[2], KULESHWARI[2]

[1]Department of Mathematics, Govt.Chandulal Chandrakar Arts and Science College Patan, Durg (C.G.) 491111, India

[2]Department of Mathematics, Govt. V.Y.T. PG Autonomous College, Durg (C.G.) 491001, India

Abstract. In this paper, we establish some common fixed point theorems in generalized interpolation for two mappings and generalize the Kannan type contraction.

Keywords: Kannan type contraction; interpolation; fixed point; rate of convergence.

2010 AMS Subject Classification: 49T99, 46N40, 47H10, 54H25.

1. INTRODUCTION/PRELIMINARIES

It is well known that fixed point theory played a central role in various scientific fields. The well-known result in this area is undoubtedly the famous Banach contraction principle (see [1]) which motivated researchers to find other forms of contractions. In this line, we cite the well-known Kannan contraction that does not require continuous mapping.

Definition 1.1. [2] *Let* (X,d) *be a metric space. A self-mapping on* $\mathfrak{T}\colon X \to X$ *is said to be a Kannan contraction if there exists* $\mu \in [0,1/2]$ *such that*

$$d(\mathfrak{T}a,\mathfrak{T}b) \leq \mu(d(a,\mathfrak{T}a)+d(b,\mathfrak{T}b)). \tag{1.1}$$

*Corresponding author
E-mail address: jayshrivas95@email.email

Received November 28, 2024

Kannan obtained the following theorem.

Theorem 1.1. *If (X,d) is a complete metric space, then every Kannan contraction on E has a unique fixed point.*

In 2018, Karapinar [3] published a new type of contraction obtained from the definition of the Kannan contraction by interpolation as follows:

Definition 1.2. [3] *Let (X,d) be a complete metric space. A mapping $\mathfrak{T}\colon X \to X$ is said to be an interpolative Kannan type contraction on X, if there exist $\mu \in [0,1)$ and $\alpha \in (0,1)$ such that*

$$d(\mathfrak{T}a,\mathfrak{T}b) \leq \mu[d(a,\mathfrak{T}a)]^{\alpha}[d(b,\mathfrak{T}b)]^{1-\alpha}, \tag{1.2}$$

for every $a,b \in X \setminus Fix(\mathfrak{T})$, where $Fix(\mathfrak{T}) = \{a \in X | \mathfrak{T}a = a\}$.

Theorem 1.2. [3] *On a complete metric space (X,d), any interpolative Kannan-contraction $\mathfrak{T}\colon X \to X$ has a fixed point.*

Now, we define the generalized interpolative condition in the following way;

Definition 1.3. *Let (X,d) be a complete metric space. A mapping $\mathfrak{T}\colon X \to X$ is said to be an generalized interpolative type contraction on X, if there exist $\mu \in [0,1)$ and $\alpha,\beta \in (0,1)$ such that*

$$d(\mathfrak{T}a,\mathfrak{T}b) \leq \mu[d(a,\mathfrak{T}a)]^{\alpha}[d(b,\mathfrak{T}b)]^{\beta}, \tag{1.3}$$

for every $a,b \in X \setminus Fix(\mathfrak{T})$, where $Fix(\mathfrak{T}) = \{a \in X | \mathfrak{T}a = a\}$.

Theorem 1.3. [4] *Let (X,d) be a complete metric space. A mapping $\mathfrak{T},\mathfrak{S}\colon X \to X$ is said to be an interpolative Kannan type contraction on X, if there exist $\mu \in [0,1)$ and $\alpha \in (0,1)$ such that*

$$d(\mathfrak{T}a,\mathfrak{S}b) \leq \mu[d(a,\mathfrak{T}a)]^{\alpha}[d(b,\mathfrak{S}b)]^{1-\alpha}, \tag{1.4}$$

is satisfied for all $a,b \in X$ such that $\mathfrak{T}a \neq a$ and $\mathfrak{S}b \neq b$. Then $\mathfrak{T}$ and $\mathfrak{S}$ have a unique common fixed point.

Definition 1.4. *Let (X,d) be a complete metric space. $\mathfrak{T},\mathfrak{S}\colon X\to X$ be a self-mappings. Assume that there are some $\mu\in[0,1),\alpha,\beta\in(0,1)$ s.t. the condition*

$$d(\mathfrak{T}a,\mathfrak{S}b)\leq\mu[d^{\alpha}(a,\mathfrak{T}a).d^{\beta}(b,\mathfrak{S}b)] \tag{1.5}$$

is satisfied $\forall\ a,b\in X$ such that $\mathfrak{T}a\neq a$ whenever $\mathfrak{S}b\neq b$. Then $\mathfrak{S}$ and $\mathfrak{T}$ have a unique common fixed point.

2. Main Results

We start this section with the Theorem of generalized interpolative Kannan type contraction for pair of mapping.

Theorem 2.1. *Let (X,d) be a complete metric space. $\mathfrak{T},\mathfrak{S}\colon X\to X$ be a self-mappings. Assume that there are some $\mu\in[0,1),\alpha,\beta\in(0,1)$ s.t. the condition*

$$d(\mathfrak{T}a,\mathfrak{S}b)\leq\mu[d^{\alpha}(a,\mathfrak{T}a).d^{\beta}(b,\mathfrak{S}b)]$$

is satisfied $\forall\ a,b\in X$ such that $\mathfrak{T}a\neq a$ whenever $\mathfrak{S}b\neq b$. Then $\mathfrak{S}$ and $\mathfrak{T}$ have a unique common fixed point.

Proof. Let $a_o\in X$, define the sequence $\{a_n\}_{n=0}^{\infty}$ by

$$a_{2n+1}=\mathfrak{T}a_{2n},a_{2n+2}=\mathfrak{S}a_{2n+1},\forall\ n=\{0,1,2,3,....\}.$$

If $\exists\ n\in\{0,1,2,3,..\}$ s.t. $a_n=a_{n+1}=a_{n+2}$ then a_n is a common fixed point of $\mathfrak{S}$ and $\mathfrak{T}$. Suppose that three consecutive identical terms in the sequence $\{a_n\}_{n=0}^{\infty}$ and that $a_0\neq a_1$. Now, using (1.5), we deduce for $a=a_{2n},b=a_{2n+1}$ that

$$\begin{aligned}d(a_{2n+1},a_{2n+2})&=d(\mathfrak{T}a_{2n},\mathfrak{S}a_{2n+1})\\&\leq\mu.d^{\alpha}(a_{2n},a_{2n+1}).d^{\beta}(a_{2n+1},a_{2n+2})\end{aligned}$$

Thus,

$$d^{1-\beta}(a_{2n+1},a_{2n+2})\leq\mu d^{\alpha}(a_{2n},a_{2n+1})$$

or,

$$d(a_{2n+1},a_{2n+2}) \leq \mu^{\frac{1}{1-\beta}}.d^{\frac{\alpha}{1-\beta}}(a_{2n},a_{2n+1})$$

$$\leq \mu^{\frac{1}{1-\beta}}.d^{\frac{\alpha}{1-\beta}}(a_{2n},a_{2n+1}),\ Since\ \frac{\alpha}{1-\beta} < 1 \tag{2.1}$$

$$\leq \mu d(a_{2n},a_{2n+1}).$$

Hence,

$$d(a_{2n+1},a_{2n+2}) \leq \mu d(a_{2n},a_{2n+1}) \leq \mu^2 d(a_{2n-1},a_{2n}) \leq$$

$$... \leq \mu^k d(a_{2n-2},a_{2n-1}) \leq ... \leq \mu^{2n+1} d(a_0,a_1)$$

or

$$d(a_{2n+1},a_{2n+2}) \leq \mu^{2n+1} d(a_0,a_1), \tag{2.2}$$

similarly, on putting $a = a_{2n}\ and\ b = a_{2n-1}$ we have

$$d(a_{2n+1},a_{2n}) = d(\mathfrak{T}a_{2n},\mathfrak{S}a_{2n-1})$$

$$\leq \mu d^{\alpha}(a_{2n},\mathfrak{T}a_{2n}).d^{\beta}(a_{2n-1},Sa_{2n-1})$$

$$\leq \mu d^{\alpha}(a_{2n},a_{2n+1}).d^{\beta}(a_{2n-1},a_{2n}).$$

Thus

$$d^{1-\alpha}(a_{2n},a_{2n+1}) \leq \mu d^{\beta}(a_{2n-1},a_{2n}),$$

or

$$d(a_{2n},a_{2n+1}) \leq \mu^{\frac{1}{1-\alpha}} d^{\frac{\beta}{1-\alpha}}(a_{2n-1},a_{2n})$$

$$\leq \mu d^{\frac{\beta}{1-\alpha}}(a_{2n-1},a_{2n})$$

$$\leq \mu(a_{2n-1},a_{2n}).$$

Hence,

$$d(a_{2n+1},a_{2n}) \leq \mu.d(a_{2n-1},a_{2n}) \leq \mu^2.d(a_{2n-2},a_{2n-1}) \leq \mu^3 d(a_{2n-3},a_{2n-2}) \leq$$

$$... \leq \mu^{2n} d(a_0,a_1).$$

Thus

$$d(a_{2n+1},a_{2n}) \leq \mu^{2n} d(a_0,a_1). \tag{2.3}$$

Unifying (2.2) and (2.3) we can deduce that

$$d(a_{2n+1},a_{2n}) \leq \mu^n d(a_0,a_1) \tag{2.4}$$

Now using (2.4) we can prove that the sequence $\{a_n\}_{n=0}^{\infty}$ is a Cauchy sequence

Let $m,r \in \{0,1,2,3...\}$,

$$\begin{aligned}
d(a_m,a_{m+r}) &\leq d(a_m,a_{m+1}) + d(a_{m+1},a_{m+2}) + ... + (a_{m+r-1},a_{m+r}) \\
&\leq \mu^m + \mu^{m+1} + ...\mu^{m+r-1} d(a_0,a_1) \\
&\leq (\mu^m + \mu^{m+1} + ...\mu^{m+r-1} + ...)d(a_0,a_1) \\
&= \frac{\mu^m}{1-\mu} d(a_0,a_1).
\end{aligned}$$

Letting $m \to \infty$ we deduce that $\{a_n\}_{n=0}^{\infty}$ is a Cauchy sequence.

As (X,d) is complete, so $\exists u \in X$ $\lim_{n\to\infty} a_n = u$. Using the contrary of the metric in its both variables we may prove that u is a fixed point of $\mathfrak{T}$, as follows:

$$\begin{aligned}
d(\mathfrak{T}u,a_{2n+2}) &= d(\mathfrak{T}u,\mathfrak{S}a_{2n+1}) \\
&\leq \mu.d^{\alpha}(u,\mathfrak{T}u).d^{\beta}(a_{2n+1},a_{2n+2}).
\end{aligned}$$

Letting $n \to \infty$ we get $d(\mathfrak{T}u,u) = 0$ so $(\mathfrak{T}u,u)$.

Similarly,

$$\begin{aligned}
d(a_{2n+1},\mathfrak{S}u) &= d(\mathfrak{T}a_{2n},\mathfrak{S}u) \\
&\leq \mu.d^{\alpha}(a_{2n},a_{2n+1}).d^{\beta}(u,\mathfrak{S}u),
\end{aligned}$$

letting $n \to \infty$ we get $u = \mathfrak{S}u$.

Thus u is a common fixed point of $\mathfrak{S}$ and $\mathfrak{T}$. To prove that u is a unique common fixed point of $\mathfrak{S}$ and $\mathfrak{T}$ suppose that $v \in X$ is another common fixed point of $\mathfrak{S}$ and $\mathfrak{T}$.

Then

$$d(u,v) = d(\mathfrak{T}u,\mathfrak{S}v) \leq \mu d^{\alpha}(u,\mathfrak{T}u).d^{\beta}(v,\mathfrak{S}v) = 0.$$

Hence $u = v$. So $\mathfrak{S},\mathfrak{T} : X \to X$ has a unique common fixed point in X. □

3. Numerical Example

$$d(a,a) = d(b,b) = d(c,c) = d(d,d) = 0$$
$$d(a,b) = d(b,a) = 3$$
$$d(c,a) = d(a,c) = 4$$
$$d(b,c) = d(c,b) = \frac{3}{2}$$
$$d(d,a) = d(a,d) = 0$$
$$d(d,b) = d(b,d) = 4$$
$$d(d,c) = d(c,d) = \frac{3}{2}$$

Define self maps $\mathfrak{T}, \mathfrak{S}$ as follows

$\mathfrak{T}: \begin{pmatrix} a & b & c & d \\ a & d & c & d \end{pmatrix} \mathfrak{S}: \begin{pmatrix} a & b & c & d \\ a & b & d & c \end{pmatrix}$

It is clear that $\mathfrak{T}, \mathfrak{S}$ satisfies (1.5) with $\mu = \frac{9}{10}$ and $\alpha = \frac{1}{2}, \beta = \frac{1}{3}$, and $\mathfrak{T}$ and $\mathfrak{S}$ has unique common fixed point a.

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] S. Banach, Sur Les Opérations Dans Les Ensembles Abstraits et Leur Application Aux Équations Intégrales, Fund. Math. 3 (1922), 133–181.

[2] R. Kannan, Some Results on Fixed Points, Bull. Calcutta Math. Soc. 60 (1968), 71–76.

[3] E. Karapinar, Revisiting the Kannan Type Contractions via Interpolation, Adv. Theory Nonlinear Anal. Appl. 2 (2018), 85–87.

[4] M. Noorwali, Common Fixed Point for Kannan Type Contractions via Interpolation, J. Math. Anal. 9 (2018), 92–94.