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DECOMPOSITION METHOD FOR SOLVING SYSTEM OF LINEAR EQUATIONS

MUHAMMAD ASLAM NOOR[1,*], KHALIDA IANAYAT NOOR[1], AND MUHAMMAD WASEEM[2]

[1]Department of Mathematics, COMSATS Institute of Information Technology, Pak Road, Islamabad, Pakistan

[2]Department of Computer Science, COMSATS Institute of Information Technology, Sahiwal, Pakistan

Abstract. In this paper, we use Adomian decomposition method for suggesting an iterative methods for solving system of linear equations. We provide several numerical examples to verify the theoretical results. Comparsion with other methods show that the results obtained in this paper are better. Our results can be viewed as an improvement and extension of the previously known results.

Keywords: decomposition method; splitting matrix; series solution; auxiliary parameter; initial approximation; Convergence.

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1. Introduction

We consider the system of linear equations of the type

$$AX = b, \tag{0.1}$$

where

$$A=\left[a_{ij}\right],\ X=\left[x_j\right] \text{ and } b=\left[b_j\right],\ i=1,2,\dots,n,\ j=1,2,\dots,n.$$

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*Corresponding author

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A wide class of problems arising in engineering and sciences can be formulated as system of linear equations. See [1-18] and the references therein. Several methods and techniques have been developed for solving system of linear equations (1.1). Babolian et al. [6] have used Adomian decomposition method to derive an iterative method, which is similar to the Jacobi iterative method for solving systems of linear equations. Allahviranloo [4] used Adomian decomposition method for fuzzy system of linear equations. Keramati [10], Liu [12] and Noor et al [17] have used homotopy perturbation to derive some iterative methods for solving system of linear equations. The Adomian decomposition was developed by Adomian [2,3] and has been successfully applied to solve a wide class of linear and nonlinear problems, see [1-6,8,15]. Adomian decomposition method gives the solution as an infinite series usually converging to an accurate solution. Noor [16] has proved the equivalence between the homotopy perturbation method and the Adomian decomposition method. In this paper, we use Adomian decomposition technique to develop some iterative methods for solving systems of linear equations (1.1). These iterative are exactly the same as in Liu [12], which were obtained using homotopy perturbation method. This is the main motivation of this paper. We give several numerical examples to illustrate the efficiency and performance of our results. Results obtained in this paper may be extended for solving nonlinear system of equations, which is another direction for future research.

2. Iterative Methods

For an auxiliary parameter $\hbar \neq 0$, any splitting matrix Q and an auxiliary matrix H, one can decompose the system of linear equation (1.1) as follows:

$$QX + (\hbar HA - Q)X = \hbar Hb. \tag{1.1}$$

Let W_0 be the initial approximation of X. Then (1.1) can be written as:

$$QX = W_0 + [(Q - \hbar HA)X + \hbar Hb - W_0], \tag{1.2}$$

which is written as

$$L(X) = C + M(X), \tag{1.3}$$

where

$$L(X) = Q(X), \tag{1.4}$$

$$C = W_0, \tag{1.5}$$

$$M(X) = [(Q - \hbar HA)X + \hbar Hb - W_0]. \tag{1.6}$$

The main idea of the Adomian decomposition technique is to look for a solution of equation (1.3) having the series form of the type:

$$X = \sum_{k=0}^{\infty} X_k, \tag{1.7}$$

and the operator $M(X)$ is decomposed as:

$$M\left(\sum_{k=0}^{\infty} X_k\right) = M(X) = \sum_{k=0}^{\infty} A_k, \tag{1.8}$$

where A_k are the functions which are known as the Adomian polynomials depend upon $X_0,\ X_1,\ X_2, \ldots$ and are given by the formula

$$A_m = \frac{1}{m!}\frac{d^m}{d\lambda^m}\left[M\left(\sum_{k=0}^{\infty} \lambda^k X_k\right)\right]_{\lambda=0}, \quad m = 0,1,2,\ldots. \tag{1.9}$$

From (1.6) and (1.9), we obtain

$$\begin{cases} A_0 = (Q - \hbar HA)X_0 + \hbar Hb - W_0, \\ A_m = (Q - \hbar HA)X_m, \quad m = 1,2,3,\ldots, \end{cases} \tag{1.10}$$

It follows from (1.3), (1.7) and (1.8) that

$$L\left(\sum_{k=0}^{\infty} X_k\right) = C + \sum_{k=0}^{\infty} A_k. \tag{1.11}$$

From (1.4) and (1.11), we have

$$Q\left(\sum_{k=0}^{\infty} X_k\right) = C + \sum_{k=0}^{\infty} A_k. \tag{1.12}$$

Also from (1.12), we have the following iterative scheme

$$\begin{cases} QX_0 = C, \\ QX_k = A_{k-1}, \quad k = 1,2,3,\ldots. \end{cases} \tag{1.13}$$

Using the Adomian polynomials given by (1.10), we obtain

$$\begin{cases} QX_0 = W_0, \\ QX_1 = (Q - \hbar HA)X_0 + \hbar Hb - W_0, \\ QX_k = (Q - \hbar HA)X_{k-1}, \quad k = 2, 3, 4, \ldots. \end{cases} \tag{1.14}$$

From (1.14), we get

$$\begin{cases} X_0 = Q^{-1}W_0, \\ X_1 = (I - \hbar Q^{-1}HA)X_0 + Q^{-1}(\hbar Hb - W_0), \\ X_k = (I - \hbar Q^{-1}HA)X_{k-1}, \quad k = 2, 3, 4, \ldots. \end{cases} \tag{1.15}$$

Taking initial approximation $W_0 = \hbar Hb,$ we have

$$\begin{cases} X_0 = \hbar(Q^{-1}H)b, \\ X_k = (I - \hbar Q^{-1}HA)^k \hbar(Q^{-1}H)b, \quad k = 1, 2, 3, \ldots. \end{cases} \tag{1.16}$$

Thus, from (1.16), we have the series solution

$$X = \sum_{k=0}^{\infty} X_k = \sum_{k=0}^{\infty} (I - \hbar Q^{-1}HA)^k \hbar(Q^{-1}H)b, \tag{1.17}$$

which is exactly the same series solution obtained by using homotopy perturbation technique [12].

For the convergence analysis of the series (1.17), see Liu [12].

3. Some Special Adomian Methods

We now discuss some special cases, which can be obtained from our results.

3.1. **Adomian Jacobi Method.** Let $Q = D = diag(a_{ii}),\ i = 1, 2, \ldots, n.$ Then, from (1.16), we have

$$\begin{cases} X_0 = \hbar D^{-1}Hb, \\ X_k = (I - \hbar D^{-1}HA)^k \hbar D^{-1}Hb, \quad k = 1, 2, 3, \ldots. \end{cases} \tag{2.1}$$

From (2.1), we have the series solution

$$X = \sum_{k=0}^{\infty} (I - \hbar D^{-1}HA)^k \hbar D^{-1}Hb, \tag{2.2}$$

which is the same as homotopy Jecobi obtained by Liu [12] using homotopy perturbation method.

3.2. Adomiam Gauss-Seidel Method.

Let $Q = D + L,$ where $D = diag(a_{ii}),\ i = 1,2,\ldots,n,$ and

$$L = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ -a_{21} & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ -a_{n1} & \cdots & -a_{n,n-1} & 0 \end{bmatrix}.$$

From (1.16), we have

$$\begin{cases} X_0 = \hbar(D-L)^{-1}Hb, \\ X_k = (I - \hbar(D-L)^{-1}HA)^k \hbar(D-L)^{-1}Hb, \quad k = 1,2,3,\ldots. \end{cases} \tag{2.3}$$

From (2.3), we have the series solution

$$X = \sum_{k=0}^{\infty}(I - \hbar(D-L)^{-1}HA)^k \hbar(D-L)^{-1}Hb, \tag{2.4}$$

which is the same as obtained by Liu [12] using homotopy perturbation method.

4. Numerical Examples

In this section, we present some numerical examples to illustrate the efficiency of the newly developed methods in this paper. We compare our methods with Jacobi method (JC) [7], Gauss-Seidel method (GS) [7], Adomian Jacobi method (AJC) and Adomian Guass-Seidel method (AGS). For each example, we calculate number of iterations, error estimate, spectral radius $\rho(D^{-1}(L+U))$ (for Jacobi method), $\rho((D-L)^{-1}U)$ (for Guass-Seidel method), $\rho(I-\hbar D^{-1}HA)$ (for Adomian Jacobi method) and $\rho(I-\hbar(D-L)^{-1}HA)$ (for Adomian Guass-Seidel method). All computations are done on Matlab and we use $\varepsilon = 10^{-15}$. The following stopping criteria is used for computer programs:

$$\frac{\left\| X^k - X^{[k-1]} \right\|_\infty}{\left\| X^{[k]} \right\|_\infty} < \varepsilon.$$

We consider the following examples to illustrate the efficiency and implementation of these new iterative methods.

Examples 4.1 [7]. Consider the following system of linear equations

$$
\begin{aligned}
&4x_1 - x_2 - x_4 = 0,\\
&-x_1 + 4x_2 - x_3 - x_5 = 5,\\
&-x_2 + 4x_3 - x_6 = 0,\\
&-x_1 + 4x_4 - x_5 = 6,\\
&-x_2 - x_4 + 4x_5 - x_6 = -2,\\
&-x_3 - x_5 + 4x_6 = 6.
\end{aligned}
$$

Example 4.2 [7]. Consider the following system of linear equations $AX = b,$ where the entries of A are

$$
a_{ij=}\begin{cases}
2i, & \text{when } j = i \text{ and } i = 1, 2, \ldots, 80,\\
0.5i, & \text{when } \begin{cases} j = i + 2 \text{ and } i = 1, 2, \ldots, 78,\\ j = i - 2 \text{ and } i = 3, 4, \ldots, 80,\end{cases}\\
0.25i, & \text{when} \begin{cases} j = i + 4 \text{ and } i = 1, 2, \ldots, 76,\\ j = i - 4 \text{ and } i = 5, 6, \ldots, 80,\end{cases}\\
0, & \text{otherwise,}
\end{cases}
$$

and those of b are $b_i = \pi,$ for each $i = 1, 2, \ldots, 80.$

Example 4.3 [7]. Consider the following system of linear equations $AX = b,$ where the entries of A are

$$
a_{ij=}\begin{cases}
4, & \text{when } j = i \text{ and } i = 1, 2, \ldots, 25,\\
-1, & \text{when } \begin{cases} j = i + 1 \text{ and } i = \begin{cases} 1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14,\\ 16, 17, 18, 19, 21, 22, 23, 24,\end{cases}\\ j = i - 1 \text{ and } i = \begin{cases} 2, 3, 4, 5, 7, 8, 9, 10, 12, 13, 14, 15,\\ 17, 18, 19, 20, 22, 23, 24, 25,\end{cases}\\ j = i + 5 \text{ and } i = 1, 2, \ldots, 20,\\ j = i - 5 \text{ and } i = 6, 7, \ldots, 25,\end{cases}\\
0, & \text{otherwise,}
\end{cases}
$$

And $b = (1, 0, -1, 0, 2, 1, 0, -1, 0, 2, 1, 0, -1, 0, 2, 1, 0, -1, 0, 2, 1, 0, -1, 0, 2)^t.$

Table 4.1 (Numerical Examples)

Exp.	Method	ρ	IT	Error
4.1	JC	0.6036	70	5.5511e-016
	GSC	0.3643	36	5.5511e-016
	AJC	0.6036	68	5.8777e-016
	AGS	0.3643	34	6.5445e-016
4.2	JC	0.7457	109	9.0190e-016
	GSC	0.2277	25	5.9976e-016
	AJC	0.7457	107	8.1209e-016
	AGS	0.2277	23	5.7894e-016
4.3	JC	0.8660	220	9.1699e-016
	GSC	0.7500	114	9.8938e-016
	AJC	0.8660	220	8.6332e-016
	AGS	0.7500	113	9.5255e-016

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